Positivity characterization of nonlinear DAEs.

Part II: A flow formula for linear and nonlinear DAEs using projections.

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Abstract We present a closed solution formula for differential-algebraic equations (DAEs) that generalizes the concept of the flow to linear and nonlinear problems of arbitrary index. This flow is stated in the original coordinate system and thus allows to study coordinate depending properties like positivity, in particular. Embedded in the concept of the strangeness-index, we separate the differential and algebraic components by a projection approach and remodel a given DAE as a semi-explicit system. Exploiting the results found in [2], we solve this system and compute a closed solution formula. Verifying that this solution is unique defined by the original DAE and uniquely related with a given consistent initial values, we construct the flow associated with a DAE with regular strangeness-index.

Contents

1 Introduction

The aim of this paper is to construct a closed solution formula of differential-algebraic equations (DAEs) that allows to study coordinate related properties like positivity. We consider general DAEs

$$F(\dot{x}, x, t) = 0, \quad x(t_0) = x_0, \tag{1}$$

where $F \in C(\mathcal{I} \times \Omega_x \times \Omega_{\dot{x}}, \mathbb{R}^n)$, $\mathcal{I} \subset \mathbb{R}$ is a (compact) interval and $\Omega_x, \Omega_{\dot{x}} \subset \mathbb{R}^n$ are open sets. DAEs model dynamical processes that are confined by auxiliary constraints, like connected joints in multibody systems, connections or loops in networks, e.g., electrical circuits, or balance equations and conservation laws in advection-diffusion equations, see e.g. [17, 18, 33, 58]. These constraints are given by algebraic equations and combined with dynamical processes, lead to an involved set of differential-algebraic equations (36) with singular Jacobian $F_{\dot{x}}$. To study system properties like solvability or uniqueness and to improve the numerical performance, the differential and algebraic components must be entangled. Necessarily, this involves differentiation, typically combined with a change of variables to filter out the regular components. The number of times (36) has to be differentiated until all constraints are explicitely given, is known as the *index* of the DAE. There are several index concepts and remodeling approaches, see e.g., [6, 25, 28, 49], [19, 20, 21, 22], [8, 9, 11, 12, 13, 14, 15], [41, 42, 43, 51, 52], [47, 50] and the references therein. We follow the concept of the strangeness-index as it was developed in [34, 35, 36, 37]. Most approaches supply the framework to study system properties like stability or contractivity, cp. [38, 40, 56]. Studying coordinate depend properties like positivity, however, i.e., every solution starting with an entrywise nonnegative initial value stays entrywise nonnegative for all its lifetime we require a decomposition procedure that filters out the differential and algebraic components without changing the coordinate system. Within the framework of the strangeness-index, we pursue a projection approach that remodels a large class of DAEs as semi-explicit system and provides a flow formula for the original coordinates. We have prepared our analysis in [2], where we have studied how to decompose and solve differential and algebraic equations in (flow invariant) subspace.

The outline is as follows. In Section 2, we give our general notation, introduce the basic concepts of projections and the Moore-Penrose inverse and we summarize standard solvability results of differential and algebraic equations. In Section 3, we recapitulate the main results developed in [2] that we need in our analysis. In Section 3.1, we consider (flow) invariant subspace in the context of projections, in Section 3.2, we present closed solution formulas for differential and algebraic equations restricted to (invariant) subspaces. Pointing out the difficulties in the treatment of DAEs in Section 4.1, we motivate and introduce the concept of the strangeness-index in Section 4.2. Based on these observations, we remodel linear and nonlinear DAEs of arbitrary index as semi-explicit systems and generalized the notion of the flow. We illustrate our results by two examples.

2 Preliminaries

In this section, we summarize the basic notation and results needed in our analysis. In Section 2.1, we give our general notation and introduce the concept of projections and Moore-Penrose inverses for time and time-state dependent matrix functions. in Section 2.2. In Section 2.3.1, 2.3.2, we summarize basic properties and solvability results of differential and algebraic equations as they are needed for our analysis.

2.1 General notation

Throughout the work, we denote by $\mathcal{I}$ a (compact) interval in $\mathbb{R}$ and by $\bar{\mathcal{I}}$ its closure. We denote by $\Omega, \Omega_x, \Omega_{\dot{x}}$ open sets in $\mathbb{R}^n$, and we set $\bar{\Omega} := \mathcal{I} \times \Omega$ and $\mathbb{D} := \mathcal{I} \times \Omega_x \times \Omega_{\dot{x}}$. The set of ℓ-times continuous differential functions from $\bar{\Omega}$ to $\mathbb{R}^{n\times n}$ is denoted by $C^\ell(\bar{\Omega}, \mathbb{R}^{n\times n})$ and the set of locally Lipschitz continuous functions by $C^{\text{Lip}}_{\text{loc}}(\bar{\Omega}, \mathbb{R}^{n\times n})$. The space of polynomials of maximal degree r is denoted by Π_r. For a function $F \in C^1(\mathcal{I} \times \Omega, \mathbb{R}^n)$, we denote by F_t, F_x the partial derivatives with respect to t and x, respectively. If $x = x(t)$, then the total time derivative is given by $\dot{F}(t, x(t)) = F_t + F_x\dot{x}$.

For $z \in \bar{\Omega}$, we call the open ball $\mathcal{B}_\delta(z) := \{\tilde{z} \in \Omega \mid \text{dist}(z - \tilde{z}) < \delta\}$ with center z and $\delta > 0$ a *neighborhood* of z in $\bar{\Omega}$. If the radius is not specified, we simply write $\mathcal{B}(z)$.

Further, we need the notion of a *differentiable manifold*, which is a subset $\mathcal{L} \subset \mathbb{R}^n$ that is locally diffeomorphic to an open set $O \subset \mathbb{R}^n$, i.e., for every $z \in \mathcal{L}$ there exists a neighbourhood $\mathcal{B}(z)$ and a differentiable, bijective mapping Φ such that $\Phi : \mathcal{B}(z) \to O$.

For a vector $x \in \mathbb{R}^n$ and a matrix $A \in \mathbb{R}^{n\times n}$, we denote the i-th and ij-th entry by x_i and A_{ij}, respectively. If $A \in \mathbb{R}^{n\times n}$ is nonsingular and partitioned as $A = [A_1\, A_2]$ with $A_1 \in \mathbb{R}^{n\times d}$, $A_2 \in \mathbb{R}^{n\times a}$, then we partition the inverse accordingly, i.e., we set $A^{-1} = [A_1^{-T}, A_2^{-T}]^T$, where $A_1^{-} \in \mathbb{R}^{d\times n}$, $A_2^{-} \in \mathbb{R}^{a\times n}$.

For a matrix $A \in \mathbb{R}^{n\times n}$, we denote by $\ker(A)$ and $\text{Rg}(A)$ the *kernel* and *range* of A, respectively, and by $\text{corange}(A) := \ker(A)^\perp$ and $\text{coker}(A) := \text{Rg}(A)^\perp$ the *corange* and *cokernel*. The rank is denoted by $\text{rank}(A)$ where $\text{rank}(A) = \dim \text{Rg}(A)$.

For a vector $x \in \mathbb{R}^n$, we denote the norm of x by $\|x\|_p$, where, e.g., $p = 2, \infty$ refers to the standard euclidean or the maximums norm, respectively. For a matrix $A \in \mathbb{R}^{n\times n}$, we denote the associated operator norm by $\|A\|_p$. If no particular norm is specified, we simply write $\|x\|$ and $\|A\|$.

A subset $\mathbb{K} \subset \mathbb{R}^n$ is a linear *subspace* if $0, u + v, cu \in \mathbb{K}$ for every $u, v \in \mathbb{K}$, $c \in \mathbb{R}$, cp. [39], p. 106. A set of vectors $v_1, ..., v_d \in \mathbb{K}$ is called a *basis* of $\mathbb{K}$ if $v_1, ..., v_d$ span $\mathbb{K}$ and are linearly independent, i.e., $\text{Rg}([v_1, ..., v_d]) = \mathbb{K}$ and $c_1v_1 + ... + c_dv_d = 0$ only if $c_i = 0$, $i = 1, ..., d$, cp. [39], p. 109. The length d of a basis $v_1, ..., v_d$ is the dimension of

$\mathbb{K}$ and we write $d = \dim(\mathbb{K})$, cp. [39], p. 109. For $\mathbb{R}^n$, we denote the standard canonical basis by $e_1, ..., e_n$ where $e_i = [\delta_{ij}]_{j=1,...,n}$. We say that a matrix T spans a subspace $\mathbb{K}$ if $\mathrm{Rg}(T) = \mathbb{K}$. If $\mathrm{rank}(T)$ is maximal, then T is called a basis of $\mathbb{K}$ and $\dim(\mathbb{K}) = \mathrm{rank}(T)$.

The *complement* of a linear subspace $\mathbb{K}$ is denoted by $\mathbb{K}^{'}$, i.e., $\mathbb{K} \cap \mathbb{K}^{'} = \{0\}$ and $\mathbb{R}^n = \mathbb{K} + \mathbb{K}^{'}$, see e.g. [23], p. 20. The orthogonal complement by $\mathbb{K}^{\perp}$, i.e., $\mathbb{K} \cap \mathbb{K}^{\perp} = \{0\}$, $\mathbb{R}^n = \mathbb{K} + \mathbb{K}^{\perp}$ and $v^T v^{'} = 0$ for every $v \in \mathbb{K}$, $v^{'} \in \mathbb{K}^{\perp}$, see e.g. [23], p. 20.

For a subspace $\mathbb{K} \subset \mathbb{R}^n$ and a vector $v \in \mathbb{K}^{'}$, we denote by $\mathbb{K}_v := \mathbb{K} + v$ the *affine subspace* $\mathbb{K} + v := \{x \in \mathbb{R}^n \mid x - v \in \mathbb{K}\}$. If $v = 0$, then $\mathbb{K}_0 = \mathbb{K}$.

For nonlinear DAEs, we consider *time* or *time-state dependent subspaces* $\mathbb{K}(t)$, $\mathbb{K}(t, x)$, $t \in \mathcal{I}$, $x \in \Omega$, as they arise, e.g., as kernel or range of the Jacobian $F_{\dot{x}}(t, x, \dot{x})$. These spaces are defined pointwise, i.e., we call $\mathbb{K}$ a *subspace on* $\mathcal{I} \times \Omega$ if $\mathbb{K}(t, x)$ is a subspace for every $(t, x) \in \mathcal{I} \times \Omega$. Accordingly, the affine subspace $\mathbb{K}_v(t, x)$ is defined pointwise by $\mathbb{K}(t, x) + v(t, x)$ for a function $v : \mathcal{I} \times \Omega \to \mathbb{R}^n$. We say that a matrix function $T : \mathcal{I} \to \Omega, \mathbb{R}^{n \times n}$ spans $\mathbb{K}$ if $T(t, x)$ is a basis of $\mathbb{K}(t, x)$ for every $(t, x) \in \mathcal{I} \times \Omega$. We write $\mathrm{span}(T) = \mathbb{K}$ on $\mathcal{I} \times \Omega$.

For a function $F : \mathcal{I} \times \Omega \to \mathbb{R}^n$, we denote the restriction of f onto a linear subspace $\mathbb{K}$ by $f_{|\mathbb{K}}(t, x) := f(t, x)$ for $(t, x) \in \mathcal{I} \times \Omega$.

2.2 Projections and Moore-Penrose inverse

To separate the differential and algebraic components in a given DAE $F(t, x, \dot{x}) = 0$, we use projections that are induced by the Moore-Penrose inverse of the Jacobian $F_{\dot{x}}$. We first summarize the basic notations of projections, then introduce the Moore-Penrose inverse for matrix functions.

2.2.1 Projections

We first introduce the basic concept of projections, see e.g. [30], p. 73, [1], p. 280. A matrix $P \in \mathbb{R}^{n \times n}$ is called *projection* if it is idempotent, i.e., $P^2 = P$, see e.g. [30], p. 73. A projection $P \in \mathbb{R}^{n \times n}$ is called *orthogonal projection* if P is symmetric, i.e., $P^T = P$, else P is called *oblique*. The *complement* $P^{'} := I_n - P$ of a projection P is again a projection that satisfies $\mathrm{Rg}(P^{'}) = \ker(P)$ and $\ker(P^{'}) = \mathrm{Rg}(P)$, see e. g., [1], p. 280.

Projections provide a useful description of linear subspaces. We call $P \in \mathbb{R}^{n \times n}$ a *projection onto a subspace* $\mathbb{K} \subset \mathbb{R}^n$ if $P^2 = P$ and $\mathrm{Rg}(P) = \mathbb{K}$, and call P a *projection along a subspace* $\mathbb{K}^{'} \subset \mathbb{R}^n$ if $P^2 = P$ and $\ker(P) = \mathbb{K}^{'}$. The complement $P^{'} := I_n - P$ is a projection along $\mathrm{coker}(P)$ onto $\ker(P)$, see e. g. [1], p. 280. Furthermore, $\ker(P)$ and $\mathrm{Rg}(P)$ are complementary subspaces and if P is orthogonal, then $\ker(P)$ and $\mathrm{Rg}(P)$ are orthogonaly complementary, cp. [23], p. 20. If $P \in \mathbb{R}^{n \times n}$ is a projection onto $\mathbb{K}$, then $\mathrm{rank}(P) = \dim(\mathbb{K})$, see e. g. [1], p. 280.

If $P \in \mathbb{R}^{n\times n}$ is a projection onto (along) $\mathbb{K}$, then $P_v(x) := Px + v$, $x \in \mathbb{R}^n$, is the *affine projection onto (along)* $\mathbb{K}_v$. The complement is given by $P_v^{'}(x) := P^{'}x - g$ and $P_v^{'}$ is a projection along $\mathbb{K}_v$ onto $\mathbb{K}_v^{'}$.

For the uniqueness of projections onto or along a subspace, we cite the following result, see [30].

Lemma 2.1. *[30]*

1. *Two projections* $P_1, P_2 \in \mathbb{R}^n$ *project onto the same subspace* $\mathbb{K} \subset \mathbb{R}^n$ *if and only if* $P_1 = P_2P_1$ *and* $P_2 = P_1P_2$. *If this is the case, then* $P_1^{'} = P_1^{'}P_2^{'}$ *and* $P_2^{'} = P_2^{'}P_1^{'}$.

2. *Two projections* $P_1, P_2P \in \mathbb{R}^n$ *project along the same subspace* $\mathbb{K} \subset \mathbb{R}^n$ *if and only if* $P_1 = P_1P_2$ *and* $\tilde{P} = \tilde{P}P$. *If this is the case, then* $P_1^{'} = P_1^{'}P_2^{'}$ *and* $P_2^{'} = P_2^{'}P_1^{'}$.

Given a complement $\mathbb{K}^{'}$, the a linear subspace $\mathbb{K} \subset \mathbb{R}^n$ uniquely defines a projection, see e.g. [23], p. 22.

Lemma 2.2. *Let* $\mathbb{K} \subset \mathbb{R}^n$ *be a subspace with* $\dim(K) = d$ *and let* $\mathbb{K}^{'}$ *be its complement. Then, there exists a unique projection* P *along* $\mathbb{K}^{'}$ *onto* $\mathbb{K}$. *The projection* P *is diagonalizable for every* $= [T_1\ T_2]$ *that satisfies span*$(T_1) = \mathbb{K}$ *and span*$(T_2) = \mathbb{K}^{'}$, *i.e.,*

$$P = T \begin{bmatrix} I_d & 0 \\ 0 & 0 \end{bmatrix} T^{-1}, \quad P^{'} = T \begin{bmatrix} 0 & 0 \\ 0 & I_{n-d} \end{bmatrix} T^{-1}. \tag{2}$$

If $\mathbb{K}^{\perp}$ *is the orthogonal complement of* $\mathbb{K}$, *then there exists a unique orthogonal projection* P *along* $\mathbb{K}^{\perp}$ *onto* $\mathbb{K}$ *and* P *is orthogonaly diagonalizable.*

Partitioning the inverse T^{-1} according to $T = [T_1\ T_2]$, i.e., $T^{-1} = [T_1^{-T}\ T_2^{T}]^T$, then (2) implies that $P = T_1T_1^{-}$ and $P^{'} = T_2T_2^{-}$. If P is orthogonal, then we have that $P = T_1T_1^T$, $P^{'} = T_2T_2^T$ in particular.

In the following, we consider *time* or *time-state dependent projections*, i.e., matrix functions $P \in C(\bar{\Omega}, \mathbb{R}^{n\times n})$ that satisfy $P^2(z) = P(z)$ for every $z \in \bar{\Omega}$. As $P(z) \in \mathbb{R}^{n\times n}$, then we can pointwise extend the properties and definitions mentioned previously. In particular, we note the following identities involving the time derivative $\dot{P}$, cp. [2], Lemma 2.3.

Lemma 2.3. 1. *Let* $P \in C^1(\mathcal{I}, \mathbb{R}^{n\times n})$ *be a projection with complement* $P^{'}$. *Then,* $\dot{P} = -\dot{P}'$, $P\dot{P} = \dot{P}P^{'}$ *and* $\dot{P}P = P^{'}\dot{P}$ *are satisfied pointwise on* $\mathcal{I}$. *In particular, then* $P\dot{P}P = 0$ *and* $P^{'}\dot{P}P^{'} = 0$ *are satisfied pointwise on* $\mathcal{I}$.

2. *Let* $P_1, P_2 \in C^1(\mathcal{I}, \mathbb{R}^{n\times n})$ *be projections with complements* $P_1^{'}, P_2^{'}$ *and* $Rg(P_1) = Rg(P_2)$. *Then,* $P_2^{'}\dot{P}_1 = \dot{P}_2P_1$ *and* $P_1^{'}\dot{P}_2 = \dot{P}_1P_2$ *are satisfied pointwise on* $\mathcal{I}$.

3. *Let $T = [T_1\ T_2] \in C^1(\mathcal{I}, \mathbb{R}^{n\times n})$ be pointwise nonsingular with span$(T_1) = Rg(P)$ on $\mathcal{I}$. For every $t \in \mathcal{I}$, the $\dot{P}$ is given by*

$$\dot{P}(t) = T(t) \begin{bmatrix} 0 & -T_1^-(t)\dot{T}_2(t) \\ T_2^-(t)\dot{T}_1(z) & 0 \end{bmatrix} T^{-1}(t), \tag{3}$$

where $T^{-1} = [T_1^-\ T_2^-]$ is partitioned according to T.

2.2.2 Moore-Penrose inverse

Now, we consider projections that are induced by the Moore-Penrose inverse, see e. g. [3, 16, 27]. For $E \in \mathbb{R}^{m\times n}$, a matrix $E^+ \in \mathbb{R}^{n\times m}$ is called *Moore-Penrose inverse* of E, if the following conditions are satisfied, i.e.,

$$(i)\ EE^+E = E, \quad (ii) E^+EE^+ = E^+, \quad (iii)\ (E^+E)^T = E^+E, \quad (iv)\ (EE^+)^T = EE^+. \tag{4}$$

For every matrix $E \in \mathbb{R}^{n\times n}$, there exists a unique Moore-Penrose inverse, see [24] and if E is nonsingular, then $E^+ = E^{-1}$, see [54]. Alternatively, the Moore-Penrose inverse can be defined by the Singular Value Decomposition (SVD), see e.g. [24]. We need following properties of the Moore-Penrose inverse, cp. [2], Lemma 2.4.

Lemma 2.4. *Consider $E \in \mathbb{R}^{m\times n}$ and let $U \in \mathbb{R}^{m\times m}$, $V \in \mathbb{R}^{n\times n}$ be orthogonal. Then, $(U^TEV)^+ = V^TE^+U$. More general, if $U_1 \in \mathbb{R}^{m\times k}$, $V_1 \in \mathbb{R}^{n\times l}$, $k \le m$, $l \le n$ have orthogonal columns, respectively, and $E_{11} \in \mathbb{R}^{k\times l}$, then $(UE_{11}V^T)^+ = VE_{11}^+U^T$.*

To extend these results to matrix functions $E \in C(\bar{\Omega}, \mathbb{R}^{n\times n})$, we use a smooth factorization that resembles the smooth singular value decomposition [7] except that the middle factor is not diagonal. For matrix functions $E \in C^\ell(\Omega, \mathbb{R}^{m\times n})$, this is a local result obtained from the implicit function theorem [37], p. 155, for functions $E \in C^\ell(\mathcal{I}, \mathbb{R}^{m\times n})$ the decomposition is globally smooth on $\mathcal{I}$ [37], p. 62.

Theorem 2.1 ([37])**.** 1. *Let $E \in C^\ell(\Omega, \mathbb{R}^{m\times n})$ have rank$(E) = d$ on $M \subset \Omega$, where M is an open set. For every $z^* \in M$ there exists a neighborhood $\mathcal{B}(z^*) \subset \Omega$ and pointwise orthogonal functions $U \in C^\ell(\mathcal{B}(z^*), \mathbb{R}^{m\times m})$, $V \in C^\ell(\mathcal{B}(z^*), \mathbb{R}^{n\times n})$, such that*

$$E(z) = U(z) \begin{bmatrix} E_{11}(z) & 0 \\ 0 & 0 \end{bmatrix} V^T(z), \tag{5}$$

is satisfied pointwise on $\mathcal{B}(z^*)$. The matrix $E_{11} \in C^\ell(\mathcal{B}(z^*), \mathbb{R}^{d\times d})$ is pointwise nonsingular.

2. *If $E \in C^\ell(\mathcal{I}, \mathbb{R}^{m\times n})$ and rank$(E) = d$ on $\mathcal{I}$, then there exists pointwise orthogonal functions $U \in C^\ell(\mathcal{I}, \mathbb{R}^{m\times m})$, $V \in C^\ell(\mathcal{I}, \mathbb{R}^{n\times n})$, such that*

$$E(t) = U(t) \begin{bmatrix} E_{11}(t) & 0 \\ 0 & 0 \end{bmatrix} V^T(t), \tag{6}$$

is satisfied pointwise on $\mathcal{I}$. The matrix $E_{11} \in C^\ell(\mathcal{I}, \mathbb{R}^{d\times d})$ is pointwise nonsingular.

The transformations U, V are partitioned as $U = [U_1\, U_2]$ and $V = [V_1\, V_2]$, where $U_1 = \operatorname{span}(\operatorname{Rg}(E))$, $U_2 = \operatorname{span}(\operatorname{corange}(E))$, and $V_1 = \operatorname{span}(\operatorname{coker}(E))$, $V_2 = \operatorname{span}(\ker(E))$. Pointwise, for a given $z \in \Omega$ we can always construct such bases $V(z)$, $U(z)$ using a standard singular value decomposition. However, if rank(E) is constant on Ω, then Theorem 2.1 ensures that U, V are *locally* as smooth as E. If $E \in C^\ell(\mathcal{I}, \mathbb{R}^{m\times n})$, then U, V are smooth on $\mathcal{I}$.

Theorem 2.1 allows to define the Moore-Penrose inverse for matrix functions.

Lemma 2.5. 1. *Let $E \in C^\ell(\Omega, \mathbb{R}^{m\times n})$ have rank$(E) = d$ on $M \subset \Omega$, where M is an open set. For every $z^* \in M$, there exists a neighborhood $\mathcal{B}(z^*) \subset \Omega$ and pointwise orthogonal functions $U \in C^\ell(\mathcal{B}(z^*), \mathbb{R}^{m\times m})$, $V \in C^\ell(\mathcal{B}(z^*), \mathbb{R}^{n\times n})$ providing a decomposition* (5)*. For every $z \in \mathcal{B}(z^*)$, then the Moore-Penrose inverse of E is given by*

$$(E(z))^+ = V(z) \begin{bmatrix} E_{11}^{-1}(z) & 0 \\ 0 & 0 \end{bmatrix} U^T(z), \tag{7}$$

and $E^+ \in C^\ell(\mathcal{B}(z^*), \mathbb{R}^{m\times n})$.

2. *Let $E \in C^\ell(\mathcal{I}, \mathbb{R}^{m\times n})$ have rank$(E) = d$ on $\mathcal{I}$ and let $U \in C^\ell(\mathcal{I}, \mathbb{R}^{m\times m})$, $V \in C^\ell(\mathcal{I}, \mathbb{R}^{n\times n})$ provide a decomposition* (6)*. Then, the Moore-Penrose inverse of E is given by*

$$(E(t))^+ = V(t) \begin{bmatrix} E_{11}^{-1}(t) & 0 \\ 0 & 0 \end{bmatrix} U^T(t) \tag{8}$$

for every $t \in \mathcal{I}$, and $E^+ \in C^\ell(\mathcal{I}, \mathbb{R}^{m\times n})$.

Proof. Using the representation (5) and (7), the characteristic properties of the Moore-Penrose inverse, cp. (4), are pointwise verified for every $z^* \in M$. By Theorem 2.1, we have that $U \in C^\ell(\mathcal{B}(z^*), \mathbb{R}^{m\times m})$, $V \in C^\ell(\mathcal{B}(z^*), \mathbb{R}^{n\times n})$ and $E_{11} \in C^\ell(\mathcal{B}(z^*), \mathbb{R}^{d\times d})$. Using Cramer's rule [32], p. 21, and noting that the determinant of a matrix is multilinear in the entries, it follows that $E_{11}^{-1} \in C^\ell(\mathcal{B}(z^*), \mathbb{R}^{d\times d})$, i.e., $E^+ \in C^\ell(\mathcal{B}(z^*), \mathbb{R}^{m\times n})$.

If $E \in C^\ell(\mathcal{I}, \mathbb{R}^{m\times n})$ has rank$(E) = d$ on $\mathcal{I}$, then Theorem 2.1 implies that $U \in C^\ell(\mathcal{I}, \mathbb{R}^{m\times m})$, $V \in C^\ell(\mathcal{I}, \mathbb{R}^{n\times n})$ and $E_{11}(z) \in C^\ell(\mathcal{I}, \mathbb{R}^{d\times d})$, i.e., we have that $E^+ \in C^\ell(\mathcal{I}, \mathbb{R}^{m\times n})$. □

Thus, provided $E \in C^\ell(\Omega, \mathbb{R}^{m\times n})$ has constant rank on a subset M, then we can treat the Moore-Penrose locally as a function as smooth as E. Properties of the Moore-Penrose inverse like uniqueness or, e.g., the assertions of Lemma 2.4, then extend pointwise to this matrix function. In particular, we can straighforward define the Moore-Penrose projections for matrix functions.

Corollary 2.1. *1. Let $E \in C^\ell(\Omega, \mathbb{R}^{m\times n})$ have rank$(E) = d$ on $M \subset \Omega$, where M is an open set. Then, on M, EE^+ is a pointwise orthogonal projection along corange(E) onto Rg(E), and E^+E is a pointwise orthogonal projection along* $\ker(E^T)$ *onto coker(E).*

2. For every $z^* \in M$, there exists a neighbourhood $\mathcal{B}(z^*) \subset \Omega$, such that $EE^+ \in C^\ell(\mathcal{B}(^*), \mathbb{R}^{m\times m})$, $E^+E \in C^\ell(\mathcal{B}(z^*), \mathbb{R}^{n\times n})$. Furthermore, there exist pointwise orthogonal functions $U \in C^\ell(\mathcal{B}(z^*), \mathbb{R}^{m\times m})$, $V \in C^\ell(\mathcal{B}(z^*), \mathbb{R}^{n\times n})$, such that

$$E^+(z)E(z) = U(z)\begin{bmatrix} I_d & 0 \\ 0 & 0 \end{bmatrix} U^T(z), \quad E(z)E^+(z) = V(z)\begin{bmatrix} I_d & 0 \\ 0 & 0 \end{bmatrix} V^T(z). \tag{9}$$

is satisfied pointwise on $\mathcal{B}(z^*)$.

3. If $E \in C^\ell(\mathcal{I}, \mathbb{R}^{m\times n})$ has rank$(E) = d$ on $\mathcal{I}$, then $EE^+ \in C^\ell(\mathcal{I}, \mathbb{R}^{m\times m})$ and $E^+E \in C^\ell(\mathcal{I}, \mathbb{R}^{n\times n})$. The decomposition (9) *is satisfied pointwise on $\mathcal{I}$.*

For the flow of a linear DAE with variable coefficients, cp. Theorem 4.5, we need the concept of a *semi inverse*,cp. eg. [4]. For $E \in \mathbb{R}^{n\times m}$, a matrix $E^{\text{ginv}} \in \mathbb{R}^{m\times n}$ is called the *semi inverse of E* if E^{ginv} satisfies

$$EE^{\text{ginv}}E = E, \quad E^{\text{ginv}}EE^{\text{ginv}} = E. \tag{10}$$

2.3 Solvability of differential and algebraic equations

We summarize the classical solvability results for differential and algebraic equations.

2.3.1 Differential equations

For a given initial value problem $\dot{x} = f(t, x)$, $x(t_0) = x_0$, we define the *maximal interval of existence* $J_{max}(t_0, x_0) = [t_0, t_+)$, where either $t_+ = \infty$, i. e., the solution exists forever, $t_+ < \infty$ and $\lim_{t\to t_+} \|x(t)\| = \infty$, i. e., the solution blows up in finite time, or, $t_+ < \infty$ and $\lim_{t\to t_+} \|x(t) - \bar{x}\| = 0$, $\bar{x} \in \partial\Omega$ or $\lim_{t\to t_+} |t - \bar{t}| = 0$, where $\bar{\mathcal{I}} = [\underline{t}, \bar{t}]$, i. e., the solution leaves the phase space $\mathcal{I} \times \Omega$ in finite time. If $f(t, x) = A(t)$, then $J_{max}(t_0, x_0) = \bar{\mathcal{I}}$, cp. [5].

For the existence and uniqueness, we cite Peano's and Picard-Lindelöf's Theorem, see e.g. [5], p. 43-44, or [59].

Theorem 2.2. *Consider $\dot{x} = f(t,x)$ with $f \in C^{Lip}_{loc}(\mathcal{I} \times \Omega, \mathbb{R}^n)$. For every $(t_0, x_0) \in \mathcal{I} \times \Omega$, the initial value problem $\dot{x} = f(t,x)$, $x(t_0) = x_0$ has a unique solution $x \in C^1(\mathcal{J}_{max}(t_0,x_0), \mathbb{R}^n)$.*

If $f \in C^1(\mathcal{I} \times \Omega, \mathbb{R}^n)$, then $f \in C^{\mathrm{Lip}}_{\mathrm{loc}}(\mathcal{I} \times \Omega, \mathbb{R}^n)$, see e. g., [5], p. 44.

The unique relation between a given initial value and its associated solution motivates the definition of the *flow*, see e. g. [5], p. 49.

Lemma 2.6. *Consider $\dot{x} = f(t,x)$, $f \in C^{Lip}_{loc}(\mathcal{I} \times \Omega, \mathbb{R}^n)$. There exists a locally defined, unique function $\Phi_f : \mathcal{I} \times \mathcal{I} \times \Omega \to \mathbb{R}^n$ that satisfies the following assertions.*

(i) For every $(t_0, x_0) \in \mathcal{I} \times \Omega$, the flow Φ_f satisfies

$$\Phi_f^{t_0}(t_0, x_0) = x_0, \tag{11a}$$

$$\Phi_f^t(s, \Phi_f^s(t_0, x_0)) = \Phi_f^t(t_0, x_0), \tag{11b}$$

$$\dot{\Phi}_f^t(t_0, x_0) = f(t, \Phi_f^t(t_0, x_0)), \tag{11c}$$

for $t \in J_{max}(t_0, x_0)$.

(ii) For every $(t_0, x_0) \in \mathcal{I} \times \Omega$, the flow satisfies $\Phi_f^{(\cdot)}(t_0, x_0) \in C^1(\mathcal{J}_{max}(t_0,x_0), \mathbb{R}^n)$. If $f(t,\cdot) \in C^m(\Omega, \mathbb{R}^n)$, then $\Phi_f^t(t_0, \cdot) \in C^m(\Omega, \mathbb{R}^n)$ for every $(t_0, t \in \mathcal{I}$ and if $f \in C^m(\mathcal{I} \times \Omega, \mathbb{R}^n)$, then $\Phi_f^{\cdot}(t_0, \cdot) \in C^m(\mathcal{I} \times \Omega, \mathbb{R}^n)$ for every $t_0 \in \mathcal{I}$.

(iv) For every $(t_0, x_0) \in \mathcal{I} \times \Omega$, the solution of $\dot{x} = f(t,x)$, $x(t_0) = x_0$ is given by $x(t) = \Phi_f^t(t_0, x_0)$ for $t \in J_{max}(t_0, x_0)$.

The function Φ_f is called the *flow* of $\dot{x} = f(t,x)$ or f. For given initial data $(t_0, x_0) \in \mathcal{I} \times \Omega$, properties (11) state that $\Phi_f^t(t_0, x_0)$ is the unique solution of $\dot{x} = f(t,x)$, $x(t_0) = x_0$ that can be uniquely extended on $\mathcal{J}_{max}(t_0, x_0)$.

For linear problems, the flow is a linear map of the initial values.

Lemma 2.7. *Consider $\dot{x} = Ax + f$ with $A \in C(\mathcal{I}, \mathbb{R}^{n \times n})$, $f \in C(\mathcal{I}, \mathbb{R}^n)$.*

1. *There exists a unique function $\Phi_A : \mathcal{I} \times \mathcal{I} \to \mathbb{R}^{n \times n}$ that satisfies the following assertions.*

   (i) For every $t_0 \in \mathcal{I}$, the flow Φ_A satisfies

$$\Phi_A^{t_0}(t_0) = I_n, \tag{12a}$$

$$\Phi_A^t(s)\Phi_A^s(t_0) = \Phi_A^t(t_0), \tag{12b}$$

$$\dot{\Phi}_A^t(t_0) = A(t)\Phi_A^t(t_0), \tag{12c}$$

   for $t \in \bar{\mathcal{I}}$.

(ii) *For every* $t_0 \in \mathcal{I}$*, the flow satisfies* $\Phi_A^{(\cdot)}(t_0) \in C^1(\mathcal{I}, \mathbb{R}^{n\times n})$*. If* $A \in C^m(\mathcal{I}, \mathbb{R}^{n\times n})$*, then* $\Phi_A^{(\cdot)}(t_0) \in C^m(\mathcal{I}, \mathbb{R}^{n\times n})$*.*

(iii) *For every* $t_0, t \in \mathcal{I}$*, the flow* Φ_A *is invertible with* $(\Phi_A^t(t_0))^{-1} = \Phi_A^{t_0}(t_0)$*.*

(iv) *For every* $(t_0, x_0) \in \mathcal{I} \times \Omega$*, the solution of* $\dot{x} = Ax$*,* $x(t_0) = x_0$ *is given by* $x(t) = \Phi_A^t(t_0)x_0$ *for* $t \in \bar{\mathcal{I}}$*.*

2. Let $f \in C(\mathcal{I}, \mathbb{R}^n)$*. The flow of* $\dot{x} = Ax + f$ *is given by*

$$\Phi_{A,f}^t(t_0, x_0) = \Phi_A^t(t_0)x(t_0) + \int_{t_0}^t \Phi_A^t(s)f(s)\,ds, \tag{13}$$

for $(t_0, x_0) \in \mathcal{I} \times \Omega$ *and satisfies the following assertions.*

(i) *For every* $(t_0, x_0) \in \mathcal{I} \times \Omega$*,* $\Phi_{A,f}$ *satisfies the assertions* (11) *of Lemma 2.6 for* $t \in \bar{\mathcal{I}}$*.*

(ii) *For every* $(t_0, x_0) \in \mathcal{I} \times \Omega$*, the flow satisfies* $\Phi_{A,f}^{(\cdot)}(t_0, x_0) \in C^1(\mathcal{I}, \mathbb{R}^{n\times n})$*. If* $A \in C^m(\mathcal{I}, \mathbb{R}^{n\times n})$ *and* $f \in C^m(\mathcal{I}, \mathbb{R}^n)$*, then* $\Phi_{A,f}^{(\cdot)}(t_0, x_0) \in C^m(\mathcal{I}, \mathbb{R}^{n\times n})$*.*

(iii) *For every* $(t_0, x_0) \in \mathcal{I} \times \Omega$*, the solution of* $\dot{x} = Ax + f$*,* $x(t_0) = x_0$ *is given by* $x(t) = \Phi_{A,f}^t(t_0, x_0)$ *for* $t \in \mathcal{I}$*.*

Proof. 1. The flow properties (12) follow from (11) as well as the solution formula $x(t) = \Phi_A^t(t_0)x_0$. The Taylor expansion of $\Phi_A^t(t_0)$ is given by

$$\Phi_A^{t_0+\tau}(t_0) = I_n + \tau A(t_0) + \tfrac{\tau^2}{2}t(\dot{A}(t_0) + A^2(t_0)) + \mathcal{O}(\tau^3),$$

i. e., $\Phi_A^t(t_0)$ is invertible for sufficiently small $\tau > 0$. For larger τ, we use (11b) to decompose $\Phi_A^{t_0+\tau}(t_0)$ into invertible factors. Moreover, (11b) proves that $(\Phi_A^t(t_0))^{-1} = \Phi_A^{t_0}(t)$.

2. To derive the formula (13), we follow the arguments given in [57], p. 163. Noting that Φ_A is invertible, we have that $\frac{d}{dt}[\Phi\Phi^{-1}] = \dot{\Phi}\Phi^{-1} + \Phi\frac{d}{dt}[\Phi^{-1}] = \frac{d}{dt}[I_n] = 0$, and this implies that

$$\frac{d}{dt}(\Phi_A^t(t_0))^{-1} = -\Phi_A^{t_0}(t)\dot{\Phi}_A^t(t_0)\Phi_A^{t_0}(t).$$

Using (11b), (11c), we get that

$$\frac{d}{dt}(\Phi_A^t(t_0))^{-1} = \Phi_A^{t_0}(t)A(t)\Phi_A^t(t_0)\Phi_A^{t_0}(t) = -\Phi_A^{t_0}(t)A(t).$$

Thus, we obtain the identity

$$\Phi_A^t(t_0)\frac{d}{dt}\big[\Phi_A^{t_0}(t)x(t)\big] = \Phi_A^t(t_0)\big(-\Phi_A^{t_0}(t)A(t)x(t) + \Phi_A^{t_0}(t)\dot{x}(t)\big) = -A(t)x(t) + \dot{x}(t),$$

and we can express $\dot{x} = A(t)x + f$ as $\Phi_A^t(t_0)\frac{d}{dt}\left[\Phi_A^{t_0}(t)x(t)\right] = f$. Multiplying by $\Phi_A^{t_0}(t)$ and integrating over $[t_0, t]$, we obtain that

$$\Phi_A^{t_0}(t)x(t) = \Phi_A^{t_0}(t_0)x(t_0) + \int_{t_0}^{t} \Phi_A^{t_0}(s)f(s)\, ds,$$

and multiplying once more by $\Phi_A^t(t_0)$ yields (13).

Using this formula, we verify that $\Phi_{A,f}$ satisfies the flow properties (11). For (11a), this follows from the respective property of Φ_A. For (11b), we use (12b) and the linearity of the integral. Finally, (11c) follows by considering $\dot{\Phi}_{A,f}$. □

The homogenous flow $\Phi_A^t(t_0)$ generalizes the concept of the *matrix exponential* $e^{(t-t_0)A}$, which is the fundamental solution of $\dot{x} = Ax$ with $A \in \mathbb{R}^{n\times n}$, see e.g. [5], p. 103. For a matrix function $A \in C^1(\mathcal{I}, \mathbb{R}^{n\times n})$ that commutes with its integral, we have that $\Phi_A^t(t_0) = e^{\int_{t_0}^t A(s)ds}$. A function $A \in C^1(\mathcal{I}, \mathbb{R}^{n\times n})$ commutes with its integral, for example if A is diagonalizable by constant transformations, see [31].

The inhomogeneous flow $\Phi_{A,f}$ generalizes Duhamel's formula [57].

We call both to Φ_A and $\Phi_{A,f}$ the flow of $\dot{x} = Ax + f$, using the subscript to indicate the homogeneous or inhomogeneous case, respectively. Besides (12), we note that Φ_A and $\Phi_{A,f}$ satisfy $\Phi_0 = I_n$, $\Phi_{A,0} = \Phi_A$ and

$$\Phi_{A,f}^t(t_0, x_0) - \Phi_{A,f}^t(t_0, \tilde{x}_0) = \Phi_A^t(t_0)(x_0 - \tilde{x}_0).$$

Remark 2.1. *For upper or lower triangular systems $\dot{x} = Ax$, $A \in C(\mathcal{I}, \mathbb{R}^{n\times n})$ with*

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

we can successively apply Duhamel's formula and obtain that

$$\Phi_A^t(t_0) = \begin{bmatrix} \Phi_{A_{11}}^t(t_0) & \int_{t_0}^t \Phi_{A_{11}}^{t_0}(s)A_{12}(s)\, ds \\ 0 & \Phi_{A_{22}}^t(t_0) \end{bmatrix}$$

for $t_0, t \in \mathcal{I}$. The functions $\Phi_{A_{11}}$, $\Phi_{A_{22}}$ are the flows induced by the block entries A_{11}, A_{22}, respectively.

A particular class of solutions of $\dot{x} = f(t,x)$ are *fixed points* of the flow Φ_f, i.e., $\hat{x} \in \Omega$ satisfying $\Phi_f^t(t_0, \hat{x}) = \hat{x}$ for $t \in \mathcal{J}_{max}(t_0, \hat{x})$, cp. e.g. [55], p. 14. Equivalently , fixed points are the zeros of f, i.e., $f(t, \hat{x}) = 0$ for $t \in \mathcal{J}_{max}(t_0, \hat{x})$. If $\dot{x} = f(t,x)$ has an equilibrium point $\hat{x}$, then the shifted system $\dot{\tilde{x}} = \tilde{f}(t, \tilde{x})$, $\tilde{x} = x - \hat{x}$ has the fixed point 0, cp. [53]. For linear systems $\dot{x} = Ax$, the origin is always an fixed point, cp. [5], p. 109.

2.3.2 Algebraic equations

For the solvability of algebraic equations, we cite the implicit function theorem, cp. e.g. [46], p. 128.

Theorem 2.3. *Consider $G \in C(\tilde{\Omega} \times \Omega, \mathbb{R}^{n\times n})$, where $\tilde{\Omega} \subset \mathbb{R}^p$, $\Omega \subset \mathbb{R}^n$ are open sets and let $(y_0, x_0) \in \tilde{\Omega} \times \Omega$ be such that $G(y_0, x_0) = 0$. Let G_x exist in a neighborhood of (t_0, x_0) and let $G_x(y_0, x_0)$ be continuous. If $G_x(t_0, x_0)$ is nonsingular, then there exist neighborhoods $\mathcal{B}(y_0) \subset \tilde{\Omega}$, $\mathcal{B}(x_0) \subset \Omega$ and a function $h \in C(\mathcal{B}(y_0), \mathcal{B}(x_0))$, such that*

$$x = h(y) \text{ solves } G(y, x) = 0 \text{ for every } y \in \mathcal{B}(y_0). \tag{14}$$

If $G_y(y_0, x_0)$ exists, then $\mathcal{D}h(y_0)$ exists and $\mathcal{D}h(y_0) = -(F_x(y_0, x_0)^{-1} F_y(y_0, x_0)$.

For $G \in C(\tilde{\Omega} \times \Omega, \mathbb{R}^{n\times n})$, we call a point $(y_0, x_0) \in \tilde{\Omega} \times \Omega$ *consistent* if $G(y_0, x_0) = 0$. We call (y_0, x_0) *regular* if it is consistent, G_x exist in a neighborhood of (t_0, x_0) and $G_x(y_0, x_0)$ is continuous and nonsingular. We denote the set of consistent points by

$$\mathcal{C}_G := \{(y, x) \in \tilde{\Omega} \times \Omega \mid G(y, x) = 0\}. \tag{15}$$

For linear problems $A(t)x = f(t)$ with $A \in C(\mathcal{I}, \mathbb{R}^{n\times n})$, $f \in C(\mathcal{I}, \mathbb{R}^n)$, we set

$$\mathcal{C}_{A,f} := \{(t, x) \in \mathcal{I} \times \Omega \mid A(t)x = f(t)\}. \tag{16}$$

In the following, we focus on equations of the form $F(t, x) = 0$ with $F \in C(\mathcal{I} \times \Omega, \mathbb{R}^n)$. In the neighborhood $\mathcal{B}(t_0) \times \mathcal{B}(x_0)$ of a regular point (t_0, x_0), we can uniquely solve $F(t, x) = 0$ for $x \in C(\mathcal{B}(t_0), \mathcal{B}(x_0))$. Setting $\mathcal{J}(t_0, x_0) := \mathcal{B}(t_0)$, we say that x solves $F(t, x) = 0$ on $\mathcal{J}(t_0, x_0)$.

3 Differential and algebraic equations restricted to linear subspaces

In [2], we analyzed the decomposition of algebraic and differential equations by projections. Using projections associated with invariant subspaces, in particular, we could decouple the system and successively compute a solution. Here, we summarize the essential results as they provide the tools to compute a flow formula for DAEs in Section 4.

3.1 Invariant and flow invariant subspaces

Invariant subspaces are linear subspaces that are invariant under the mapping of a function. For $F \in C(\mathcal{I} \times \Omega, \mathbb{R}^n)$, two linear subspaces $\mathbb{K}, \mathbb{K}$ are called F *invariant* if $F(t,x) \in \mathbb{K}(t)$ for every $(t,x) \in \mathcal{I} \times \mathbb{K} \cap \Omega$. If $\mathbb{K} = \mathbb{K}$, then $\mathbb{K}$ is F *invariant.* Accordingly, two affine subspaces $\mathbb{K}_v, \mathbb{K}_w \subset \mathbb{R}^n$, are F *affine invariant* if $F(t,x) \in \mathbb{K}_w(t)$ for every$(t,x) \in \mathcal{I} \times \mathbb{K}_v \cap \Omega$.

We characterize invariant subspaces in terms of projections.

Lemma 3.1. *Consider $F \in C(\mathcal{I} \times \Omega, \mathbb{R}^n)$. Let $\mathbb{K}, \mathbb{K}$ be linear subspaces and $P, Q \in C(\mathcal{I}, \mathbb{R}^{n\times n})$ projections onto $\mathbb{K}, \mathbb{K}$, respectively, with complements $P^{'}, Q^{'}$. Then, $\mathbb{K}, \mathbb{K}$ are F invariant if and only if*

$$Q^{'}(t)F(t, P(t)x) = 0 \tag{17}$$

is satisfied pointwise on $\mathcal{I} \times \Omega$. Accordingly, the affine subspaces $\mathbb{K}_v, \mathbb{K}_w$ are F affine invariant if and only if

$$Q^{'}(t)F(t, P_v(t,x)) = w, \tag{18}$$

is satisfied pointwise on $\mathcal{I} \times \Omega$. Condition (17), (17) *are independent of the choice of projections P, Q.*

For algebraic equations $F(t,x) = 0$, we consider invariant subspaces as a concept of the function F. For differential equations $\dot{x} = f(t,x)$, invariant subspaces refer to the flow Φ_f and Lemma 3.1 reads as follows.

Corollary 3.1. *Consider $\dot{x} = f(t,x)$, $f \in C^{Lip}_{loc}(\mathcal{I} \times \Omega, \mathbb{R}^n)$, with flow Φ_f. Let $\mathbb{K}$ be a linear subspace and $P \in C(\mathcal{I}, \mathbb{R}^{n\times n})$ a projection onto $\mathbb{K}$ with complement $P^{'}$. Then, $\mathbb{K}$ is Φ_f invariant if and only if*

$$P^{'}(t)\Phi^t_f(t_0, P(t_0)x_0) = 0 \tag{19}$$

for $t \in \mathcal{I}_{max}(t_0, x_0)$ *for every* $(t_0, x_0) \in \mathcal{I} \times \mathbb{K}$*. The affine subspace* $\mathbb{K}_v$ *is* Φ_f *affine invariant if and only if*

$$P^{'}(t)\Phi_f^t(t_0, P_v(t_0, x_0)) = v \tag{20}$$

for $t \in \mathcal{I}_{max}(t_0, x_0)$ *for every* $(t_0, x_0) \in \mathcal{I} \times \mathbb{K}_v$*. Condition* (19) *and* (20) *are independent of the projection* P*.*

For linear problems, condition (19), (20) are independent of the initial values and we observe a close relation between Φ_A, Φ_A affine and $\Phi_{A,f}$ invariant subspaces.

Lemma 3.2. *Consider* $\dot{x} = Ax$, $A \in C(\mathcal{I}, \mathbb{R}^{n\times n})$*,with flow* Φ_A*. Let* $\mathbb{K}$ *be a linear subspace and* $P \in C(\mathcal{I}, \mathbb{R}^{n\times n})$ *a projection onto* $\mathbb{K}$ *with complement* $P^{'}$*. Then,* $\mathbb{K}$ *is* Φ_A *invariant if and only if*

$$P^{'}(t)\Phi_A^t(t_0)P(t_0) = 0 \tag{21}$$

on $\mathcal{I} \times \mathcal{I}$*. Moreover,* $\mathbb{K}$ *is* $\Phi_{A,f}$ *invariant for every* $f \in C(\mathcal{I}, \mathbb{K})$ *if and only if* $\mathbb{K}$ *is* Φ_A *invariant. The affine subspace* $\mathbb{K}_v$ *is* Φ_A *affine invariant if and only if* $\mathbb{K}$ *is* Φ_A *invariant.*

In [2], Lemma 4.5, we have derived a sufficient condition to characterize Φ_f invariant subspaces in terms of the function f.

Lemma 3.3. *Consider* $\dot{x} = f(t, x)$, $f \in C_{loc}^{Lip}(\mathcal{I} \times \Omega, \mathbb{R}^n)$*, with flow* Φ_f*. Let* $\mathbb{K}$ *be a linear subspace and* $P \in C(\mathcal{I}, \mathbb{R}^{n\times n})$ *a projection onto* $\mathbb{K}$ *with complement* $P^{'}$*. If* $P^{'}(t)\Phi_f^t(t_0, 0) = 0$ *for* $t \in \mathcal{I}_{max}(t_0, 0)$ *and every* $t_0 \in \mathcal{I}$ *and*

$$P^{'}(t)\Phi_f^t(t_0, P^{'}(t_0)x_0) = P^{'}(t)\Phi_f^t(t_0, x_0), \tag{22}$$

for $t \in \mathcal{I}_{max}(t_0, x_0)$ *and every* $(t_0, x_0) \in \mathcal{I} \times \Omega$*, then* $\mathbb{K}$ *is* Φ_f *invariant. If* f_x *exists on* $\mathcal{I} \times \Omega$*, then condition* (22) *is equivalent to*

$$P^{'}(t)f_x(t, x)P(t) = \dot{P}(t)P(t). \tag{23}$$

For $v \in \mathbb{R}^n$*, provided* (22) *and* $P^{'}(t)\Phi_f^t(t_0, v) = 0$ *is satisfied for* $t \in \mathcal{I}_{max}(t_0, v)$ *and* $t_0 \in \mathcal{I}$*, then* $\mathbb{K}_v$ *is* Φ_f *affine invariant. Condition* (22) *is independent of the considered projection* P*.*

Exploiting that $\dot{P}P = P^{'}\dot{P}$, cp. Lemma 2.3, 1., condition (23) reads $P^{'}(t)(f_x(t, x) - \dot{P})P = 0$. Regarding (21), this means that $\mathbb{K}$ is $f_x - \dot{P}$ invariant, i.e., the Jacobian f_x captures the change $\dot{P}$ of the subspace $\mathbb{K}$. If, in addition, the solution starting in 0 remains in $\mathbb{K}$ for all its lifetime, then $\mathbb{K}$ is Φ_f invariant. For $v \in \mathbb{K}^{'}$, if the solution starting in v lies in $\mathbb{K}$ for all its lifetime, then $\mathbb{K}_v$ is affine Φ_f invariant.

Corollary 3.2. *Consider $\dot{x} = f(t,x)$, $f \in C^{Lip}_{loc}(\mathcal{I} \times \Omega, \mathbb{R}^n)$, with flow Φ_f. Let $\mathbb{K}$ be a linear subspace and $P \in C(\mathcal{I}, \mathbb{R}^{n\times n})$ a projection onto $\mathbb{K}$ with complement P'.*

1. *If f satisfies condition* (22) *and there exists $(t_0, x_0) \in \mathbb{K}\times\mathcal{I}$ such that $P'(t)\Phi^t_f(t_0, x_0) = 0$ for $t \in \mathcal{I}_{max}(t_0, x_0)$, then $P'(t)\Phi^t_f(t_0, 0) = 0$ for $t \in \mathcal{I}_{max}(t_0, 0)$.*
2. *For $v \in \mathbb{K}'$, if $f(t, v) = 0$ on $\mathcal{I}$, then $P'(t)\Phi^t_f(t_0, v) = 0$ for $t \in \mathcal{I}_{max}(t_0, v)$ and $t_0 \in \mathcal{I}$.*

For $\dot{x} = Ax$, $A \in C(\mathcal{I}, \mathbb{R}^{n\times n})$, the origin always is an equilibrium. Since the flow Φ_A is linear in the initial values, then condition (22) and (21) are equivalent, i.e., For linear problems, in particular, then Lemma 3.3 reads as follows. $\mathbb{K}$ is Φ_A invariant, if and only if

$$P'AP = \dot{P}P \tag{24}$$

on $\mathcal{I}$. Thus, for linear problems $\dot{x} = Ax$, the system matrix A must capture the change of $\mathbb{K}$ if $\mathbb{K}$ is supposed to be invariant. In terms of a basis representation, condition (24) is reflected as follows.

Remark 3.1. *For a basis $T = [T_1\, T_2] \in C^1(\mathcal{I}, \mathbb{R}^{n\times n})$ with span$(T_1) = \mathbb{K}$, we can diagonalize P and $\dot{P}$ with respect to T, cp. Lemma 2.2 and Lemma 2.3, 3.. Partitioning T^{-1} accordingly, i.e., $T^{-1} = [T_1^{-T} T_2^{-T}]^T$, and setting $A_{ij} = T_i^- AT_j$, $i, j = 1, 2$, then condition* (24) *reads*

$$\begin{aligned} P'AP - \dot{P}P &= T \begin{bmatrix} 0 & 0 \\ 0 & I_a \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} I_d & 0 \\ 0 & 0 \end{bmatrix} T^{-1} - T \begin{bmatrix} 0 & -T_1^- \dot{T}_2 \\ T_2^- \dot{T}_1 & 0 \end{bmatrix} \begin{bmatrix} I_d & 0 \\ 0 & 0 \end{bmatrix} T^{-1} \\ &= T \begin{bmatrix} 0 & 0 \\ A_{21} - T_2^- \dot{T}_1 & 0 \end{bmatrix} T^{-1}. \end{aligned} \tag{25}$$

3.2 Solving differential and algebraic equations in subspaces using projections

On a flow invariant subspace $\mathbb{K}$, the dynamics of $\dot{x} = f(t, x)$ are closed in the sense that every solution starting in $\mathbb{K}$ remains in $\mathbb{K}$ for all its lifetime. This allows to restrict Φ_f onto $\mathbb{K}$ without loosing the characteristic flow properties (11). To preserve the original coordinate systems, we implement the restriction using a projection.

Lemma 3.4. *Consider $\dot{x} = f(t,x)$, $f \in C^{Lip}_{loc}(\mathcal{I} \times \Omega, \mathbb{R}^n)$, with flow Φ_f. Let $\mathbb{K} \subset \mathbb{R}^n$ be a linear subspace and $P \in C(\mathcal{I}, \mathbb{R}^{n\times n})$ a projection onto $\mathbb{K}$. If $\mathbb{K}$ Φ_f invariant, then $\Phi_f \circ P = \Phi_{f\circ P}$, where $f \circ P(t, x) := f(t, P(t)x)$ and $\Phi_f \circ P(t : t_0, x_0) := \Phi^t_f(t_0, P(t_0)x_0)$. Thus, the restriction $\Phi_f \circ P$ is the flow associated with $f \circ P$ and satisfies the following properties.*

(i) For every $(t_0, x_0) \in \mathcal{I} \times \Omega$, $\Phi_f \circ P$ *satisfies*

$$\Phi_f^{t_0}(t_0, P(t_0)x_0) = P(t_0)x_0, \tag{26a}$$

$$\Phi_f^t(s, \Phi_f^s(t_0, P(t_0)x_0)) = \Phi_f^t(t_0, P(t_0)x_0), \tag{26b}$$

$$\dot{\Phi}_f^t(t_0, P(t_0)x_0) = f_{|\mathbb{K}}(t, \Phi_f^t(t_0, P(t_0)x_0)), \tag{26c}$$

for $t \in J_{max}(t_0, P(t_0)x_0)$. *In particular,* $\Phi_f P$ *satisfies*

$$P^{'}(t)\Phi_f^{t_0}(t_0, P(t_0)x_0) = 0, \tag{27}$$

for $t \in J_{max}(t_0, P(t_0)x_0)$ *and every* $(t_0, x_0) \in \mathcal{I} \times \Omega$.

(ii) For every $(t_0, x_0) \in \mathcal{I} \times \Omega$, $\Phi_f \circ P\,(\cdot, t_0, x_0) \in C^1(\mathcal{I}_{max}(t_0, P(t_0)x_0), \mathbb{R}^n)$. *If* $f(t, \cdot) \in C^m(\Omega, \mathbb{R}^n)$, *then* $\Phi_f \circ P\,(t, t_0, \cdot) \in C^m(\Omega, \mathbb{R}^n)$ *and if* $f \in C^m(\mathcal{I} \times \Omega, \mathbb{R}^n)$ *and* $P \in C^m(\mathcal{I}, \mathbb{R}^{n\times n})$, *then for every* $t_0 \in \mathcal{I}$ $\Phi_f \circ P\,(\cdot, t_0, \cdot) \in C^m(\mathcal{I} \times \Omega, \mathbb{R}^n)$.

(iii) For every $(t_0, x_0) \in \mathcal{I} \times \Omega$, *the solution of* $\dot{x} = f(t, Px)$, $x(t_0) = P(t_0)x_0$ *is given by* $x(t) = \Phi_f^t(t_0, P(t_0)x_0)$ *for* $t \in J_{max}(t_0, P(t_0)x_0)$.

We use Lemma 3.4 to compute a solution of the differential components in a DAE. To filter out the corresponding differential equation as well as the equation for the algebraic components, we have adjusted the implicit function theorem, cp. [2], Theorem 4.4.

Theorem 3.1. *Consider* $F(t, x) = 0$, $F \in C(\mathcal{I}\times\Omega, \mathbb{R}^n)$. *Let* $\mathbb{K}, \mathbb{K}$ *be linear subspaces with* $\dim(\mathbb{K}) = \dim(\mathbb{K})$ *on* $\mathcal{I} \times \Omega$ *and let* $P, Q \in C(\mathcal{I} \times \Omega, \mathbb{R}^{n\times n})$ *be the orthogonal projections onto* $\mathbb{K}, \mathbb{K}$, *respectively, with complements* $P^{'}, Q^{'}$. *Let* $(t_0, x_0) \in \mathcal{I} \times \Omega$ *be consistent and set* $x_{d,0} := P(t_0, x_0)x_0$, $x_{a,0} := P^{'}(t_0, x_0)x_0$. *If*

$$\big(Q(t_0, x_0)F_x(t_0, x_0)P(t_0, x_0)\big)^+ Q(t_0, x_0)F_x(t_0, x_0)P(t_0, x_0) = P(t_0, x_0), \tag{28a}$$

$$Q(t_0, x_0)F_x(t_0, x_0)P(t_0, x_0)\,\big(Q(t_0, x_0)F_x(t_0, x_0)P(t_0, x_0)\big)^+ = Q(t_0, x_0), \tag{28b}$$

then there exist neighborhoods $\mathcal{B}(t_0, x_{a,0}) \subset \mathcal{I} \times \mathbb{R}^n$, $\mathcal{B}(x_{a,0}) \subset \mathbb{R}^n$ *and a function* $h \in C(\mathcal{B}(t_0, x_{a,0}), \mathcal{B}(x_{d,0}))$, *such that*

$$P(t, x_0)x = h(t, P^{'}(t, x_0)x) \text{ solves } Q(t, x_0)F(t, x) = 0$$
$$\text{for every } (t, P^{'}(t, x_0)x) \in \mathcal{B}(t_0, x_{a0}). \tag{29}$$

The partial derivatives are given by

$$h_{x_a,0} = -(Q_0F_{x,0}P_0)^+Q_0F_{x,0}P_0^{'}, \tag{30a}$$

$$h_{t,0} = -(Q_0F_{x,0}P_0)^+(Q_0F_{t,0} + \dot{Q}_0F_0) + \dot{P}_0h_0, \tag{30b}$$

where the subscript 0 *denotes the function evaluation in* (t_0, x_0).

On a neighborhood $\mathcal{B}(t_0, x_0)$ *with* $P^{'}(t, x_0) \cdot \mathcal{B}(t_0, x_0) \subset \mathcal{B}(t_0, x_{a0})$, *then* $\mathbb{K}^{'}(\cdot, x_0)$, $\mathbb{K}(\cdot, x_0)$ *are* h *invariant.*

The partial derivatives are given by

$$h_{x_a,0} = -(Q_0 F_{x,0} P_0)^{+} Q_0 F_{x,0} P_0^{'}, \tag{31a}$$

$$h_{t,0} = -(Q_0 F_{x,0} P_0)^{+} (Q_0 F_{t,0} + \dot{Q}_0 F_0) + \dot{P}_0 h_0, \tag{31b}$$

where the subscript 0 *denotes the function evaluation in* (t_0, x_0).

On a neighborhood $\mathcal{B}(t_0, x_0)$ *with* $P^{'}(t, x_0) \cdot \mathcal{B}(t_0, x_0) \subset \mathcal{B}(t_0, x_{a0})$, *then* $\mathbb{K}^{'}(\cdot, x_0)$, $\mathbb{K}(\cdot, x_0)$ *are* h *invariant.*

Theorem 3.1 characterizes the solvability of algebraic equations restricted to a subspace and locally supplies a solution. Stated in terms of projections, it allows to filter out the regular components while keeping the original coordinate system. More exactly, the implicit function h filters out the parametrizing components in $\mathbb{K}^{'}$ being restricted by a projection. Then, the solution of $Q(t, x_0)F(t, x) = 0$ is given by $(t, P^{'}(t, x_0)x + h(t, P^{'}(t, x_0)x)$ for every $(t, x) \in \mathcal{B}(t_0, P^{'}(t_0, x_0)x_0) \subset \mathbb{R}^n$. As $\mathcal{B}(t_0, P^{'}(t_0, x_0)x_0) \subset \mathcal{I} \times \mathbb{R}^n$, this allows to specify a solution working in the original coordinate system on $\mathbb{R}^n$.

By construction, the spaces $\mathbb{K}^{'}$, $\mathbb{K}$ are locally h invariant. We exploit this to compute a solution of the algebraic equations in DAEs.

If $P = Q = I_n$, then condition (28) coincides with the condition of Theorem 2.3. If the projections are such that $P = F_x^{+} F_x$ and $Q = F_x F_x^{+}$, then (28) is satisfied on $\mathcal{I} \times \Omega$.

For linear systems, the solution (32) is globally defined on $\mathbb{R}^n$ and we can compute the implicit function explicitly, cp. [2], Corollary 4.8.

Corollary 3.3. *Consider* $Ax = f$ *with* $A \in C(\mathcal{I}, \mathbb{R}^{n \times n})$, $f \in C(\mathcal{I}, \mathbb{R}^n)$. *Let* $\mathbb{K}, \mathbb{K}$ *be linear subspaces with* $\dim(\mathbb{K}) = \dim(\mathbb{K})$ *on* $\mathcal{I} \times \Omega$ *and let* $P, Q \in C(\mathcal{I}, \mathbb{R}^{n \times n})$ *be the orthogonal projections onto* $\mathbb{K}, \mathbb{K}$, *respectively, with complements* $P^{'}, Q^{'}$. *Let* (t_0, x_0) *be consistent. If*

$$\big(Q(t_0)A(t_0)P(t_0)\big)^{+} Q(t_0)A(t_0)P(t_0) = P(t_0), \tag{32a}$$

$$Q(t_0)A(t_0)P(t_0) \big(Q(t_0)A(t_0)P(t_0)\big)^{+} = Q(t_0), \tag{32b}$$

then there exist a neighborhood $\mathcal{B}(t_0) \subset \mathcal{I}$, *such that (om.arg.)*

$$Px = -(QAP)^{+} QAP^{'} x + (QAP)^{+} Qf \tag{33}$$

solves $QAx = f$ *for every* $t \in \mathcal{B}(t_0)$ *and* $P^{'}(t)x \in \mathbb{R}^n$. *In particular,* $(QAP)^{+} QAP^{'} e_i$, $(QAP)^{+} Qf \in C(\mathcal{B}(t_0), \mathbb{K})$ *for* $i = 1, ..., n$.

For DAEs, Theorem 3.1 allows to solve $F(t, x, \dot{x}) = 0$ for particular components x_a, $\dot{x}_d$ of x, $\dot{x}$, respectively, where $x_a := P_d^{'}x$, $x_d := P_d x$ and P_d is the projection onto the differential components. Solving for $\dot{x}_d$, in particular, we ask if the implicit defined ODE $\dot{x}_d = h(t, x_d, x_a, \dot{x}_a)$ preserves $\mathrm{Rg}(P_d)$ as Φ_h invariant subspace. As for purely algebraic equations, we denote the set of consistent initial values by

$$\mathcal{C}_F = \{(t_0, x_0, \dot{x}_0) \in \mathbb{D} \mid F(t_0, x_0, \dot{x}_0) = 0\}.$$

In [2], Corollary 4.9, then we have obtained the following result.

Corollary 3.4. *Consider $F(t, x, \dot{x}) = 0$, $F \in C(\mathbb{D}, \mathbb{R}^n)$. Let $\mathbb{K}, \mathbb{K}$ be linear subspaces with $\dim(\mathbb{K}) = \dim(\mathbb{K})$ on $\mathbb{D}$ and let $P, Q \in C^1(\mathbb{D}, \mathbb{R}^{n\times n})$ be the orthogonal projections onto $\mathbb{K}, \mathbb{K}$ with complements $P^{'}, Q^{'}$, respectively. Let $z_0 := (t_0, x_0, \dot{x}_0) \in \mathbb{D}$ be consistent and set $x_{d,0} := P(z_0)x_0$, $x_{a,0} := P^{'}(z_0)x_0$ and $\dot{x}_{d,0} := \dot{P}(z_0)x_0 + P(z_0)\dot{x}_0$, $\dot{x}_{a,0} := \dot{P}^{'}(z_0)x_0 + P^{'}(z_0)\dot{x}_0$. If*

$$\big(Q(z_0)F_{\dot{x}}(z_0)P(z_0)\big)^+ Q(z_0)F_{\dot{x}}(z_0)P(z_0) = P(z_0), \tag{34a}$$

$$Q(z_0)F_{\dot{x}}(z_0)P(z_0)\big(Q(z_0)F_{\dot{x}}(z_0)P(z_0)\big)^+ = Q(z_0), \tag{34b}$$

then there exist neighborhoods $\mathcal{B}(t_0, x_{d0}, x_{a0}, \dot{x}_{a,0})$, $\mathcal{B}(\dot{x}_{d,0})$ and a function $h \in C(\mathcal{B}(t_0, x_{d0}, x_{a0}, \dot{x}_{a,0}), \mathcal{B}(\dot{x}_{d,0}))$, such that

$$\dot{x}_d = h(t, x_d, x_a, \dot{x}_a) \text{ solves } Q(t, x_0, \dot{x}_0)F(t, x_d + x_a, \dot{x}_d + \dot{x}_a) = 0 \quad \text{for every } (t, x_d, x_a, \dot{x}_a) \in \mathcal{B}(t_0, x_{d0}, x_{a0}, \dot{x}_{a,0}). \tag{35}$$

For $(y_1, y_2) \in \mathcal{B}(x_{a0}, \dot{x}_{0,a})$, if $h(\cdot, \cdot, y_1, y_2) \in C^{Lip}_{loc}(\mathcal{B}(t_0, x_{d0}), \mathbb{R}^n)$, then $\mathbb{K}(\cdot, x_0, \dot{x}_0)$ is Φ_h invariant.

Corollary 3.4 allows to solve $F(t, x, \dot{x}) = 0$ for particular components of $\dot{x}$ while keeping the original coordinate system. Using a projection P onto the desired components $\frac{d}{dt}[Px]$, then provided (34), there exists an implicit defined function h that parametrizes $\frac{d}{dt}[Px]$ with Px, $P^{'}x$, $\frac{d}{dt}[P^{'}x]$ and the solution of $Q(t, x_0, \dot{x}_0)F(t, x, \dot{x}) = 0$ is given by $(t, x_d + x_a, h(t, x_d, x_a, \dot{x}_a) + \dot{x}_a)$ for every $(t, x_d, x_a, \dot{x}_a) \in \mathcal{B}(t_0, x_{d0}, x_{a0}, \dot{x}_{a,0})$.

For given $(y_1, y_2) \in \mathcal{B}(x_{a0}, \dot{x}_{0,a})$, then h provides an ODE $\dot{x} = h(t, x, y_1, y_2)$ on $\mathcal{B}(t_0, x_{d0})$. Provided $h(\cdot, \cdot, y_1, y_2) \in C^{\mathrm{Lip}}_{\mathrm{loc}}(\mathcal{B}(t_0, x_{d0}), \mathbb{R}^n)$, then there exists a flow Φ_h and $\mathbb{K}(\cdot, x_0, \dot{x}_0)$ is Φ_h invariant.

4 A flow formula for DAEs

We use the projection approach presented in Section 3 to remodel implicit differential equations as semi-explicit systems. Pointing out the difficulties in the treatment of DAEs in Section 4.1, we motivate and introduce the concept of the strangeness-index in Section 4.2. This theory provides a decomposition of a large class of DAEs into a set of uniquely solvable differential and algebraic equations. Based on these results, in Section 4.3 we separate the differential and algebraic variables using the decomposition by projection.

4.1 The problem with DAEs

To solve the implicit system

$$F(t, x, \dot{x}) = 0, \quad x(t_0) = x_0, \tag{36}$$

with $(t_0, x_0) \in \mathcal{J} \times \Omega$, by the classical ODE theory, the function F must provide a unique relation between x and its derivative $\dot{x}$. If the Jacobian $F_{\dot{x}}$ is pointwise nonsingular in a neighborhood of $\dot{x}$, then we can use the implicit function theorem and resolve (36) as ODE

$$\dot{x} = h(t, x), \quad x(t_0) = x_0. \tag{37}$$

For such systems, solvability and uniqueness is characterized by Theorem 2.6 that assures a unique solution for every $(t_0, x_0) \in \mathcal{J} \times \Omega$ if the implicit function h is locally Lipschitz continuous in x. This solution can be computed numerically using e.g. a Runge-Kutta or multistep discretization. For special cases, like linear problems, also an analytical solution is available.

However, if (36) contains both dynamic and algebraic equations, then the Jacobian $F_{\dot{x}}$ is singular. To solve such a coupled system, we have to separate the differential and algebraic equations and variables and decompose $F(t, x, \dot{x}) = 0$ with respect to the regular and singular components of $F_{\dot{x}}$, e.g., by a change of coordinates. Then, we can locally reformulate (36) as a system of the form

$$F_1(t, x_1, x_2, \dot{x}_1) = 0, \tag{38a}$$
$$F_2(t, x_1, x_2) = 0, \tag{38b}$$

where F_{1,x_1} is pointwise nonsingular. If F_{2,x_2} is pointwise nonsingular as well, then, applying the implicit function theorem again, we can solve (38b) locally for x_2. Inserting the solution x_2 into (38a), then we obtain an ODE for x_1 and solvablity again is covered by Theorem 2.6.

However, if F_{2,x_2} is singular, then we have to decompose the algebraic equation (38b) with respect to the regular and singular components of F_{2,x_2} and we obtain the system

$$F_1(t, x_1, x_{2_1}, x_{2_2}, \dot{x}_1) = 0, \tag{39a}$$
$$F_{2_1}(t, x_1, x_{2_1}, x_{2_2}) = 0, \tag{39b}$$
$$F_{2_2}(t, x_1) = 0, \tag{39c}$$

where $F_{2_1,x_{2_2}}$ is pointwise nonsingular. Now, we observe a strange coupling between the differential and algebraic equations (39a) and (39c): where (39a) determines x_1 by a dynamic relation, (39c) fixes components of x_1 algebraically. In particular, the associated derivatives are fixed by an algebraic equation. To solve this coupling, we must decompose (39c) with respect to the regular and singular components of F_{2_2,x_1}, i.e., we get that

$$F_1(t, x_{1_1}, x_{1_2}, x_{2_1}, x_{2_2}, \dot{x}_{1_1}, \dot{x}_{1_2}) = 0, \tag{40a}$$

$$F_{2_1}(t, x_{1_1}, x_{1_2}, x_{2_1}, x_{2_2}) = 0, \tag{40b}$$

$$F_{2_{21}}(t, x_{1_1}, x_{1_2}) = 0, \tag{40c}$$

with $F_{2_{21},x_{1_1}}$ nonsingular. Then, there exists a locally implicitly defined function h_{1_1}, such that

$$x_{1_1} = h_{1_1}(t, x_{1_2}). \tag{41a}$$

The derivatives are given by

$$\dot{x}_{1_1} = \dot{h}_{1_1}(t, x_{1_2}), \tag{41b}$$

requiring additional smoothness of the function $F_{2_{21}}$. Inserting (41) into (40a) and noting that $\dot{h}_{1_1}(t, x_{1_2}) = h_{1_1,t} + h_{1_1,x_{1_2}}\dot{x}_{1_2}$, then we observe that (40a) is uniquely solvable for x_{1_2} if and only if $F_{1,\dot{x}_{1_2}} = F_{1,\dot{x}_{1_2}} + F_{1,\dot{x}_{1_1}} h_{1_1,x_{1_2}}$ is nonsingular. If $F_{1,\dot{x}_{1_2}}$ is singular, then we must repeat the decomposition procedure for (40a) until we get a uniquely solvable system.

The process of differentiating and decoupling may be repeated several times until the differential and algebraic equations are clearly separated and uniquely solvable. The number of times, the original system (36) has to be differentiated determines the *index* of the DAE. There are different index concepts that differ in the smoothness and regularity assumptions on the function $F(t, x, \dot{x})$, for example the differentiation index [8, 10, 6], the tractability index [25, 41, 42, 52], the perturbation index [5, 28, 29] or the structural index [47, 48]. A comparison of the different concepts is given in [15] or recently in [45].

4.2 The concept of the strangeness-index

We follow the concept of the *strangeness-index* as it was developed in [34, 35, 36] and [37]. The strangeness index [37] is a remodeling framework that allows to classify general DAEs with respect to their solvability properties, analytically and numerically.

4.2.1 Nonlinear systems

The strangeness index is based on the so-called *derivative array* [10], which is the inflated DAE obtained from $F(t,x,\dot{x})=0$ by successive differentiation, i. e.,

$$F_\ell(t,x,\dot{x},...,x^{(\ell+1)})=0, \tag{42}$$

for $\ell\in\mathbb{N}$, where

$$F_\ell(t,x,\dot{x},...,x^{(\ell+1)})=\begin{bmatrix} F(t,x,\dot{x})\\ \frac{d}{dt}F(t,x,\dot{x})\\ \vdots\\ \frac{d^\ell}{dt^\ell}F(t,x,\dot{x})\end{bmatrix}.$$

The derivative array also provides the starting point for the differentiation index, see [8, 10, 6] and [37] for a comparison of the differentiation and strangeness-index.

Every sufficiently smooth solution of $F(t,x,\dot{x})=0$ solves the derivative array (42). Vice versa, if $(t,x,\dot{x},...,x^{(\ell+1)})$ solves (42), then $(t,x,\dot{x})$ also solves $F(t,x,\dot{x})=0$. For suitable $\ell\in\mathbb{N}$, the idea of the strangeness-index is to filter out a set of linearly independent differential and algebraic equations that uniquely determine this solution $(t,x,\dot{x},...,x^{(\ell+1)})$. This may include algebraic equations for derivatives of x, so we consider (42) formally as an algebraic equation for $z:=(t,x,\dot{x},...,x^{(\ell+1)})$ and define the solution set of (42) by

$$\mathcal{L}_\ell:=\{z\in\mathcal{I}\times\mathbb{R}^n\times\ldots\times\mathbb{R}^n \mid F_\ell(z)=0\}. \tag{43}$$

For a given $z\in\mathcal{L}_\ell$, we define the Jacobians

$$M_\ell(t,x,\dot{x},...,x^{(\ell+1)}):=F_{\ell;\dot{x},...,x^{(\ell+1)}}(t,x,\dot{x},.,x^{(\ell+1)}), \tag{44a}$$
$$N_\ell(t,x,\dot{x},...,x^{(\ell+1)}):=F_{\ell;x}(t,x,\dot{x},.,x^{(\ell+1)}), \tag{44b}$$

where $F_{\ell;\dot{x},...,x^{(\ell+1)}}=\left[\frac{\partial}{\partial x}F_\ell\ldots\frac{\partial}{\partial x^{(\ell+1)}}F_\ell\right]$ and $F_{\ell;x}=\frac{\partial}{\partial x}F_\ell$ denote the partial derivatives with respect to the variables $\dot{x},...,x^{(\ell+1)}$ and x, respectively. To solve (42) at least locally for $(t,x,\dot{x})$, we cite the following assumptions on the derivative array [37], p. 155.

Hypothesis 4.1 ([37])**.** *Consider $F(t,x,\dot{x})=0$. Let $F\in C^m(\mathbb{D},\mathbb{R}^n)$ be sufficiently smooth, such that there exist $\mu,d,a\in\mathbb{N}_0$ with $\mathcal{L}_\mu\neq\emptyset$. Furthermore, for every $z_0\in\mathcal{L}_\mu$ there exist a sufficiently small neighborhood $\mathcal{B}(z_0)\subset\mathcal{L}_\mu$, such that the following properties hold.*

1. *On $\mathcal{L}_\mu$, rank$(M_\mu(z_0))=(\mu+1)n-a$ and there exists a matrix function Z_2 with orthonormal columns and maximal rank$(Z_2)=a$ on $\mathcal{L}_\mu$, such that $Z_2^TM_\mu=0$ on $\mathcal{L}_\mu$. Locally, we have that $Z_2\in C^\mu(\mathcal{B}(z_0),\mathbb{R}^{(\mu+1)n\times a})$.*

2. *On $\mathcal{L}_\mu$, rank$(Z_2^T \bar{N}_\mu) = a$ on $\mathcal{L}_\mu$, where $\bar{N}_\mu = N_\mu[I_n\, 0]$, and there exists T_1 with orthonormal columns and maximal rank$(T_1) = d$, $d = n - a$, on $\mathcal{L}_\mu$, such that $Z_2^T \bar{N}_\mu T_1 = 0$ on $\mathcal{L}_\mu$. Locally, we have that $T_1 \in C^\mu(\mathcal{B}(z_0), \mathbb{R}^{n\times d})$.*

3. *On $\mathcal{L}_\mu$, rank$(F_{\dot{x}}(t,x,\dot{x})T_1(z_\mu)) = d$ on $\mathcal{L}_\mu$ and there exists $Z_1 \in \mathbb{R}^{n\times d}$ with orthonormal columns and maximal rank$(Z_1) = d$ on $\mathcal{L}_\mu$, such that rank$(Z_1^T F_{\dot{x}} T_1) = d$ on $\mathcal{L}_\mu$.*

Hypothesis 4.1 gives rise to the definition of the strangeness-index.

Definition 4.1. *Consider $F(t,x,\dot{x}) = 0$. The minimal μ for which $F(t,x,\dot{x}) = 0$ satisfies Hypothesis 4.1, is called the* strangeness index *or* s-index *of F or $F(t,x,\dot{x}) = 0$, respectively.*

If $\mu = 0$, then $F(t,x,\dot{x}) = 0$ is called *strangeness-free* or *s-free.* In particular, ordinary differential equations, purely algebraic equations and semi-explicit systems of the form $\dot{x}_1 = f(t,x_1,x_2)$, $0 = g(t,x_1,x_2)$ with g_{x_2} nonsingular are s-free.

Provided Hypothesis 4.1, the set $\mathcal{L}_\mu$ is a manifold with $\dim(\mathcal{L}_\mu) = n+1$, cp. [36], and for every $z_0 \in \mathcal{L}_\mu$, there exist neighborhoods $\mathcal{B}(t_0,x_0) \subset \mathbb{R}^n$, $\mathcal{B}(z_0) \subset \mathcal{L}_\mu$ and a local parametrization $\varphi \in C^1(\mathcal{B}(t_0,x_0), \mathcal{B}(z_0))$ such that $z = \varphi(t,x)$ for every $z \in \mathcal{B}(z_0)$. For every sufficiently smooth solution x^* of $F(t,x,\dot{x}) = 0$, then the associated trajectory

$$z^*(t) := (t, x^*, \dot{x}^*, ..., x^{*(\mu+1)}) \in \mathcal{L}_\mu$$

can locally be parametrized by $(t, x(t))$ via $z(t) = \varphi(t, x(t))$. Then, the derivative array (42) can locally be remodeled as s-free system, cp. [36, 37], p. 163.

Theorem 4.1. *[36, 37] Let $F(t,x,\dot{x}) = 0$, $F \in C^\mu(\mathbb{D}, \mathbb{R}^n)$ satisfy Hypothesis 4.1 with μ, d, a. Every sufficiently smooth solution of $F(t,x,\dot{x}) = 0$ locally solves*

$$\hat{F}_1(t,x,\dot{x}) = 0, \tag{45a}$$

$$\hat{F}_2(t,x) = 0, \tag{45b}$$

where $\hat{F}_1(t,x,\dot{x}) := Z_1^T F(t,x,\dot{x})$ and $\hat{F}_2(t,x) = Z_2^T(z(t,x))F_\mu(z(t,x))$ for $z(t,x) \in \mathcal{L}_\mu$.

If $F \in C^{\mu+1}(\mathbb{D}, \mathbb{R}^n)$ satisfies Hypothesis 4.1 with $\mu+1, d, a$, then (45) has a unique solution for every $z_0 \in \mathcal{L}_\mu$ and this solution locally solves $F(t,x,\dot{x}) = 0$, $x(t_0) = x_0$. Then, $\hat{F}_1 \in C^1(\mathcal{B}(t,x,\dot{x}), \mathbb{R}^d)$ and $\hat{F}_2 \in C^1(\mathcal{B}(t,x), \mathbb{R}^a)$.

We call (45) the *s-free formulation* of $F(t,x,\dot{x}) = 0$. It consists of d differential equations and a algebraic equations. In neighborhoods $\mathcal{B}(t,x)$, $\mathcal{B}(t,x,\dot{x})$ of a sufficiently smooth solution of $F(t,x,\dot{x}) = 0$, the functions $\hat{F}_1, \hat{F}_2$ satisfy $\hat{F}_1 \in C(\mathcal{B}(t,x,\dot{x}), \mathbb{R}^d)$ and $\hat{F}_2 \in C(\mathcal{B}(t,x), \mathbb{R}^a)$. If $F \in C^{\mu+1}(\mathbb{D}, \mathbb{R}^n)$, then $\hat{F}_1 \in C^1(\mathcal{B}(t,x,\dot{x}), \mathbb{R}^d)$ and $\hat{F}_2 \in$

$C^1(\mathcal{B}(t,x),\mathbb{R}^a)$, in particular. The Jacobians $\hat{F}_{1,\dot{x}}$, $\hat{F}_{2,x}$ satisfy $\operatorname{rank}(\hat{F}_{1,\dot{x}}) = d$, $\operatorname{rank}(\hat{F}_{2,x}) = a$ and $\ker(\hat{F}_{1,\dot{x}}) \cap \ker(\hat{F}_{2,x}) = 0$ pointwise on $\mathcal{L}_\mu$.

The remodeling (45) allows to treat DAEs with higher, but well-defined s-index, locally in the framework of s-free systems.

Definition 4.2. *Let $F \in C^\mu(\mathbb{D},\mathbb{R}^n)$ satisfy Hypothesis 4.1 with μ, d, a.*

1. *We call an initial value $(t_0, x_0, \dot{x}_0) \in \mathbb{D}$* consistent *if there exists $\ddot{x}_0, ..., x_0^{(\ell+1)} \in \mathbb{R}^n$, such that $(t_0, x_0, \dot{x}_0, \ddot{x}_0..., x_0^{(\mu+1)}) \in \mathcal{L}_\mu$. We write $\hat{z}_0 := (t_0, x_0, \dot{x}_0)$ and $z_0 := (\hat{z}_0, \dot{x}_0, ..., x_0^{(\mu+1)})$ and denote the set of consistent initial values by*

$$\mathcal{C}_F := \{\hat{z}_0 \in \mathbb{D} \mid \hat{z}_0 \text{ is consistent } \}. \tag{46}$$

   For a fixed initial time $t_0 \in \mathcal{I}$, we set $\mathcal{C}_F(t_0) := \{(x_0, \dot{x}_0) \in \mathbb{D} \setminus \mathcal{I} \mid (t_0, x_0, \dot{x}_0) \in \mathcal{C}_F\}$.

2. *We call $F(t, x, \dot{x}) = 0$* regular with s-index μ *if $F \in C^{\mu+1}(\mathbb{D},\mathbb{R}^n)$ and $F(t, x, \dot{x}) = 0$ satisfies Hypothesis 4.1 with μ, d, a and $\mu + 1, d, a$, respectively.*

3. *If $F(t, x, \dot{x}) = 0$ is regular with s-index μ and $\hat{z}_0 \in \mathcal{C}_F$, then we denote by $\mathcal{I}_{max}(\hat{z}_0)$ the* maximal interval of existence *of the solution associated with $\hat{z}_0$.*

If F is regular with s-index μ and $\hat{z}_0 \in \mathcal{C}_F$, then x solves $F(t, x, \dot{x}) = 0$ if and only if x locally solves $\hat{F}(t, x, \dot{x}) = 0$. Then, we call $F(t, x, \dot{x}) = 0$ and $\hat{F}(t, x, \dot{x}) = 0$ *locally equivalent* on $\mathcal{L}_\mu$.

Necessarily, $F \in C^\mu(\mathbb{D},\mathbb{R}^n)$ if $F(t, x, \dot{x}) = 0$ has s-index μ. As Hypothesis 4.1 is a local result, the domain of definition $\mathbb{D}$ may be reduced to match the smoothness assumption.

As the differential and algebraic equations (45a), (45a) are linear independent, all constraints are explicitly given and the s-free formulation (45) can be solved numerically with the same order of accuracy as ODEs, see [37], p. 251.

Constructing the s-free formulation can be incorporated in the numerical integration, see [37], ch. 6. Computing the derivative array, e.g., by automatic differentiation [26], then along a numerical solution z_Δ of (45), the assertions of Hypothesis 4.1 can be checked numerically by an SVD of the Jacobians $M_\mu(z_\Delta)$, $N_\mu(z_\Delta)$ [24, 37]. But, as $M_\mu(z_\Delta)$, $N_\mu(z_\Delta)$ only approximate the Jacobians $M_\mu(z)$, $N_\mu(z)$ along the exact solution z, the computed values $\mu_\Delta, d_\Delta, a_\Delta$ only indicate the true characteristic values μ, d, a. Additionally, computing the characteristic values $\mu_\Delta, d_\Delta, a_\Delta$ involves numerical rank decisions, so in cases of doubt a higher value of μ should be choosen to ensure that all hidden constraints are explicitly given, see [37], p. 281 and [44].

To compute a consistent initial value $z_0 \in \mathcal{L}_F$, one can either use a fixpoint iteration on the derivative array, the Gauss-Newton method [36], or decompose the variables with a time-varying transformation, cp. [38].

4.2.2 Linear systems

For linear initial value problems

$$E\dot{x} = Ax + f, \quad x(t_0) = x_0 \tag{47}$$

with $E, A \in C^\ell(\mathcal{I}, \mathbb{R}^{n\times n})$, $f \in C^\ell(\mathcal{I}, \mathbb{R}^n)$, $\ell \in \mathbb{N}$ and $(t_0, x_0) \in \mathcal{I} \times \mathbb{R}^n$, the derivative array is given by

$$M_\ell \dot{z}_\ell = N_\ell z_\ell + g_\ell, \tag{48}$$

where

$$M_\ell = \begin{bmatrix} E & & & & \\ \dot{E} - A & E & & & \\ \ddot{E} - 2\dot{A} & 2\dot{E} - A & E & & \\ \vdots & & \ddots & \ddots & \\ E^{(\ell)} - \ell \dot{A}^{(\ell-1)} & & \dots & \ell\dot{E} - A & E \end{bmatrix}, \tag{49a}$$

$$N_\ell = \begin{bmatrix} A & 0 & \dots & 0 \\ \dot{A} & 0 & \dots & 0 \\ \ddot{A} & 0 & \dots & 0 \\ \vdots & 0 & \dots & 0 \\ A^{(\ell)} & 0 & \dots & 0 \end{bmatrix}, \tag{49b}$$

and

$$z_\ell = \left[x^T,\ \dot{x}^T,\ \ddot{x}^T,\ \dots\ x^{(\ell)T}\right]^T, \quad g_\ell = \left[f^T,\ \dot{f}^T,\ \ddot{f}^T,\ \dots\ f^{(\ell)T}\right]^T.$$

The derivative array (48) is linear in the state z_ℓ, such that the Jacobians M_ℓ, N_ℓ only depend on time. The assertions of Hypothesis 4.1 are verified for every sufficiently smooth functions E, A and f, cp. [37], p. 108.

Theorem 4.2. *[37] Consider $E\dot{x} = Ax + f$. Let $E, A \in C^m(\mathcal{I}, \mathbb{R}^{n\times n})$, $f \in C^m(\mathcal{I}, \mathbb{R}^n)$ be sufficiently smooth and consider M_ℓ, N_ℓ, $\ell \le m$ as given in (48). Then, there exist $\mu, d, a \in \mathbb{N}_0$, such that the following properties hold.*

1. *On $\mathcal{I}$, rank$(M_\mu) = (\mu+1)n - a$ and there exists $Z_2 \in C^\mu(\mathcal{I}, \mathbb{R}^{(\mu+1)n\times a})$ with orthonormal columns and maximal rank$(Z_2) = a$, such that $Z_2^T M_\mu = 0$.*
2. *On $\mathcal{I}$, rank$(Z_2^T \bar{N}_\mu) = a$, where $\bar{N}_\mu = N_\mu[I_n\, 0]$, and there exists $T_1 \in C^\mu(\mathcal{I}, \mathbb{R}^{n\times d})$ with orthonormal columns and maximal rank$(T_1) = d$, such that $Z_2^T \bar{N}_\mu T_1 = 0$.*

3. *On $\mathcal{I}$, rank$(ET_1) = d$ and there exists $Z_1 \in \mathbb{R}^{n\times d}$ with orthonormal columns and maximal rank$(Z_1) = d$, such that rank$(Z_1^T ET_1) = d$.*

For linear problems, the Jacobians M_μ, N_μ are defined globally on $\mathbb{R}^n$ and hence, the assertions of the Hypothesis 4.1 are ensured globally on $\mathbb{R}^n$ for every sufficiently smooth matrix pair $E, A \in C^\mu(\mathcal{I}, \mathbb{R}^{n\times n})$. On every interval, on which the rank conditions of Theorem 4.2 are satisfied, $E\dot{x} = Ax + f$ can be remodeled as a s-free system, cp. [37], p. 109, 111.

Theorem 4.3 ([37])**.** *Let $E, A \in C^{\mu+1}(\mathcal{I}, \mathbb{R}^{n\times n})$ satisfy Theorem 4.2 with μ, d, a. Then, x solves $E\dot{x} = Ax + f$, $f \in C^{\mu+1}(\mathcal{I}, \mathbb{R}^n)$ on $\mathcal{I}$ if and only if x solves*

$$\begin{bmatrix} \hat{E}_1 \\ 0 \end{bmatrix} \dot{x} = \begin{bmatrix} \hat{A}_1 \\ \hat{A}_2 \end{bmatrix} x + \begin{bmatrix} \hat{f}_1 \\ \hat{f}_2 \end{bmatrix}, \tag{50}$$

on $\mathcal{I}$, where $\hat{E}_1 = Z_1^T E$, $\hat{A}_1 = Z_1^T A$, $\hat{A}_2 = Z_2^T N_\mu [I_n \ 0]$ and $\hat{f}_1 = Z_1^T f$, $\hat{f}_2 = Z_2^T g_\mu$.

If $(t_0, x_0) \in \mathcal{I} \times \Omega_x$ satisfies the consistency condition

$$0 = \hat{A}_2(t_0)\, x_0 + f_2(t_0), \tag{51}$$

then the initial value problem $E\dot{x} = Ax + f$, $x(t_0) = x_0$ has a unique solution.

Again, as the Jacobians M_ℓ, N_ℓ are independent of the state, the remodeling (50) is globally defined on $\mathbb{R}^n$. Furthermore, $E\dot{x} = Ax+f$ is *regular* with s-index μ if $E\dot{x} = Ax+f$ has s-index μ and $E, A \in C^{\mu+1}(I, \mathbb{R}^{n\times n})$, $f \in C^{\mu+1}(I, \mathbb{R}^n)$.

Regarding (51), the set of consistent initial values is the affine subspace

$$\mathcal{C}_{E,A,f} = \{(t_0, x_0) \in \mathcal{I} \times \mathbb{R}^n \mid 0 = \hat{A}_2(t_0)\, x_0 + \hat{f}_2(t_0)\}.$$

For given initial time $t_0 \in \mathcal{I}$, we set $\mathcal{C}_{E,A,f}(t_0) = \{x_0 \in \mathbb{R}^n \mid (t_0, x_0) \in \mathcal{C}_{E,A,f}\}$.

Concerning the linearization of $F(t, x, \dot{x}) = 0$ and its s-free formulation, we make the following observation.

Remark 4.1. *The* time varying linearization *of $F(t, x, \dot{x}) = 0$ along a given solution $x^*(t)$ is given by*

$$F_{\dot{x}}(t, x^*, \dot{x}^*)\dot{x} = -F_x(t, x^*, \dot{x}^*)x + F_{\dot{x}}(t, x^*, \dot{x}^*)\dot{x}^* + F_x(t, x^*, \dot{x}^*)x^*, \tag{52}$$

where $F_{\dot{x}}, F_x$ denote the partial derivatives of F, respectively, cp. [12]. Comparing the derivative arrays (42) *and* (48) *and the associated Jacobians* (44) *and* (49)*, then we observe that the subspaces $Rg(Z_1)$, $Rg(Z_2)$, $Rg(T_1)$ associated with $F(t, x, \dot{x}) = 0$ correspond to those associated with* (52)*. Consequently, the operations of remodeling and linearizations commute. We obtain the same linearized s-free system if $F(t, x, \dot{x}) = 0$ is first remodeld, then linearized or first linearized and then remodeled, cp. [12, 36].*

4.3 The projection concept

Having separated the differential and algebraic equations using the concept of the s-index, now we filter out the differential and algebraic variables and remodel $F(t,x,d\dot{x})=0$ as a semi-explicit system. Wishing to preserve the original coordinate system to study properties like positivity, we apply the projection approach presented in [2].

Considering the s-free remodeling (45) and assuming that the Jacobians $\hat{F}_x$, $\hat{F}_{\dot{x}}$ exist on $\mathcal{C}_F$, we observe that the solution components lying in $\text{coker}(\hat{F}_{\dot{x}})$ are fixed by a differential equation while the components in $\ker(\hat{F}_{\dot{x}})$ are determined algebraically. Setting

$$\mathbb{X}_d(t_0,x_0,\dot{x}_0) := \text{coker}(\hat{F}_{\dot{x}}(t_0,x_0,\dot{x}_0)), \tag{53a}$$

$$\mathbb{X}_a(t_0,x_0,\dot{x}_0) := \ker(\hat{F}_{\dot{x}}(t_0,x_0,\dot{x}_0)), \tag{53b}$$

for $(t_0,x_0,\dot{x}_0)\in\mathcal{C}_F$, we can locally partition the variables with respect to $\mathbb{X}_d$ and $\mathbb{X}_a$. The associated projection is given by the Moore-Penrose projection

$$P_d(t_0,x_0,\dot{x}_0) = \hat{F}_{\dot{x}}^+(t_0,x_0,\dot{x}_0)\hat{F}_{\dot{x}}(t_0,x_0,\dot{x}_0), \tag{54}$$

cp. Corollary 2.1.

Lemma 4.1. *Let $F(t,x,\dot{x})=0$ have s-index μ and let the Jacobians $\hat{F}_x$, $\hat{F}_{\dot{x}}$ exist on $\mathcal{C}_F$. Consider P_d given by (54). The projection P_d is uniquely defined by $F(t,x,\dot{x})=0$ and independent of the chosen remodeling $\hat{F}$. On $\mathcal{C}_F$, rank$(P_d)=d$. For every $\hat{z}_0\in\mathcal{C}_F$, there exists a neighborhood $\mathcal{B}(\hat{z}_0)\subset\mathcal{C}_F$ such that $P_d\in C^1(\mathcal{B}(\hat{z}_0),\mathbb{R}^n)$.*

Proof. If $F(t,x,\dot{x})=0$ has s-index μ, then F induces a s-free system $\hat{F}(t,x,\dot{x})=0$ that is locally defined on $\mathcal{L}_\mu$, cp. Theorem 4.1. The matrices $Z_1\in\mathbb{R}^{n\times d}$, $Z_2\in C^\mu(\mathcal{L}_\mu,\mathbb{R}^{n\times a})$ specifying $\hat{F}(t,x,\dot{x})=0$ are determined up to a change of basis only. If Z_1,Z_2 locally satisfy Hypothesis 4.1, then so do Z_1U_1,Z_2U_2 for every pointwise orthogonal matrix functions $U_1\in\mathbb{R}^{d\times d}$, $U_2\in C^\mu(\mathcal{L}_\mu,\mathbb{R}^{a\times a})$. Setting $U=\left[\begin{smallmatrix}U_1 & 0\\ 0 & U_2\end{smallmatrix}\right]$ and $\tilde{F}=[\tilde{Z}_1^TF,\tilde{Z}_2^TF_\mu]$, then $\tilde{F}=U^T\hat{F}$. Noting that $U_1\in\mathbb{R}^{d\times d}$ and $\partial_{\dot{x}}[Z_2F_\mu]=0$ on $\mathcal{L}_\mu$, cp. Theorem 4.1, then $\tilde{F}_{\dot{x}}=U^T\hat{F}_{\dot{x}}$ on $\mathcal{C}_F$, in particular. Then, $(\tilde{F}_{\dot{x}})^+=\hat{F}_{\dot{x}}U$ pointwise on $\mathcal{C}_F$, cp. Lemma 2.4, and it follows that

$$\tilde{F}_{\dot{x}}^+\tilde{F}_{\dot{x}} = \hat{F}_{\dot{x}}U^TU\hat{F}_{\dot{x}} = \hat{F}_{\dot{x}}^+\hat{F}_{\dot{x}}$$

pointwise on $\mathcal{C}_F$. Thus, the projections P_d and the spaces $\mathbb{X}_d$, $\mathbb{X}_a$ provided by $\hat{F}$ and $\tilde{F}$ coincide, meaning that P is independent of the considered remodeling and unique for F. As rank$(\hat{F}_{\dot{x}})=d$ on $\mathcal{C}_F$, then also rank$(P_d)=d$ on $\mathcal{C}_F$, cp. Corollary 2.1, and $\dim\mathbb{X}_d=d$, $\dim\mathbb{X}_a=n-d=a$ on $\mathcal{C}_F$. By Theorem 2.1, then P_d is locally as smooth as $\hat{F}_{\dot{x}}$ and since $\hat{F}\in C^1(\mathcal{B}(\hat{z}_0),\mathbb{R}^n)$ for every $\hat{z}_0\in\mathcal{C}_F$, then also $P\in C^1(\mathcal{B}(\hat{z}_0),\mathbb{R}^n)$. □

If $\hat{F}_{\dot{x}}$ exists, then $\operatorname{rank}(\hat{F}_{1,\dot{x}})$ is maximal on $\mathcal{C}_F$ and the projection onto $\operatorname{Rg}(\hat{F}_{\dot{x}})$ is given by

$$Q := \hat{F}_{\dot{x}}(t_0, x_0, \dot{x}_0)\hat{F}_{\dot{x}}(t_0, x_0, \dot{x}_0)^+ = \begin{bmatrix} I_d & 0 \\ 0 & 0 \end{bmatrix} \tag{55}$$

for every $(t_0, x_0, \dot{x}_0) \in \mathcal{C}_F$ and every remodeling $\hat{F}$. We denote the complements of P_d, Q by $P_d^{'} = I - P_d$, $Q^{'} = I - Q$, respectively.

Using the projections P_d, Q, we decompose $\hat{F}(t, x, \dot{x}) = 0$ into its differential and algebraic components and give a closed solution formula. As $\hat{F}(t, x, \dot{x}) = 0$ and $F(t, x, \dot{x}) = 0$ are locally equivalent, this solution also solves the original system. We consider linear systems first and then turn to nonlinear problems.

4.3.1 Linear systems

For regular systems $E\dot{x} = Ax + f$ with s-index μ, the s-free remodeling (50) is given by $\hat{E}\dot{x} = \hat{A}x + \hat{f}$. The spaces (53a) are given by $\mathbb{X}_d = \operatorname{coker}(\hat{E})$, $\mathbb{X}_a = \ker(\hat{E})$, and the associated projections by

$$P_d = \hat{E}^+\hat{E}, \quad P_d^{'} = I - \hat{E}^+\hat{E}. \tag{56}$$

As $\hat{E} \in C^1(\mathcal{I}, \mathbb{R}^{n\times n})$ and $\operatorname{rank}(\hat{E}) = d$ on $\mathcal{I}$, then $P_d \in C^1(\mathcal{I}, \mathbb{R}^{n\times n})$, cp. Theorem 2.1. In the following, on $\mathcal{I} \times \mathbb{R}^n$, we set

$$x_d := P_d x, \quad x_a := P_d^{'} x. \tag{57}$$

By construction, the algebraic equation $\hat{A}_2 x + \hat{f}_2 = 0$ in (50) is unique for x_a. To solve this equation for x_a, we verify the assertion of Corollary 3.3.

Lemma 4.2. *Let $E\dot{x} = Ax + f$ have s-index μ. Let Q, P be the projections defined in* (55), (56)*. Then, $(Q^{'}\hat{A}P_d^{'})^+Q^{'}\hat{A}P_d^{'} = P_d^{'}$ and $Q^{'}\hat{A}P_d^{'}(Q^{'}\hat{A}P_d^{'})^+ = Q^{'}$ are satisfied pointwise on $\mathcal{I}$ for every $\hat{A}$ constructed as in Theorem* (4.3)*.*

Proof. To obtain an explicit formula of the Moore-Penrose inverse, we represent $Q^{'}\hat{A}P_d^{'}$ with respect to a pointwise orthogonal basis $T = [T_1\, T_2] \in C^1(\mathcal{I}, \mathbb{R}^{n\times n})$ with $\operatorname{span}(T_1) = \mathbb{X}_d$ and $\operatorname{span}(T_2) = \mathbb{X}_a$. Diagonalizing P_d with respect to T and setting $\hat{A}_{ij} = \hat{A}_i T_j$, $i, j = 1, 2$, we have that

$$Q^{'}\hat{A}P_d^{'} = \begin{bmatrix} 0 & 0 \\ 0 & \hat{A}_{22} \end{bmatrix} T^T.$$

By the construction of $\hat{A}$, then $\hat{A}_{22}$ is pointwise nonsingular and applying Lemma 2.5, we get that

$$(Q'\hat{A}P_d')^+ = T \begin{bmatrix} 0 & 0 \\ 0 & \hat{A}_{22}^{-1} \end{bmatrix}.$$

Then, $(Q'\hat{A}P_d')^+ Q'\hat{A}P_d' = P_d'$ and $Q'\hat{A}P_d'(Q'\hat{A}P_d')^+ = Q'$ are satisfied pointwise on $\mathcal{I}$. □

Noting that $Q'\hat{A} = [0 \ \hat{A}_2^T]^T$, $Q'\hat{f} = [0 \ \hat{f}_2^T]^T$ and $x_a \in \mathrm{Rg}(P_d')$, then Lemma 4.2 allows to solve $hE\dot{x} = \hat{A}x + \hat{f}$ for x_a. Then, we can remodel $E\dot{x} = Ax + f$ as semi-explicit system.

Theorem 4.4. *Let $E\dot{x} = Ax + f$ be regular with s-index μ. Let P_d be given by* (54) *and consider* (57)*. Then, x solves $E\dot{x} = Ax + f$ on $\mathcal{I}$ if and only if x_d, x_a solve the semi-explicit system*

$$\dot{x}_d = D_{\mathcal{S}} x_d + f_d, \tag{58a}$$

$$x_a = -D_{\mathcal{C}} x_d - f_a, \tag{58b}$$

on $\mathcal{I}$, where

$$D_{\mathcal{C}} := (Q'\hat{A}P_d')^+ Q'\hat{A}P_d, \quad D_{\mathcal{S}} := D(P_d - D_{\mathcal{C}}), \quad D := \hat{E}^+\hat{A} + \dot{P}_d, \tag{59}$$
$$f_a := (Q'\hat{A}P_d')^+\hat{f}, \quad f_d := \hat{E}^+\hat{f} - Df_a.$$

An initial value $(t_0, x_0) \in \mathcal{I} \times \mathbb{R}^n$ is consistent, if and only if it satisfies

$$(P_d'(t_0) + D_{\mathcal{C}}(t_0))x_0 = -f_a(t_0). \tag{60}$$

Proof. If $E\dot{x} = Ax + f$ has regular s-index μ, then it has the same solution set as the remodeling $\hat{E}\dot{x} = \hat{A}x + \hat{f}$, cp. Theorem 4.3. Thus, it is sufficient to prove that x solves $\hat{E}\dot{x} = \hat{A}x + \hat{f}$ if and only if x_d, x_a solve (58).

⇒ Let x solve the $\hat{E}\dot{x} = \hat{A}x + \hat{f}$. Projecting $\hat{E}\dot{x} = \hat{A}x + \hat{f}$ by Q', we obtain that

$$0 = Q'\hat{A}(x_a + x_d) + Q'\hat{f}. \tag{61}$$

By Lemma 4.2, then $(Q'\hat{A}P_d')^+ Q'\hat{A}P_d' = P_d'$ and $Q'\hat{A}P_d'(Q'\hat{A}P_d')^+ = Q'$ pointwise on $\mathcal{I}$ and using Corollary 3.3, we solve (61) for x_a, i.e.,

$$x_a = -(Q'\hat{A}P_d')^+ Q'\hat{A}x_d - (Q'\hat{A}P_d')^+ Q'\hat{f}.$$

Noting that $(Q'\hat{A}P_d')^+ = (Q'\hat{A}P_d')^+ Q'$ and setting $D_{\mathcal{C}} := (Q'\hat{A}P_d')^+ Q'\hat{A}P$, $f_a := (Q'\hat{A}P_d')^+\hat{f}$, then we obtain (58b). For x_d, we multiply $\hat{E}\dot{x} = \hat{A}x + \hat{f}$ by $\hat{E}^+$ and obtain that

$$P_d(\dot{x}_d + \dot{x}_a) = \hat{E}^+\hat{A}(x_d + x_a) + \hat{E}^+\hat{f}. \tag{62}$$

Since $P_d x_a = 0$, we have that $\dot{x}_a = P_d^{\prime}\dot{x}_a + \dot{P}_d^{\prime} x_a$. Noting that $\dot{P}_d^{\prime} = -\dot{P}_d$, cp. Lemma 2.3, 1., then $P_d\dot{x}_a = -\dot{P}_d x_a$. Similarly, $\dot{x}_d = P_d\dot{x}_d + \dot{P}_d x_d$, and we conclude from (62) that

$$\dot{x}_d = (\hat{E}^+\hat{A} + \dot{P}_d)(x_d + x_a) + \hat{E}^+\hat{f}.$$

Inserting $x_a = -D_{\mathcal{C}} x_d - f_a$, then we obtain that

$$\dot{x}_d = (\hat{E}^+\hat{A} + \dot{P}_d)(P_d - D_{\mathcal{C}})x_d + \hat{E}^+\hat{f} - (\hat{E}^+\hat{A} + \dot{P}_d)f_a.$$

Setting $D := \hat{E}^+\hat{A} + \dot{P}_d$, $D_{\mathcal{S}} := D(P_d - D_{\mathcal{C}})$, $f_d := \hat{E}^+\hat{f} - Df_a$, we get (58a). Thus, every solution of $E\dot{x} = Ax + f$ solves (58) with x_d, x_a.

$\Leftarrow$ Let x solve (58). By Lemma 4.2, we have that $Q^{\prime}\hat{A}P_d^{\prime}(Q^{\prime}\hat{A}P_d^{\prime})^+ = Q^{\prime}$ pointwise on $\mathcal{I}$. Then, (58b) yields that

$$Q^{\prime}\hat{A}P_d^{\prime} x_a = -Q^{\prime}\hat{A}P_d x_d - Q^{\prime}\hat{f},$$

implying that $x_d + x_a$ solves

$$0 = Q^{\prime}\hat{A}(x_d + x_a) + Q^{\prime}\hat{f}. \tag{63}$$

Inserting (58b) into (58a), then we get that

$$\dot{x}_d = (\hat{E}^+\hat{A} + \dot{P}_d)(x_d + x_a) + \hat{E}^+\hat{f}.$$

Using that $\dot{P}_d x_a = -P_d\dot{x}_a$ and $\dot{x}_d = P_d\dot{x}_d + \dot{P}_d x_d$, again, we find that

$$P_d(\dot{x}_d + \dot{x}_a) = \hat{E}^+\hat{A}(x_d + x_a) + \hat{E}^+\hat{f}.$$

Multiplying by $\hat{E}^+$, it follows that $x_d + x_a$ solves

$$\hat{E}(\dot{x}_d + \dot{x}_a) = Q\hat{A}(x_d + x_a) + Q\hat{f}. \tag{64}$$

Regarding (63) and (64), then we verify that x solves $\hat{E}\dot{x} = \hat{A}x + \hat{f}$ if x_d, x_a solve (58).

For the initial values, we prove that the consistency conditions (51) and (60) are equivalent.

$\Rightarrow$ Let $(t_0, x_0) \in \mathcal{I} \times \mathbb{R}^n$ solve (51). Exploiting that $Q^{\prime} = \left[\begin{smallmatrix} 0 & 0 \\ 0 & I_a \end{smallmatrix}\right]$ and setting $x_{d0} := P(t_0)x_0$, $x_{a0} := P^{\prime}(t_0)x_0$, then (51) reads

$$0 = Q^{\prime}\hat{A}(t_0)(x_{d0} + x_{a0}) + Q^{\prime}\hat{f}(t_0).$$

Multiplying by $(Q^{\prime}\hat{A}P_d^{\prime})^+(t_0)$ and noting that $(Q^{\prime}\hat{A}P_d^{\prime})^+(Q^{\prime}\hat{A}P_d^{\prime}) = P_d^{\prime}$ by Lemma 4.2, this implies that

$$x_{a0} = -D_{\mathcal{C}}(t_0)x_{d0} - f_a(t_0).$$

With $x_0 = x_{d0} + x_{a0}$, then we obtain (60).

$\Leftarrow$ Let $(t_0, x_0) \in \mathcal{I} \times \mathbb{R}^n$ solve (60). By Lemma 4.2, we have that $P_d'(Q'\hat{A}P_d')^+ = (Q'\hat{A}P_d')^+$, implying that $P_d' D_\mathcal{C} = D_\mathcal{C}$ and $P_d' f_a = f_a$. Projecting (60) by $P_d'(t_0)$, then we verify that $x_{a0} = -D_\mathcal{C}(t_0)x_{d0} - f_a(t_0)$. Multiplying by $Q'\hat{A}P_d'$ and exploiting that $Q'\hat{A}P_d'(Q'\hat{A}P_d')^+ = Q'$ on $\mathcal{I}$, cp. Lemma 4.2, then we find that

$$0 = Q'\hat{A}(t_0)(x_{d0} + x_{a0}) + Q'\hat{f}(t_0) = [0\ \hat{A}_2^T]^T x_0 + [0\ \hat{f}_2^T]^T,$$

i.e., x_0 satisfies (51).

□

We call (58a) the *inherent ODE obtained by projection* and (58b) the *algebraic equation obtained by projection*. Since $x_d, x_a \in \mathbb{R}^n$, the differential and algebraic equation (58a), (58b) both have dimension n. The differential equation corresponds to the restriction of the ODE $\dot{y} = D_\mathcal{S} y + f_d$ onto the subspace $\mathbb{X}_d$, where $y \in C^1(\mathcal{I}, \mathbb{R}^n)$ is a function on $\mathbb{R}^n$. We use this observation to compute a solution of the remodeling (58), cp. Lemma 4.5.

The remodeling (58) is unique for a given problem $E\dot{x} = Ax + f$.

Lemma 4.3. *Let $E\dot{x} = Ax + f$ be regular with s-index μ. The matrices given in* (59) *are independent of the s-free remodeling $\hat{E}, \hat{A}, \hat{f}$. Moreover, $D \in C(\mathcal{I}, \mathbb{R}^{n\times n})$, $D_\mathcal{C} \in C^1(\mathcal{I}, \mathbb{R}^{n\times n})$ and $f_d \in C(\mathcal{I}, \mathbb{R}^n)$, $f_a \in C^1(\mathcal{I}, \mathbb{R}^n)$.*

Proof. Let $\hat{E}\dot{x} = \hat{A}x + \hat{f}$ and $\tilde{E}\dot{x} = \tilde{A}x + \tilde{f}$ be s-free formulations of $E\dot{x} = Ax + f$ obtained by Z_1, Z_2 and $\tilde{Z}_1$, $\tilde{Z}_2$, respectively. Then, there exist pointwise orthogonal functions $U_1 \in \mathbb{R}^{d\times d}$, $U_2 \in C^\mu(\mathbb{R}, \mathbb{R}^{a\times a})$ such that $\tilde{Z}_1 = Z_1 U_1$, $\tilde{Z}_2 = Z_2 U_2$. Setting $U = \left[\begin{smallmatrix} U_1 & 0 \\ 0 & U_2 \end{smallmatrix}\right]$, we have that $\tilde{E} = U^T\hat{E}$, $\tilde{A} = U^T\hat{A}$, $\tilde{f} = U^T\hat{f}$. Since $Q' = \left[\begin{smallmatrix} 0 & 0 \\ 0 & I_a \end{smallmatrix}\right]$, then $Q'U^T = U^T Q'$, in particular, implying that $Q'\tilde{A}P_d' = U^T Q'\hat{A}P_d'$. Noting that $\tilde{E}^+ = \hat{E}^+ U$ and $(Q'\tilde{A}P_d')^+ = (Q'\hat{A}P_d')^+ U$ by Lemma 2.4, then we have that $\tilde{E}^+\tilde{A} = \hat{E}^+\hat{A}$, $\tilde{E}^+\tilde{f} = \hat{E}^+\hat{f}$ and $(Q'\tilde{A}P_d')^+\tilde{f} = (Q'\hat{A}P_d')^+\hat{f}$. Thus, $D_\mathcal{C}$ and f_a are unique and as P_d is unique, also $D = \hat{E}^+\hat{A} + \dot{P}_d$ and f_d are unique.

If $E\dot{x} = Ax + f$ is regular with s-index μ, then $E, A \in C^{\mu+1}(\mathcal{I}, \mathbb{R}^{n\times n})$ and $f \in C^{\mu+1}(\mathcal{I}, \mathbb{R}^n)$. By the construction via the derivative array, then $\hat{E}, \hat{A} \in C^1(\mathcal{I}, \mathbb{R}^{n\times n})$ and $\hat{f} \in C^{\mu+1}(\mathcal{I}, \mathbb{R}^n)$. Noting that $P_d \in C^1(\mathcal{I}, \mathbb{R}^{n\times n})$, then we verify that $D \in C(\mathcal{I}, \mathbb{R}^{n\times n})$, $D_\mathcal{C} \in C^1(\mathcal{I}, \mathbb{R}^{n\times n})$ and $f_d \in C(\mathcal{I}, \mathbb{R}^n)$, $f_a \in C^1(\mathcal{I}, \mathbb{R}^n)$.

□

The consistency condition (60) illustrates that only the components $x_{d0} \in \mathbb{X}_d$ can be chosen arbitrarily, whereas $x_{a0} \in \mathbb{X}_a$ is fixed by

$$P_d'(t_0)x_0 = -D_\mathcal{C}(t_0)x_0 - f_a(t_0). \tag{65}$$

The set of consistent initial values is given by

$$\mathcal{C}_{E,A,f} = \{(t_0, x_0) \in \mathcal{I} \times \mathbb{R}^n \mid (P^{'}(t_0) + D_{\mathcal{C}}(t_0))x_0 = -f_a(t_0)\}. \tag{66}$$

The set $\mathcal{C}_{E,A,f}$ is an affine linear subspace, i.e., $\mathcal{C}_{E,A,f} = \mathcal{C}_{E,A} - f_a$. We write $\mathcal{C}_{E,A} := \mathcal{C}_{E,A,0}$. Setting

$$P := P_d - D_{\mathcal{C}}, \qquad P_{f_a}(t,x) := P(t)x - f_a(t), \tag{67}$$

we have found a (affine) projection onto $\mathcal{C}_{E,A}$ and $\mathcal{C}_{E,A,f}$, respectively.

Lemma 4.4. *Let $E\dot{x} = Ax + f$ be regular with s-index μ and d, a as in Theorem 4.2, in particular. Consider the remodeling* (58) *and let P, P_{f_a} be given by* (67)*.*

(i) The matrix P is a projection with complement $P^{'} = P_d^{'} + D_{\mathcal{C}}$.

(ii) $Rg(P) = \mathcal{C}_{E,A}$ and $Rg(P) = Rg(P_d)$.

(iv) $rank(P) = d$ on $\mathcal{I}$ and $P \in C^1(\mathcal{I}, \mathbb{R}^{n\times n})$.

For $f_a \neq 0$, then P_{f_a} is the affine projection onto $\mathcal{C}_{E,A,f}$ and $P_{f_a} \in C^1(\mathcal{I} \times \mathbb{R}^n, \mathbb{R}^{n\times n})$.

Proof. (i) Since $D_{\mathcal{C}} = (Q^{'}\hat{A}P_d^{'})^{+}Q^{'}\hat{A}P_d$, we have that $P_d D_{\mathcal{C}} = D_{\mathcal{C}} P_d^{'} = 0$. Then, $(P_d - D_{\mathcal{C}})^2 = P_d - D_{\mathcal{C}}$, i.e., P is idempotent. The complement is given by $I-(P-D_{\mathcal{C}}) = P_d^{'} + D_{\mathcal{C}}$.
(ii) Noting that $\mathcal{C}_{E,A} = \{x \in \mathbb{R}^n \mid (P_d^{'} + D_{\mathcal{C}})x = 0\}$, we find that $\mathcal{C}_{E,A} = \ker(P^{'})$. Since $\ker(I - P) = \mathrm{Rg}(P)$ if P is a projection, cp. [1], p. 280, it follows that P projects onto $\mathcal{C}_{E,A}$. In particular, as $\mathcal{C}_{E,A,f} = \mathcal{C}_{E,A} - f_a$, then P_{f_a} is the affine projection onto $\mathcal{C}_{E,A,f}$. Regarding $P_d D_{\mathcal{C}} = D_{\mathcal{C}} P_d^{'} = 0$, once more, we verify that $PP_d = P$ and $P_d P = P_d$. By Lemma 2.1, 1., then $\mathrm{Rg}(P) = \mathrm{Rg}(P_d)$.

(iii) Since $\mathrm{rank}(P_d) = \mathrm{rank}(\hat{E}) = d$, where d is given by Theorem 4.2, we verify that $\mathrm{rank}(P) = d$. Noting that $P_d, D_{\mathcal{C}} \in C^1(\mathcal{I}, \mathbb{R}^{n\times n})$ and $f_a \in C^1(\mathcal{I}, \mathbb{R}^n)$, cp. Lemma 4.3, we verify that $P \in C^1(\mathcal{I}, \mathbb{R}^{n\times n})$ and $P_{f_a} \in C^1(\mathcal{I} \times \mathbb{R}^n, \mathbb{R}^{n\times n})$. □

Being able to present the set of admissiable initial values as linear or affine subspace and having identified an (affine) projection accessing this space, we can treat linear DAEs in the setting of [2] where we have studied differential and algebraic equations restricted to linear subspaces. In the remodeling (58), the algebraic equation (58b) has already been solved applying the solution formula of Corollary 3.3. For the differential equation (58a), we verify that $\mathbb{X}_d$ is an flow invariant subspace. Using Lemma 3.4, providing the solution on a flow invariant subspace, then we compute a solution of (58).

Lemma 4.5. *Let $E\dot{x} = Ax + f$ be regular with s-index μ and consider the remodeling (58). For every $(t_0, x_0) \in \mathcal{C}_{E,A,f}$, the solution of $E\dot{x} = Ax + f$, $x(t_0) = x_0$ is given by*

$$x(t) = P(t)\Phi^t_{D_\mathcal{S}}(t_0)P_d(t_0)x_0 + P(t)\int_{t_0}^t \Phi^t_{D_\mathcal{S}}(s)f_d(s)\,ds - f_a(t) \tag{68}$$

for $t \in \bar{\mathcal{I}}$, where $\Phi_{D_\mathcal{S}}$ is the flow induced by $D_\mathcal{S}$. Moreover, $x \in C^1(\mathcal{I}, \mathbb{R}^n)$.

Proof. Consider the remodeling (58). For (58a), we observe that $\dot{x}_d = D_\mathcal{S}x_d + f_d$ is the restriction of the ODE $\dot{y} = D_\mathcal{S}y + f_d$ onto $\mathbb{X}_d$, where $y \in C^1(\mathcal{I}, \mathbb{R}^n)$ is a function on $\mathbb{R}^n$. Noting that

$$P_d' D_\mathcal{S} = P_d'(E^+A + \dot{P}_d)P = \dot{P}_d P_d(P_d - D_\mathcal{C}) = \dot{P}_d P_d \tag{69}$$

is satisfied pointwise on $\mathcal{I}$, then we verify that $\mathbb{X}_d$ is $\Phi_{D_\mathcal{S}}$ invariant on $\mathcal{I}$, cp. (24). As $f_d \in \mathbb{X}_d$ on $\mathcal{I}$ by construction, then also $\mathbb{X}_d$ is $\Phi_{D_\mathcal{S},f_d}$ invariant, cp. Lemma 3.2. Then, the restriction $\Phi_{D_\mathcal{S}} \circ P(t_0, t) := \Phi^t_{D_\mathcal{S}}(t_0)P(t_0)$ satisfies $\Phi_{D_\mathcal{S}} \circ P = \Phi_{D_\mathcal{S}P}$, i.e., $\Phi_{D_\mathcal{S}} \circ P$ is the flow associated with $x_d = D_\mathcal{S}x_d + f_d$, cp. Lemma 3.4. Thus, the solution of (58a) is given by

$$x_d(t) = \Phi^t_{D_\mathcal{S}}(t_0)P_d(t_0)x_0 + \int_{t_0}^t \Phi^t_{D_\mathcal{S}}(s)f_d(s)\,ds. \tag{70}$$

The algebraic equation (58b) is already resolved for x_a. Inserting (70), then the solution of (58b) is given by

$$x_a(t) = -D_\mathcal{C}(t)\Phi^t_{D_\mathcal{S}}(t_0)P_d(t_0)x_0 - D_\mathcal{C}(t)\int_{t_0}^t \Phi^t_{D_\mathcal{S}}(s)f_d(s)\,ds - f_a(t). \tag{71}$$

Since x solves $\hat{E}\dot{x} = \hat{A}x + \hat{f}$ if and only if x_d, x_a solve (58), then, noting that $P = P_d - D_\mathcal{C}$, the solution of $\hat{E}\dot{x} = \hat{A}x + \hat{f}$, $x(t_0) = x_0$ is given by (68). If $E\dot{x} = Ax + f$ has regular s-index μ and $(t_0, x_0) \in \mathcal{C}_{E,A,f}$, then x solves $\hat{E}\dot{x} = \hat{A}x + \hat{f}$, $x(t_0) = x_0$ if and only if x solves $E\dot{x} = Ax + f$, $x(t_0) = x_0$. Thus, (68) also solves $E\dot{x} = Ax + f$ provided $(t_0, x_0) \in \mathcal{C}_{E,A,f}$.

The derivative of (68) is given by

$$\dot{x}(t) = \big(P(t)D_\mathcal{S}(t) + \dot{P}(t)\big)\big(\Phi^t_{D_\mathcal{S}}(t_0)P_d(t_0)x_0 + \int_{t_0}^t \Phi^t_{D_\mathcal{S}}(s)f_d(s)\,ds\big) + P(t)f_d(t) - \dot{f}_a(t) \tag{72}$$

for $t, t_0 \in \bar{\mathcal{I}}$. Since $P, P_d, D_\mathcal{C} \in C^1(\mathcal{I}, \mathbb{R}^{n\times n})$, $f_a \in C^1(\mathcal{I}, \mathbb{R}^n)$ and $D_\mathcal{S} \in C(\mathcal{I}, \mathbb{R}^{n\times n})$, $f_d \in C(\mathcal{I}, \mathbb{R}^n)$, cp. Lemma 4.3 and Lemma 4.4, then we find that $x \in C^1(\mathcal{I}, \mathbb{R}^n)$.

□

For linear DAEs with regular s-index, Lemma 4.5 supplies a closed solution formula for every consistent initial value. Formula (68) illustrates that only the components $x_{d0} \in \mathbb{X}_d(t_0)$ are evolved by a flow, while $x_a \in \mathbb{X}_a$ are fixed by the inhomogeneity f_a. For the inhomogeneity, accordingly, only the components $f_d \in \mathbb{X}_d$ are evolved. Regarding equation (69), we note the following.

Corollary 4.1. *Let $E\dot{x} = Ax + f$ be regular with s-index μ and consider the remodeling* (58)*. Then, $\mathbb{X}_d$ is $\Phi_{D_\mathcal{S}}$ invariant.*

If $\mathbb{X}_d$ is $\Phi_{D_\mathcal{S}}$ invariant, then $P_d'(t)\Phi^t_{D_\mathcal{S}} P_d(t_0) = 0$ for $t \in \bar{\mathcal{I}}$, cp. 3.1. Projecting (68) by P_d, P_d', respectively, then we can recover the differential and algebraic components (70), (71).

We summarize the results of Theorem 4.4, Lemma 4.3 and Lemma 4.5 and define the *flow* associated with a regular system $E\dot{x} = Ax + f$.

Theorem 4.5. *Let $E\dot{x} = Ax + f$ be regular with s-index μ, consider the remodeling* (58)*.*

1. *The flow associated with $E\dot{x} = Ax$ is given by*

$$\Phi^t_{E,A}(t_0) := P(t)\Phi^t_{D_\mathcal{S}}(t_0)P_d(t_0) \tag{73}$$

for $t_0, t \in \mathcal{I}$, where $\Phi_{D_\mathcal{S}}$ is the flow induced by $D_\mathcal{S}$. The flow $\Phi_{E,A}$ satisfies

$$\Phi^{t_0}_{E,A}(t_0) = P(t_0), \tag{74a}$$
$$\Phi^t_{E,A}(s)\Phi^s_{E,A}(t_0) = \Phi^t_{E,A}(t_0), \tag{74b}$$
$$\hat{E}(t)\dot{\Phi}^t_{E,A}(t_0) = \hat{A}(t)\Phi^t_{E,A}(t_0), \tag{74c}$$

for $t_0, t \in \mathcal{I}$, and moreover,

$$P_d(t)\Phi^t_{E,A}(t_0) = \Phi^t_{D_\mathcal{S}}(t_0)P_d(t_0), \tag{75a}$$
$$P_d'(t)\Phi^t_{E,A}(t_0) = -D_\mathcal{C}(t)\Phi^t_{D_\mathcal{S}}(t_0)P_d(t_0). \tag{75b}$$

The flow $\Phi_{E,A}$ is unique and $\Phi^{(\cdot)}_{E,A}(t_0) \in C^1(\mathcal{I}, \mathbb{R}^n)$ for every $t_0 \in \mathcal{I}$. Moreover, $\Phi_{E,A}$ possesses the semi inverse

$$(\Phi^t_{E,A}(t_0))^{ginv} = \Phi^{t_0}_{E,A}(t) \tag{76}$$

that satisfies

$$\Phi^{t_0}_{E,A}(t)\Phi^t_{E,A}(t_0) = P(t_0), \quad \Phi^t_{E,A}(t_0)\Phi^{t_0}_{E,A}(t) = P(t), \tag{77a}$$

for $t_0, t \in \mathcal{I}$.

For every $(t_0, x_0) \in \mathcal{C}_{E,A}$, the solution of $E\dot{x} = Ax$ is given by $x(t) = \Phi^t_{E,A}(t_0)x_0$ for $t \in \mathcal{I}$.

2. For $f \in C^{\mu+1}(\mathcal{I}, \mathbb{R}^n)$, the flow associated with $E\dot{x} = Ax + f$ is given by

$$\Phi^t_{E,A,f}(t_0, x_0) := \Phi^t_{E,A}(t_0)P(t_0)x_0 + \int_{t_0}^{t} \Phi^t_{E,A}(s)f_d(s)\,ds - f_a(t), \tag{78}$$

for $(t_0, x_0) \in \mathcal{C}_{E,A,f}$ and $t \in \mathcal{I}$. The flow $\Phi_{E,A,f}$ satisfies

$$\Phi^{t_0}_{E,A,f}(t_0, x_0) = P(t_0)x_0 - f_a(t_0), \tag{79a}$$

$$\Phi^t_{E,A,f}(t_0, \Phi^s_{E,A,f}(t_0, x_0)) = \Phi^t_{E,A,f}(t_0, x_0), \tag{79b}$$

$$\hat{E}(t)\dot{\Phi}^t_{E,A,f}(t_0, x_0) = \hat{A}(t)\Phi^t_{E,A,f}(t_0, x_0) + \hat{f}(t), \tag{79c}$$

for $t_0, t \in \mathcal{I}$, and moreover,

$$P_d(t)\Phi^t_{E,A,f}(t_0, x_0) = \Phi_{D_\mathcal{S},f_d}(t_0, P_d(t_0)x_0) \tag{80a}$$

$$P_d'(t)\Phi^t_{E,A,f}(t_0, x_0) = -D_\mathcal{C}(t)\Phi_{D_\mathcal{S},f_d}(t_0, P_d(t_0)x_0) - f_a(t). \tag{80b}$$

The flow $\Phi_{E,A,f}$ is unique and $\Phi^{(\cdot)}_{E,A,f}(t_0, x_0) \in C^1(\mathcal{I}, \mathbb{R}^n)$ for every $(t_0, x_0) \in \mathcal{C}_{E,A,f}$. For every $(t_0, x_0) \in \mathcal{C}_{E,A,f}$, the solution of $E\dot{x} = Ax + f$, $x(t_0) = x_0$ is given by $x(t) = \Phi^t_{E,A,f}(t_0, x_0)$ for $t \in \mathcal{I}$.

Proof. If $E\dot{x} = Ax + f$ is regular with s-index μ and $(t_0, x_0) \in \mathcal{C}_{E,A,f}$, then the solution of $E\dot{x} = Ax + f$, $x(t_0) = x_0$ is given by (68). This motivates the definition of $\Phi_{E,A,f}$ as in (78). If $f = 0$, then also $f_d = f_a = 0$ and we obtain $\Phi_{E,A}$ as in (73).

In particular, Lemma 4.5 implies that $x(t) = \Phi^t_{E,A}(t_0)x_0$ solves $\hat{E}\dot{x} = \hat{A}x + f$, $x(t_0) = x_0$ for every $(t_0, x_0) \in \mathcal{C}_{E,A}$.

To prove (74) and (79), we use the flow properties of $\Phi_{D_\mathcal{S}}$ and exploit that $\mathbb{X}_d$ is $\Phi_{D_\mathcal{S}}$ invariant, cp. Corollary 4.1, i.e.,

$$P_d(t)\Phi^t_{D_\mathcal{S},f_d}(t_0)P_d(t_0) = \Phi^t_{D_\mathcal{S},f_d}(t_0)P_d(t_0), \tag{81}$$

is satisfied for $t_0, t \in \mathcal{I}$, cp. Corollary 3.1. Furthermore, we have that $\mathrm{Rg}(P) = \mathrm{Rg}(P_d)$, cp. Lemma 4.4 on $\mathcal{I}$, yielding that

$$P_d P = P_d, \tag{82}$$

is satisfied pointwise on $\mathcal{I}$, cp. Lemma 2.1.

1. Since $\Phi^{t_0}_{D_\mathcal{S}}(t_0) = I_n$, then (82) implies that $\Phi^{t_0}_{E,A}(t_0) = P(t_0)$ and we verify (74a). Projecting (73) by P_d and P_d', respectively, and using (81), (82), then we we verify (75a) and (75b). For (74b), with (82), we note that

$$\begin{aligned}\Phi^t_{\hat{E},\hat{A}}(s)\Phi^s_{\hat{E},\hat{A}}(t_0) &= P(t)\Phi^t_{D_\mathcal{S}}(s)P_d(s)\,P(s)\,\Phi^s_{D_\mathcal{S}}(t_0)\\ &= P(t)\Phi^t_{D_\mathcal{S}}(s)P_d(s)\Phi^s_{D_\mathcal{S}}(t_0),\end{aligned}$$

and (81) implies that

$$\Phi^t_{D_\mathcal{S}}(s)P_d(s)\Phi^s_{D_\mathcal{S}}(t_0) = \Phi^t_{D_\mathcal{S}}(s)\Phi^s_{D_\mathcal{S}}(t_0).$$

Since $\Phi^t_{D_\mathcal{S}}(s)\Phi^s_{D_\mathcal{S}}(t_0) = \Phi^t_{D_\mathcal{S}}(t_0)$, then it follows that

$$\Phi^t_{\hat{E},\hat{A}}(s)\Phi^s_{\hat{E},\hat{A}}(t_0) = P_d(t)\Phi^t_{D_\mathcal{S}}(t_0) = \Phi^t_{\hat{E},\hat{A}}(t_0).$$

For (74c), we note that

$$\dot{\Phi}^t_{E,A}(t_0) = \big(\dot{P}(t) + P(t)D_\mathcal{S}(t)\big)\Phi^t_{D_\mathcal{S}}(t_0)P_d(t_0). \tag{83}$$

Noting that $\hat{E}\dot{P}_dP_d = 0$, cp. Lemma 2.3, 1., and $\hat{E}\dot{D}_\mathcal{C} = -\dot{\hat{E}}D_\mathcal{C}$ due to $\hat{E}D_\mathcal{C} = 0$, we have that

$$\hat{E}\dot{P}P_d = \hat{E}(\dot{P} - \dot{D}_\mathcal{C})P_d = \dot{\hat{E}}D_\mathcal{C}.$$

Similarly, $\hat{E}P_d' = 0$ implies that $\hat{E}\dot{P}_d = \dot{\hat{E}}P_d'$ and $\hat{E}P = \hat{E}P_d = \hat{E}$, and we get that

$$\hat{E}PD_\mathcal{S} = \hat{E}(\hat{E}^+\hat{A} + \dot{P})P = Q\hat{A}P - \hat{E}\dot{P}_dD_\mathcal{C} = Q\hat{A}P - \dot{\hat{E}}D_\mathcal{C}.$$

Then, (83) implies that (om.arg.)

$$\hat{E}\dot{\Phi}_{E,A}P = \big(\dot{\hat{E}}D_\mathcal{C} + Q\hat{A}P - \dot{\hat{E}}D_\mathcal{C}\big)\Phi_{D_\mathcal{S}}P_d = Q\hat{A}\Phi_{E,A}. \tag{84}$$

Exploiting that $Q' = Q'\hat{A}P_d'(Q'\hat{A}P_d')^+$, cp. Lemma 4.2 and $D_\mathcal{C} = P'D_\mathcal{C}$, then we observe that (om.arg.)

$$Q'\hat{A}D_\mathcal{C}\Phi_{D_\mathcal{S}}P_d = Q'\hat{A}\Phi_{D_\mathcal{S}}P_d$$

implying that (om.arg.)

$$Q'\hat{A}\Phi_{E,A} = Q'\hat{A}(P_d - D_\mathcal{C})\Phi_{D_\mathcal{S}}P = 0.$$

Then, $\hat{A}\Phi_{E,A} = Q\hat{A}\Phi_{E,A}$ and (84) implies that $\hat{E}\dot{\Phi}_{\hat{E},\hat{A}} = \hat{A}\Phi_{\hat{E},\hat{A}}$.

If $E\dot{x} = Ax$ has s-index μ, then the projection P_d and the remodeling (58) are unique, cp. Lemma 4.1 and Lemma 4.3, implying that $\Phi_{E,A}$ is unique. Since $D \in C(\mathcal{I}, \mathbb{R}^{n\times n})$ and $D_\mathcal{C} \in C^1(\mathcal{I}, \mathbb{R}^{n\times n})$, cp. Lemma 4.3, then $\Phi^{(\cdot)}_{E,A}(t_0) \in C^1(\mathcal{I}, \mathbb{R}^n)$ for every $t_0 \in \mathcal{I}$.

For (76), we note that

$$\begin{aligned}\Phi^{t_0}_{E,A}(t)\Phi^t_{E,A}(t_0) &= P(t_0)\Phi^{t_0}_{D_\mathcal{S}}(t)P_d(t)\,P(t)\Phi^t_{D_\mathcal{S}}(t_0)P(t_0)\\ &= P(t_0)\Phi^{t_0}_{D_\mathcal{S}}(t)\Phi^t_{D_\mathcal{S}}(t_0)P_d(t_0),\end{aligned}$$

due to (81) and (82). Using that $(\Phi^t_{D_\mathcal{S}}(t_0))^{-1} = \Phi^{t_0}_{D_\mathcal{S}}(t)$, cp. Lemma 2.7, then we get that

$$\Phi^{t_0}_{E,A}(t)\Phi^t_{E,A}(t_0) = P(t_0)P_d(t_0) = P(t_0).$$

In the same manner, we verify that

$$\begin{aligned}\Phi^t_{E,A}(t_0)\Phi^{t_0}_{E,A}(t) &= P(t)\Phi^t_{D_\mathcal{S}}(t_0)P_d(t_0)\,P(t_0)\Phi^{t_0}_{D_\mathcal{S}}(t)P_d(t)\\ &= P(t)\Phi^t_{D_\mathcal{S}}(t_0)\Phi^{t_0}_{D_\mathcal{S}}(t)P_d(t)\\ &= P(t)P_d(t)\\ &= P(t).\end{aligned}$$

Noting that $P(t_0)\Phi^{t_0}_{E,A}(t) = \Phi^{t_0}_{E,A}(t)$ and $P(t)\Phi^t_{E,A}(t_0) = \Phi^t_{E,A}(t_0)$, then we observe that

$$\Phi^t_{E,A}(t_0)\Phi^{t_0}_{E,A}(t)\Phi^t_{E,A}(t_0) = P(t)\Phi^t_{E,A}(t_0) = \Phi^t_{E,A}(t_0)$$

and

$$\Phi^{t_0}_{E,A}(t)\Phi^t_{E,A}(t_0)\Phi^{t_0}_{E,A}(t) = P(t_0)\Phi^{t_0}_{E,A}(t) = \Phi^{t_0}_{E,A}(t),$$

i.e., $\Phi^{t_0}_{E,A}(t)$ is the semi-inverse of $\Phi^t_{E,A}(t_0)$, cp. (10).

2. Since $\Phi^{t_0}_{E,A}(t_0) = P(t_0)$, we immediately verify (79a). Projecting (78) by P_d, P_d', respectively, and noting that $f_a \in \mathbb{X}_a$, then (75) implies (80). For (79b), we observe that

$$\begin{aligned}\Phi^t_{E,A,f}(s,\Phi^s_{E,A,f}(t_0,x_0)) &= P(t)\Phi^t_{D_\mathcal{S},f_d}\big(s,P_d(s)\Phi^s_{E,A,f}(t_0,x_0)\big) - f_a(t)\\ &= P(t)\Phi^t_{D_\mathcal{S},f_d}\big(s,\Phi^s_{D_\mathcal{S},f_d}(t_0,P_d(t_0)x_0)\big) - f_a(t)\end{aligned}$$

by (80a) and using that $\Phi^t_{D_\mathcal{S},f_d}(s,\Phi^s_{D_\mathcal{S},f_d}(t_0,x_0)) = \Phi^t_{D_\mathcal{S},f_d}(t_0,x_0)$, cp. (11b), this implies that

$$\Phi^t_{E,A,f}(s,\Phi^s_{E,A,f}(t_0,x_0)) = P(t)\Phi^t_{D_\mathcal{S},f_d}(t_0,P_d(t_0)x_0) - f_a(t) = \Phi^t_{E,A,f}(t_0,x_0).$$

To prove the assertions, we write (78) as

$$\Phi^t_{E,A,f}(t_0,x_0) = P(t)\Phi^t_{D_\mathcal{S},f_d}(t_0,P(t_0)x_0) - f_a(t), \tag{85}$$

To prove (79c), we note that

$$\Phi^t_{E,A}(s) = P(t)\Phi^t_{D_\mathcal{S}}(s) = P(t)\Phi^t_{D_\mathcal{S}}(t_0)\Phi^{t_0}_{D_\mathcal{S}}(s) = \Phi^t_{E,A}(t_0)\Phi^{t_0}_{D_\mathcal{S}}(s)$$

for $s \in [t_0,t]$ as $(\Phi^t_{D_\mathcal{S}}(t_0))^{-1} = \Phi^{t_0}_{D_\mathcal{S}}(t)$. Then,

$$\Phi^t_{E,A,f}(t_0,x_0) = \Phi^t_{E,A}(t_0)x_0 + \Phi^t_{E,A}(t_0)\int_{t_0}^t \Phi^{t_0}_{D_\mathcal{S}}(s)f_d(s)\,ds - f_a(t),$$

and we get that

$$\begin{aligned}\dot\Phi^t_{E,A,f}(t_0,x_0) &= \dot\Phi^t_{E,A}(t_0)x_0 + \dot\Phi^t_{E,A}(t_0)\int_{t_0}^t \Phi^{t_0}_{D_\mathcal{S}}(s)f_d(s)\,ds + \Phi^t_{E,A}(t_0)\Phi^{t_0}_{D_\mathcal{S}}(t)f_d(t) - \dot f_a(t)\\ &= \dot\Phi^t_{E,A}(t_0)x_0 + \dot\Phi^t_{E,A}(t_0)\int_{t_0}^t \Phi^{t_0}_{D_\mathcal{S}}(s)f_d(s)\,ds + P(t)f_d(t) - \dot f_a(t).\end{aligned}$$

Using (74c) and $\hat E P_d' = 0$, this implies that

$$\hat E(t)\dot\Phi^t_{E,A,f}(t_0,x_0) = \hat A(t)\Big(\Phi^t_{E,A}(t_0)x_0 + \int_{t_0}^t \Phi^t_{E,A}(s)f_d(s)\,ds\Big) + \hat E(t)\Big(f_d(t) - \dot f_a(t)\Big).$$

Noting that $\hat E\dot f_a = -\hat E\dot P_d f_a$ due to $\hat E f_a = \hat E P_d f_a = 0$ and $\hat E f_d = \hat E(\hat E^+\hat f - Df_a)$, then we get that

$$\begin{aligned}\hat E(f_d - \dot f_a) &= Q\hat f - \hat E D f_a + \hat E\dot P_d f_a\\ &= Q\hat f - (Q\hat A + \hat E\dot P_d)f_a + \hat E\dot P_d f_a.\\ &= Q\hat f - Q\hat A f_a.\end{aligned}$$

Using that $Q'\hat A f_a = Q'\hat f$ as $Q' = (Q'\hat A P_d')(Q'\hat A P_d')^+$, then it follows that $\hat E(f_d - \dot f_a) = \hat f - \hat A f_a$, and we verify that

$$\begin{aligned}\hat E(t)\dot\Phi^t_{E,A,f}(t_0,x_0) &= \hat A(t)\big(\Phi^t_{E,A}(t_0)x_0 + \int_{t_0}^t \Phi^t_{E,A}(s)f_d(s)\,ds - f_a(t)\big) + \hat f(t)\\ &= \hat A(t)\Phi^t_{E,A,f}(t_0,x_0) + \hat f(t).\end{aligned}$$

By Lemma 4.3, f_d and f_a are uniquely determined by $E\dot x = Ax + f$ and since $\Phi_{E,A}$ is unique, then also $\Phi_{E,A,f}$ is unique. With $f_d \in C(\mathcal{I},\mathbb{R}^n)$, $f_a \in C^1(\mathcal{I},\mathbb{R}^n)$, cp. Lemma 4.3, and $\Phi^{(\cdot)}_{E,A}(t_0,x_0) \in C^1(\mathcal{I},\mathbb{R}^n)$, it follows that $\Phi^{(\cdot)}_{E,A,f}(t_0,x_0) \in C^1(\mathcal{I},\mathbb{R}^n)$.

□

Theorem 4.5 extends the notion of a flow to linear DAEs with regular s-index. Properties (79) ensure the existence of a unique solution $x(t) = \Phi^t_{E,A,f}(t_0,x_0)$ associated with the initial value problem $E\dot x = Ax + f$, $x(t_0) = x_0$, $(t_0,x_0) \in \mathcal{C}_{E,A,f}$. Property (80) allows to access the solutions x_d, x_a of the semi-explicit remodeling (58) for every $t \in \mathcal{I}$. The semi inverse $(\Phi^t_{E,A}(t_0))^{ginv} = \Phi^{t_0}_{E,A}(t)$ allows to recover the initial value x_0 from a given solution x for every time $t \in \mathcal{I}$.

The specifying matrices $D_\mathcal{S}, f_d, f_a$ are obtained from the original data E, A, f via the s-free formulation $\hat E, \hat A, \hat f$, cp. Theorem 4.3 and the remodeling (59).

If $\hat{E} = I_n$, then $P = P_d = I_n$, $f_d = f$ and $D_{\mathcal{C}} = 0$, $f_a = 0$ and (74) and (79) reduce to (12) and (11), respectively. By the uniqueness stated in Lemma 2.7 and Theorem 4.5, then we find that $\Phi_{I_n,A,f} = \Phi_{A,f}$. The projection properties (80) are trivially satisfied, in particular, and the semi inverse (76) satisfies $(\Phi^t_{E,A}(t_0))^{ginv} = (\Phi^{t_0}_A(t))^{-1}$. Thus, Theorem 4.5 indeed covers the assertions of Lemma 2.7 and supplies a unified description of the flow of linear systems, constrained or unconstrained.

To conclude the analysis of linear problems, we note that the set $\mathcal{C}_{E,A,f}$ is affine $\Phi_{E,A,f}$ invariant.

Corollary 4.2. *Let $E\dot{x} = Ax + f$ be regular with s-index μ and let $\Phi_{E,A,f}$ be the associated flow. Let $\mathcal{C}_{E,A,f}$ be the set of consistent initial values. Then, $\mathcal{C}_{E,A,f}$ is affine $\Phi_{E,A,f}$ invariant.*

Proof. We prove the assertion using Lemma 3.2. We first consider the homogeneous problem, i.e., let $f = 0$. A projection onto $\mathcal{C}_{E,A}$ is given by $P = P_d - D_{\mathcal{C}}$, cp. Lemma 4.4. Projecting $\Phi_{E,A}$ by P and noting that $P = PP_d$, then (75a) implies that

$$P(t)\Phi^t_{E,A}(t_0) = P(t)P_d(t)\Phi^t_{E,A}(t_0) = P(t)\Phi^t_{D_{\mathcal{S}}}(t_0)P_d(t_0) = \Phi^t_{E,A}(t_0),$$

i.e., $P^{'}(t)\Phi^t_{E,A}(t_0) = 0$. Since $\Phi^t_{E,A}(t_0) = \Phi^t_{E,A}(t_0)\Phi^{t_0}_{E,A}(t_0) = \Phi^t_{E,A}(t_0)P(t_0)$ due to (74a), (74b), it follows that $P^{'}(t)\Phi^t_{E,A}(t_0)P(t_0) = 0$. Then, $\mathcal{C}_{E,A}$ is $\Phi_{E,A}$ is invariant, cp. Lemma 3.2.

If $f \neq 0$, then $\mathcal{C}_{E,A,f} = \mathcal{C}_{E,A} - f_a$ is an affine subspace and an associated projection is given by P_{f_a}, cp. Lemma 4.4. Since $\mathcal{C}_{E,A}$ is $\Phi_{E,A}$ is invariant, then $P^{'}(t)\Phi^t_{E,A}(t_0)P(t_0) = 0$ and $P^{'}(t)\Phi^t_{E,A}(t_0) = P^{'}(t)\Phi^t_{E,A}(t_0)P^{'}(t_0)$, in particular, and it follows that

$$\begin{aligned} P^{'}(t)\Phi^t_{E,A,f}(t_0, P_{f_a}(t_0, x_0)) &= P^{'}(t)\Phi^t_{E,A}(t_0)(P(t_0)x_0 - f_a(t_0)) \\ &\quad + P^{'}(t)\Big(\int_{t_0}^t \Phi^t_{E,A}(s)f_d(s)\, ds - f_a(t)\Big) \\ &= -f_a(t). \end{aligned}$$

According to Corollary 3.1, condition (20), then $\mathcal{C}_{E,A,f}$ is $\Phi_{E,A}$ affine invariant. □

Thus, the concept of flow invariant subspaces occurs twofold in constrained linear systems. First, we exploit that the inherent ODE obtained by projection is the restriction of an ODE to a flow invariant subspace, more exactly, to the subspace $\mathbb{X}_d$ of the differential components cp. Corollary 4.1. This allows to compute a solution for these components and for the DAE, in particular. Second, the set $\mathcal{C}_{E,A,f}$ of consistent initial values is invariant for the DAE flow $\Phi_{E,A,f}$, cp. Corollary 4.2. This permits to study properties like positivity and stability for linear DAEs in the same framework as ODEs that are constrained to linear subspaces.

4.3.2 Nonlinear systems

For nonlinear DAEs, the projection P_d onto the differential components is defined locally by the Jacobian $\hat{F}_{\dot{x}}$ of the s-free formulation $\hat{F}(t,x,\dot{x})=0$, cp. Lemma 4.1. Consequently, the variable decomposition with respect to $\mathbb{X}_d$, $\mathbb{X}_a$ is defined locally. In the neighborhood of a solution x^*, we set

$$x_d := P_d(t,x^*,\dot{x}^*)x, \quad x_a := P_d^{'}(t,x^*,\dot{x}^*)x \tag{86}$$

for $x \in \mathbb{R}^n$ and $(t,x^*,\dot{x}^*) \in \mathcal{C}_F$.

By construction, the algebraic equation $\hat{F}_2(t,x)=0$ is unique for the components in $\mathbb{X}_a$, cp. Theorem 4.1. To solve this equation locally for x_a, we verify the assertion of Theorem 3.1.

Lemma 4.6. *Let $F(t,x,\dot{x})=0$ have s-index μ and let $\hat{F}_x, \hat{F}_{\dot{x}}$ exist on $\mathcal{C}_F$. Consider the projections P_d, Q defined in* (54), (55). *For every $\hat{z}_0 \in \mathcal{C}_F$ with $z_0 \in \mathcal{L}_\mu$, then*

$$(Q^{'}\hat{F}_x(z_0)P_d^{'}(\hat{z}_0))^+(Q^{'}\hat{F}_x(z_0)P_d^{'}(\hat{z}_0)) = P_d^{'}(\hat{z}_0), \tag{87a}$$

$$(Q^{'}\hat{F}_x(z_0)P_d^{'}(\hat{z}_0))(Q^{'}\hat{F}_x(z_0)P_d^{'}(\hat{z}_0))^+ = Q^{'}, \tag{87b}$$

is satisfied for every $\hat{F}$ constructed as in Theorem 4.1.

Proof. To get an explicit formula of the Moore-Penrose inverse, we represent $Q^{'}\hat{F}_xP_d^{'}$ with respect to an orthogonal basis of $\mathbb{X}_d$, $\mathbb{X}_a$. Since $\operatorname{rank}(P_d)=d$ on $\mathcal{L}_\mu$, cp. Lemma 4.1, there exists a neighborhood $\mathcal{B}(\hat{z}_0)$ for every $\hat{z}_0 \in \mathcal{C}_F$ and a pointwise orthogonal function $T=[T_1\,T_2] \in C^1(\mathcal{B}(\hat{z}_0),\mathbb{R}^{n\times n})$ with $\operatorname{span}(T_1)=\mathbb{X}_d$, $\operatorname{span}(T_2)=\mathbb{X}_a$ on $\mathcal{B}(\hat{z}_0)$. Then, P_d is diagonalizable on $\mathcal{B}(\hat{z}_0)$ with respect to T, cp. Corollary 2.1, and we get (om. arg.)

$$Q^{'}\hat{F}_xP_d^{'} = \begin{bmatrix} 0 & 0 \\ 0 & \hat{F}_{x,2}T_2 \end{bmatrix} T^T.$$

As $\hat{F}_{2,x}T_2$ is pointwise nonsingular on $\mathcal{C}_F$ by construction of $\hat{F}$, then the Moore-Penrose projection is given by (om. arg.)

$$(Q^{'}\hat{F}_xP_d^{'})^+ = T\begin{bmatrix} 0 & 0 \\ 0 & (\hat{F}_{x,2}T_2)^{-1} \end{bmatrix}, \tag{88}$$

cp. Lemma 2.5, and we verify (87). □

As $Q^{'}\hat{F}(t,x,\dot{x}) = [0,\, \hat{F}_2^T(t,x)]^T$, Lemma 4.6 allows to solve the algebraic equation $\hat{F}_2^T(t,x)=0$ locally for x_a. This permits to remodel the s-free formulation (45) locally as semi-explicit system.

Theorem 4.6. *Consider $F(t,x,\dot{x})=0$, $F\in C^{\mu+1}(\mathbb{D},\mathbb{R}^n)$, with s-index μ. Every sufficiently smooth solution of $F(t,x,\dot{x})=0$ locally solves the semi-explicit system*

$$\dot{x_d}=h(t,x_d), \tag{89a}$$
$$x_a=g(t,x_d), \tag{89b}$$

where $x_d=P_d(t,x,\dot{x})x$ and $x_a=P_d^{'}(t,x,\dot{x})x$. The functions h,g are locally defined on a neighborhood $\mathcal{B}(t,x,\dot{x})\subset\mathcal{C}_F$ and satisfy $\hat{F}(t,x_d+g(t,x_d),h(t,x_d)+\dot{g}(t,x_d))=0$ on $\mathcal{B}(t,x,\dot{x})$.

If $F(t,x,\dot{x})=0$ is regular with s-index μ and $(t_0,x_0,\dot{x}_0)\in\mathcal{C}_F$, then (89) has a unique solution x_d, x_a that satisfy $x_d(t_0)=P(t_0,x_0,\dot{x}_0)x_0$. Locally, this solution solves $F(t,x,\dot{x})=0$, $x(t_0)=x_0$.

Proof. If F has s-index μ, then along a sufficiently smooth solution, $F(t,x,\dot{x})=0$ can locally be remodeled as s-free system $\hat{F}(t,x,\dot{x})=0$, cp. Theorem 4.1. The Jacobian $\hat{F}_{\dot{x}}$ induces the Moore-Penrose projections P_d, Q defined in (54), (55), and provided $F\in C^{\mu+1}(\mathbb{D},\mathbb{R}^n)$, then $\hat{F}_{\dot{x}},P_d,Q\in C^1(\mathcal{B}(z_0),\mathbb{R}^{n\times n})$ for every $z_0\in\mathcal{C}_F$. For $\hat{z}_0:=(t_0,x_0,\dot{x}_0)\in\mathcal{C}_F$ with associated $z_0:=(\hat{z}_0,\ddot{x}_0,...,x_0^{(\mu+1)})\in\mathcal{L}_\mu$, let x^* be a sufficiently smooth solution of $F(t,x,\dot{x})=0$ with $x^*(t_0)=x_0$, $\dot{x}^*(t_0)=\dot{x}_0$. Setting $z^*:=(t,x^*,\dot{x}^*)$, then we consider the projection P_d along the trajectory of x^* in $\mathcal{C}_F$ and set $x_d=P(t,x^*,\dot{x}^*)x$ and $x_a=P_d^{'}(t,x^*,\dot{x}^*)x$ for $x\in\mathcal{B}(x^*(t))$, where $\mathcal{B}(x^*(t))$ is a neighborhood of $x^*(t)$ in $\mathcal{C}_F$. For ${x_d}^*=P(t,x^*,\dot{x}^*)x^*$ and ${x_a}^*=P_d^{'}(t,x^*,\dot{x}^*)x^*$, in particular, then $x_d\in\mathcal{B}({x_d}^*(t))$ and $x_a\in\mathcal{B}({x_a}^*(t))$ provided $x\in\mathcal{B}(x^*(t))$.

We show that there exist locally defined functions g,h that satisfy

$$\hat{F}(t,x_d+g(t,x_d),h(t,x_d)+\dot{g}(t,x_d))=0$$

for every $x_d\in\mathcal{B}({x_d}^*(t))$. Then, we verify that x solves $F(t,x,\dot{x})=0$ in a neighborhood of x^* if and only if x_d, x_a locally solve (89).

First, we consider $Q^{'}\hat{F}(t,x,\dot{x})=0$. Since $Q^{'}\hat{F}(t,x,\dot{x})=[0,\ \hat{F}_2^T(t,x)]^T$, then $Q^{'}\hat{F}(t,x,\dot{x})$ is independent of $\dot{x}$. Choosing $\dot{x}_0$, in particular, then $Q^{'}\hat{F}(t,x,\dot{x})=0$ reads

$$Q^{'}\hat{F}(t,x_d+x_a,\dot{x}_0)=0. \tag{90}$$

As $(Q^{'}\hat{F}_xP_d^{'})^+(Q^{'}\hat{F}_xP_d^{'})=P_d^{'}$ and $(Q^{'}\hat{F}_xP_d^{'})(Q^{'}\hat{F}_xP_d^{'})^+=Q^{'}$ are satisfied pointwise on $\mathcal{C}_F$, cp. Lemma 4.6 and $\hat{F}_2\in C^1(\mathcal{I}_{max}(t_0,x_0)\times\mathcal{B}(x^*(t))$, we can solve (90) locally for x_a. More exactly, there exist neighborhoods $\tilde{\mathcal{B}}({x_d}^*(t))$, $\tilde{\mathcal{B}}({x_d}^*(t))$ in $\mathcal{C}_F$, respectively, and a function $g\in C^1(\tilde{\mathcal{B}}({x_d}^*(t)),\tilde{\mathcal{B}}({x_d}^*(t))$, such that

$$x_a(t)=g(t,x_d)\text{ solves (90) for every }x_d\in\tilde{\mathcal{B}}(t,{x_d}^*(t)), \tag{91}$$

cp. Theorem 3.1.

Similarly, we proceed for $Q\hat{F}(t,x,\dot{x})=0$. Since

$$\dot{x}_a = P_d^{'}(t,x^*,\dot{x}^*)\dot{x}_a - \dot{P}(t,x^*,\dot{x}^*)x_a,$$

we have that

$$\hat{F}_{\dot{x}_a}(t,x,\dot{x}) = \hat{F}_{\dot{x}}(t,x,\dot{x})P_d^{'}(t,x^*,\dot{x}^*).$$

As $P_d = \hat{F}_{\dot{x}}^{+}\hat{F}_{\dot{x}}$, then $\hat{F}_{\dot{x}_a}(t,x,\dot{x}) = 0$ and $Q^{'}\hat{F}(t,x_d+x_a,\dot{x}_d+\dot{x}_a)$ is independent of $\dot{x}_a$. Choosing $\dot{x}_{a,0} = P^{'}(\hat{z}_0)\dot{x}_0 - \dot{P}(\hat{z}_0)x_0$, in particular, and noting that $x_a = g(t,x_d)$ on $\tilde{\mathcal{B}}(t,{x_d}^*)$, then $Q\hat{F}(t,x,\dot{x})=0$ reads

$$Q\hat{F}(t,x_d+g(t,x_d),\dot{x}_d+\dot{g}(t_0,x_{d0}))=0. \tag{92}$$

As P_d, Q are the Moore-Penrose projections induced by $\hat{F}_{\dot{x}}$, we have that $Q^{'}\hat{F}_{\dot{x}}P_d^{'} = \hat{F}_{\dot{x}}$ and $(Q^{'}\hat{F}_{\dot{x}}P_d^{'})^{+}(Q^{'}\hat{F}_{\dot{x}}P_d^{'}) = P$, $(Q^{'}\hat{F}_{\dot{x}}P_d^{'})(Q^{'}\hat{F}_{\dot{x}}P_d^{'})^{+} = Q$ are pointwise satisfied on $\mathcal{L}_F$. Moreover, $\hat{F}_1 \in C^1(\mathcal{I}_{max}(t_0,x_0)\times\mathcal{B}(x^*(t))\times\mathcal{B}(\dot{x}^*(t)),\mathbb{R}^n)$ and $P \in C^1(\mathcal{I}_{max}(t_0,x_0)\times\mathcal{B}(x^*)\times\mathcal{B}(\dot{x}^*(t)),\mathbb{R}^n)$, where $\mathcal{B}(\dot{x}^*(t))$ is a neighborhood of $x^*(t)$ in $\mathcal{C}_F$. Then, we can solve (92) locally for $\dot{x}_d$. More exactly, there exist neighborhoods $\mathcal{B}({x_d}^*(t)) \subset \tilde{\mathcal{B}}({x_d}^*(t))$, $\mathcal{B}(\dot{x}_d^*(t))$ in $\mathcal{C}_F$, respectively, and a function $h \in C^1(\mathcal{B}({x_d}^*(t)),\mathcal{B}(\dot{x}_d^*(t)))$, such that

$$\dot{x}_d = h(t,x_d) \text{ solves (92) for every } x_d \in \mathcal{B}({x_d}^*(t)), \tag{93}$$

cp. Corollary 3.4.

Since $Q+Q^{'}=I$, then (91), (93) prove that in a neighborhood of a sufficiently smooth solution x^* of $F(t,x,\dot{x})=0$, there exist locally defined functions g,h that satisfy

$$\hat{F}(t,P(z^*)x+g(t,P(z^*)x),h(t,P(z^*)x)+\dot{g}(t,P(z^*)x))=0,$$

for every $x\in\mathcal{B}(x^*(t))$. By construction of h,g, then every function $x \in C^1(\mathcal{I}_{max}(t_0,x_0),\mathcal{B}(x^*)$ that satisfies (89) also solves $\hat{F}(t,x,\dot{x})=0$. Conversely, if $x\in C^1(\mathcal{I}_{max}(t_0,x_0),\mathcal{B}(x^*)$, then $P(z^*)x$, $P^{'}(z^*)x$ solve (89).

If $F(t,x,\dot{x})=0$ has regular s-index, then $F(t,x,\dot{x})=0$ and $\hat{F}(t,x,\dot{x})=0$ are locally equivalent, i.e., every solution of (89) and $\hat{F}(t,x,\dot{x})=0$ locally solves the original system $F(t,x,\dot{x})=0$.

□

We call (89a) the *inherent ODE obtained by projection* and (89b) the *algebraic equation obtained by projection.* Both (89a), (89b) have dimension n as $x_d, x_a \in \mathbb{R}^n$. The differential equation is the restriction of the ODE $\dot{y}=h(t,y)$ onto $\mathbb{X}_d(t,x^*,\dot{x}^*)$, where $y\in C^1(\mathcal{I},\mathbb{R}^n)$ is a function on $\mathbb{R}^n$.

The remodeling (89) is unique for a given DAE $F(t,x,\dot{x})=0$ and h,g are locally as smooth as $\hat{F}$.

Lemma 4.7. *Let $F(t,x,\dot{x}) = 0$ be regular with s-index μ and consider the remodeling* (89)*. The implicit functions h, g are uniquely defined by $F(t,x,\dot{x}) = 0$ and independent of the s-free remodeling $\hat{F}(t,x,\dot{x}) = 0$. For every $z_0 = (t_0, x_0, \dot{x}_0) \in \mathcal{C}_F$, there exists a neighborhood $\mathcal{B}(t_0, P_d(z_0)x_0)$, such that $h, g \in C^1(\mathcal{B}(t_0, P_d(z_0)x_0), \mathbb{R}^n)$.*

Proof. Let $\hat{F}(t,x,\dot{x}) = 0$ and $\tilde{F}(t,x,\dot{x}) = 0$ be two s-free formulations of $F(t,x,\dot{x}) = 0$ obtained by matrix functions Z_1, Z_2 and $\tilde{Z}_1$, $\tilde{Z}_2$, respectively, cp. Hypothesis 4.1 and Theorem 4.1. Then, there exist pointwise orthogonal matrix functions $U_1 \in C^{\mu+1}(\mathbb{R}, \mathbb{R}^{d\times d})$, $U_2 \in C^{\mu+1}(\mathbb{R}, \mathbb{R}^{a\times a})$, such that $\tilde{Z}_1 = Z_1U_1$, $\tilde{Z}_2 = Z_2U_2$. Setting $U = \left[\begin{smallmatrix} U_1 & 0 \\ 0 & U_2 \end{smallmatrix}\right]$, implies that

$$\tilde{F}(t,x,\dot{x}) = U^T\hat{F}(t,x,\dot{x}). \tag{94}$$

on $\mathcal{C}_F$. Then, every sufficiently smooth solution x^* of $F(t,x,\dot{x}) = 0$ locally solves both $\hat{F}(t,x,\dot{x}) = 0$ and $\tilde{F}(t,x,\dot{x}) = 0$. Setting $x_d = P(t,x^*,\dot{x}^*)x$, then there exist locally defined functions g, h and $\tilde{g}, \tilde{h}$ that satisfy

$$\hat{F}(t, x_d + g(t,x_d), h(t,x_d) + \dot{g}(t,x_d)) = 0$$

and

$$\tilde{F}(t, x_d + \tilde{g}(t,x_d), \tilde{h}(t,x_d) + \dot{\tilde{g}}(t,x_d)) = 0$$

for every $x_d \in \mathcal{B}({x_d}^*(t))$, respectively, where $\mathcal{B}({x_d}^*(t))$ is a neighborhood of ${x_d}^* = P(t,x^*,\dot{x}^*)x^*$ in $\mathcal{C}_F$, cp. Theorem 4.6. Regarding (94), then $\tilde{h}, \tilde{g}$ also solve

$$\hat{F}(t, x_d + \tilde{g}(t,x_d), \tilde{h}(t,x_d) + \dot{\tilde{g}}(t,{x_d}^*)) = 0, \tag{95}$$

since U is pointwise orthogonal. Noting that g, h are the unique solutions of $Q'\hat{F}(t, x_d + g(t,x_d), h(t,x_d) = 0$, cp. Theorem 3.1 and Corollary 3.4, it follows that $\tilde{h} = h$ and $\tilde{g} = g$, i.e., the remodeling (89) is independent of the s-free formulation (45).

By construction, h, g are locally as smooth as $\hat{F}_1, \hat{F}_2$. As $\hat{F} \in C^1(\mathcal{B}(\hat{z}_0, \mathbb{R}^n)$ for $\hat{z}_0 \in \mathcal{C}_F$, this proves the assertion. □

In the semi-explicit system (89), the algebraic equation (89b) has already been solved for the algebraic variables x_a. To solve the differential equation (89a), we show that (89a) is the restriction of an ODE to a flow invariant subspace.

Corollary 4.3. *Let $F(t,x,\dot{x}) = 0$, $F \in C^{\mu+1}(\mathbb{D}, \mathbb{R}^n)$, have s-index μ and consider the remodeling* (89)*. For $\hat{z}_0 := (t_0, x_0, \dot{x}_0) \in \mathcal{C}_F$, let x^* be a sufficiently smooth solution of $F(t,x,\dot{x}) = 0$ with $x^*(t_0) = x_0$, $\dot{x}^*(t_0) = \dot{x}_0$. Set $z^* := (t, x^*, \dot{x}^*)$. On a neighborhood $\mathcal{B}(x^*)$ of x^* in $\mathcal{C}_F$, then $\mathbb{X}_a(z^*)$, $\mathbb{X}_d(z^*)$ are g invariant and $\mathbb{X}_d(z^*)$ is Φ_h invariant.*

Proof. If F has s-index μ, then along a sufficiently smooth solution, $F(t,x,\dot{x}) = 0$ can locally be remodeled as semi explicit system 89, cp. Theorem 4.6. More exactly, for $\hat{z}_0 := (t_0, x_0, \dot{x}_0) \in \mathcal{C}_F$, let x^* be a sufficiently smooth solution of $F(t,x,\dot{x}) = 0$ with $x^*(t_0) = x_0$, $\dot{x}^*(t_0) = \dot{x}_0$. Setting $z^* := (t, x^*, \dot{x}^*)$, then in a neighborhood $\mathcal{B}(x^*(t))$ of x^* in $\mathcal{C}_F$, the functions g, h are defined as implicit functions satisfying

$$\hat{F}(t, P(z^*)x + g(t, P(z^*)x), h(t, P(z^*)x) + \dot{g}(t, P(z^*)x)) = 0,$$

for every $x \in \mathcal{B}(x^*(t))$. More exactly, for neighborhoods $\mathcal{B}(t, {x_d}^*(t))$, $\mathcal{B}({x_a}^*(t))$ in $\mathcal{C}_F$, then $g \in C^1(\mathcal{B}(t, {x_d}^*(t)), \mathcal{B}({x_d}^*(t))$ is the implicit function solving

$$Q^{'}\hat{F}(t, P(z^*)x + g(t, P(z^*)x), \dot{x}^*) = 0$$

for every $x \in \mathcal{B}(x^*(t))$, cp. (91). Accordingly, $h \in C^1(\mathcal{B}(t, {x_d}^*(t)), \mathcal{B}(\dot{x}_d^*(t))$ is the implicit function solving

$$Q\hat{F}(t, P(z^*)x + g(t, P(z^*)x), h(t, P(z^*) + \dot{g}(t, P(z^*)) = 0$$

for every $x \in \mathcal{B}(x^*(t))$, cp. (93). Then, $\mathbb{X}_a(z^*)$, $\mathbb{X}_d(z^*)$ are g invariant on $\mathcal{B}(x^*)$, cp. Theorem 3.1, and $\mathbb{X}_d(z^*)$ is Φ_h invariant on $\mathcal{B}(x^*)$, cp. Corollary 3.4, where Φ_h is the flow of h as a function on $\mathcal{B}(x^*)$. □

Exploiting Corollary 4.3 and Lemma 3.4, then we can specify the solution formula for $F(t,x,\dot{x}) = 0$.

Lemma 4.8. *Let $F(t,x,\dot{x}) = 0$ be regular with s-index μ and consider the remodeling (89). For every $z_0 := (t_0, x_0, \dot{x}_0) \in \mathcal{C}_F$, the solution of $F(t,x,\dot{x}) = 0$, $x(t_0) = x_0$ is locally given by*

$$x(t) = \Phi_h^t(t_0, P(z_0)x_0) + g(t, \Phi_h^t(t_0, P(z_0)x_0)) \tag{96}$$

for $t \in \mathcal{I}_{max}(t_0, x_0)$, where Φ_h is the flow induced by h. Moreover, $x \in C^1(\mathcal{I}_{max}(t_0, x_0), \mathbb{R}^n)$.

Proof. For $\hat{z}_0 := (t_0, x_0, \dot{x}_0) \in \mathcal{C}_F$, let x^* be a sufficiently smooth solution of $F(t,x,\dot{x}) = 0$ with $x^*(t_0) = x_0$, $\dot{x}^*(t_0) = \dot{x}_0$. Setting $z^* := (t, x^*, \dot{x}^*)$, then in a neighborhood $\mathcal{B}(x^*(t))$ of x^* in $\mathcal{C}_F$, $F(t,x,\dot{x}) = 0$ can locally be remodeled as semi explicit system 89, cp. Theorem 4.6. By Corollary 4.3, $\mathbb{X}_d(z^*)$ is Φ_h invariant on $\mathcal{B}(x^*)$. Then, the restriction $\Phi_h \circ P\,(t_0, t) := \Phi_h^t(t_0, P(z_0)x_0)$ satisfies $\Phi_h \circ P = \Phi_{h \circ P}$, where $h \circ P\,(t, x) := h(t, P(z^*)x)$. Thus, $\Phi_h \circ P$ is the flow associated with $x_d = h(t, x_d)$, cp. Lemma 3.4, and the solution of (89a) is given by

$$x_d(t) = \Phi_h^t(t_0, P(z^*)x_0). \tag{97a}$$

The algebraic equation (89b) is already resolved for x_a. Inserting (97a), then we obtain that

$$x_a(t) = g(t, \Phi_h^t(t_0, P(z_0)x_0)). \tag{97b}$$

Since x solves $\hat{F}(t, x, \dot{x}) = 0$ if and only if x_d, x_a solve (89), then the solution of $\hat{F}(t, x, \dot{x}) = 0$, $x(t_0) = x_0$ is given by (96). If $F(t, x, \dot{x}) = 0$ has regular s-index μ, then $F(t, x, \dot{x}) = 0$ and $\hat{F}(t, x, \dot{x}) = 0$ are locally equivalent, implying that (96) locally solves $F(t, x, \dot{x}) = 0$, $x(t_0) = x_0$ for $(t_0, x_0, \dot{x}_0) \in \mathcal{C}_F$.

Noting that $h \in C(\mathcal{I}_{max}(t_0, x_0) \times \mathcal{B}(x_d{}^*), \mathbb{R}^n)$, $g \in C^1(\mathcal{I}_{max}(t_0, x_0) \times \mathcal{B}(x_a{}^*), \mathbb{R}^n)$, cp. Lemma 4.7, then

$$\dot{x}(t) = h(t, \Phi_h^t(t_0, P(z_0)x_0)) + \dot{g}(t, \Phi_h^t(t_0, P(z_0)x_0))$$

implies that $x \in C^1(\mathcal{I}_{max}(t_0, x_0), \mathbb{R}^n)$. □

For DAEs with regular s-index, Lemma 4.8 supplies a locally defined, though closed solution formula for every consistent initial value. Formula (96) illustrates that only the components $P(z_0)x_0 \in \mathbb{X}_d(z_0)$ are evolved by a flow, while the components in $\in \mathbb{X}_a$ are specified by the constraints g. If $\mathbb{X}_a, \mathbb{X}_d$ are g invariant and $\mathbb{X}_d$ is Φ_h invariant, then projecting (96) by P_d, P_d', respectively, returns the differential and algebraic components (97a), (97b).

We summarize the results of Theorem 4.6, Lemma 4.7 and Lemma 4.8 and specify the *flow* associated with a regular DAE $F(t, x, \dot{x}) = 0$.

Theorem 4.7. *Let $F(t, x, \dot{x}) = 0$ be regular with s-index μ and consider the remodeling (89). For every $z_0 = (t_0, x_0, \dot{x}_0) \in \mathcal{C}_F$, the flow associated with $F(t, x, \dot{x}) = 0$ is locally given by*

$$\Phi_F^t(t_0, x_0) := \Phi_h^t(t_0, P(z_0)x_0) + g(t, \Phi_h^t(t_0, P(z_0)x_0)), \tag{98}$$

for $t \in \mathcal{I}_{max}(t_0, x_0)$, where Φ_h is the flow induced by h. The flow Φ_F satisfies

$$\Phi_{\hat{F}}^{t_0}(t_0, x_0) = P(z_0)x_0 + g(t_0, P(z_0)x_0), \tag{99a}$$

$$\Phi_{\hat{F}}^t(t_0, \Phi_{\hat{F}}^s(t_0, x_0)) = \Phi_{\hat{F}}^t(t_0, x_0), \tag{99b}$$

$$\hat{F}(t, \Phi_{\hat{F}}^t(t_0, x_0), \dot{\Phi}_{\hat{F}}^t(t_0, x_0)) = 0, \tag{99c}$$

for $t \in \mathcal{I}_{max}(t_0, x_0)$, and moreover,

$$P_d(t, x_0, \dot{x}_0)\Phi_{\hat{F}}^t(t_0, x_0) = \Phi_h(t_0, x_{d0}), \tag{100a}$$

$$P_d'(t, x_0, \dot{x}_0)\Phi_{\hat{F}}^t(t_0, x_0) = g(t_0, x_{d0}). \tag{100b}$$

The flow Φ_F is unique and $\Phi_F^{(\cdot)}(t_0, x_0) \in C^1((\mathcal{I}_{max}(t_0, x_0), \mathbb{R}^n)$ for every $(z_0) \in \mathcal{C}_F$. For every $z_0 \in \mathcal{C}_F$, the solution of $F(t, x, \dot{x}) = 0$, $x(t_0) = x_0$ is locally given by $x(t) = \Phi_{\hat{F}}^t(t_0, x_0)$.

Proof. If $F(t, x, \dot{x}) = 0$ is regular with s-index μ and $z_0 = (t_0, x_0, \dot{x}_0) \in \mathcal{C}_F$, then the solution of $F(t, x, \dot{x}) = 0$, $x(t_0) = x_0$ is locally given by (96). This motivates the flow formula (98). To verify the properties of Φ_F, we consider P_d, h, g along the solution x^* of $F(t, x, \dot{x}) = 0$, $x^*(t_0) = x_0$ and set $z^* := (t, x^*, \dot{x}^*)$.

Considering Φ_F in $t = t_0$ and noting that $\Phi_h^{t_0}(t_0, P(z_0)x_0) = P(z_0)x_0$, then we verify (99a). Since $\mathbb{X}_d, \mathbb{X}_a$ are g invariant and $\mathbb{X}_d$ is Φ_h invariant, cp. Corollary 4.3, then $P_d'(z^*)\Phi_h^t(t_0, P(z_0)x_0) = 0$ and $P_d(z^*)g(t_0,\ P(z_0)x_0) = 0$ for $t \in \mathcal{I}_{max}(t_0, x_0)$ and we verify (100). Using (100a), then we obtain that

$$\Phi_F^t(s, \Phi_F^s(t_0, x_0)) = \Phi_h^t(s, \Phi_h^s(t_0, P(z_0)x_0)) + g(t, \Phi_h^t(s, \Phi_h^s(t_0, P(z_0)x_0))),$$

and since $\Phi_h^t(s, \Phi_h^s(t_0, P(z_0)x_0)) = \Phi_h^t(t_0, P(z_0)x_0)$, we verify (99b). For (99c), we note that

$$\dot{\Phi}_F^t(t_0, x_0) = h(t, \Phi_h^t(t_0, P(z_0)x_0)) + \dot{g}(t, \Phi_h^t(t_0, P(z_0)x_0)).$$

Since $\mathbb{X}_d$ is Φ_h invariant, we have that $P(z^*)\Phi_h^t(t_0, P(z_0)x_0) = \Phi_h^t(t_0, P(z_0)x_0)$ on $\mathcal{I}_{max}(t_0, x_0)$. Noting that $x^*(t) = \Phi_h^t(t_0, P(z_0)x_0)$, then we get that

$$\hat{F}(t, \Phi_F^t(t_0, x_0), \dot{\Phi}_F^t(t_0, x_0)\hat{F}(t, {x_d}^* + g(t, {x_d}^*), h(t, {x_d}^*) + \dot{g}(t_0, {x_d}^*)),$$

where ${x_d}^* = P(z^*)x^*$. By construction of h, g, cp. (91), (93), then we find that

$$\hat{F}(t, {x_d}^* + g(t, {x_d}^*), h(t, {x_d}^*) + \dot{g}(t_0, {x_d}^*)) = 0,$$

i.e., $\Phi_F^t(t_0, x_0)$ satisfies $\hat{F}(t, \Phi_F^t(t_0, x_0), \dot{\Phi}_F^t(t_0, x_0) = 0$.

As P_d, h, g are uniquely defined by F, cp. Lemma 4.1, Lemma 4.7, also Φ_F is uniquely defined.

In combination with properties (99), this proves that for every $(t_0, x_0, \dot{x}_0) \in \mathcal{C}_F$, then prove that the solution of $F(t, x, \dot{x}) = 0$, $x(t_0) = x_0$ is locally given by $x(t) = \Phi_{\hat{F}}^t(t_0, x_0)$.

As every solution of $F(t, x, \dot{x}) = 0$, $x(t_0) = x_0$ satisfies $x \in C^1(\mathcal{I}_{max}(t_0, x_0), \mathbb{R}^n)$, cp. Lemma 4.8, this implies that $\Phi_F^{(\cdot)}(t_0, x_0) \in C^1((\mathcal{I}_{max}(t_0, x_0), \mathbb{R}^n)$ for every $(t_0, x_0, \dot{x}_0) \in \mathcal{C}_F$. □

For $\dot{x} = f(t, x)$ with $F_{\dot{x}} = I_n$, we have that $P_d = I_n$, $h = f$ and $g = 0$. Then, (99) correspond to (11) and $\Phi_F = \Phi_f$.

Theorem 4.7 extends the notion of a flow to nonlinear DAEs with regular s-index. The properties (99) ensure the existence of a unique solution $x(t) = \Phi_F^t(t_0, x_0)$ associated with the initial value problem $F(t, x, \dot{x}) = 0$, $x(t_0) = x_0$, $(t_0, x_0, \dot{x}_0) \in \mathcal{C}_F$. Properties (100) allow to access the solutions x_d, x_a of the semi-explicit remodeling (58) for every $t \in \mathcal{I}$. The specifying functions h, g are obtained by computing first the s-free formulation $\hat{F}$ effective in the neighborhood of a consistent value $z_0 \in \mathcal{L}_\mu$, cp. Theorem 4.1, then locally solving $\hat{F}(t, x, \dot{x}) = 0$ for h, g as described in Theorem 4.6.

If $F(t, x, \dot{x}) = \dot{x} - f(t, x)$ $F_{\dot{x}} = I_n$, then $P_d = I_n$ and Φ_F covers the assertions of Lemma 2.6. By the uniqueness of the flow, then we find that $\Phi_F = \Phi_f$. Thus, Theorem 4.7 generalizes the assertions of Lemma 2.7 and indeed supplies a unified description of the flow of constrained or unconstrained systems.

Remark 4.2. *Stated for general nonlinear problems $F(t, x, \dot{x}) = 0$, the local nature of Theorem 4.1 and Theorem 4.6 is inevitably. First, the remodeling as s-free system $\hat{F}(t, x, \dot{x}) = 0$ is defined locally as the matrices Z_1, Z_2 are constructed from the linearization of the derivative array, cp. Theorem 4.1. Second, the remodeling* (89) *as semi-explicit system is based on the implicit function theorem, and hence only locally defined. Consequently, the flow is only locally defined in the neighborhood of $\hat{z}_0 \mathcal{C}_F$.*

These results may be considerably relaxed for problems with a given structure. For quasilinear systems $E(t)\dot{x} = f(t, x)$, e.g., the remodeling matrices Z_1, Z_2 are globally defined on every interval $\mathcal{I}$ on which the derivative array satisfies the assertions of Hypothesis 4.1. Also, the projection $P_d = E^+ E$ is globally defined on $\mathcal{I}$ and the domain of definition of the remodeling (89) *depends on the implicit function g describing the constraints. If, e.g., the constraints are linear, then* (89) *is globally defined on $\mathcal{I}$.*

For many applications in biological or chemical engineering, where the constraints arise from steady state assumptions or equilibrium conditions, the Jacobian $F_{\dot{x}}$ is of the structure $\operatorname{diag}(I, 0)$*, yielding a constant projection $P_d = \operatorname{diag}(I, 0)$. Then, again, the domain of definition of* (89) *depends on the solvability of the constraints.*

4.3.3 Alternative decompositions

Following the arguments of Lemma 4.1, we observe that the subspaces $\operatorname{coker}(\hat{F}_x)$, $\ker(\hat{F}_x)$ and the projections $\hat{F}_x^+ \hat{F}_x$, $\hat{F}_x \hat{F}_x^+$ are likewise uniquely defined by F, provided $F(t, x, \dot{x}) = 0$ has s-index μ and $\hat{F}_x$ exists on $\mathcal{C}_F$. Thus, we also could decompose the variables with respect to these spaces. To illustrate the difference between these decompositions and motivate the use of the projection $\hat{F}_{\dot{x}}^+ \hat{F}_{\dot{x}}$, we study the basis representations of a linear DAE with respect to these spaces.

Lemma 4.9. *Let $E\dot{x} = Ax + f$ have s-index μ and consider the s-free remodeling $\hat{E}\dot{x} = \hat{A}x + \hat{f}$. Let $T = [T_1\, T_2] \in C^1(\mathcal{I}, \mathbb{R}^{n \times n})$ be pointwise nonsingular with $T^{-1} = [T_1^{-T}\, T_2^{-T}]^T$. Set $\hat{E}_{11} = \hat{E}_1 T_1$, $\hat{A}_{ij} = \hat{A}_i T_j$ and $\tilde{x}_i = T_i^- x$ for $j = 1, 2$ and $x \in \mathbb{R}^n$.*

1. *If* $span(T_1) = coker(\hat{E})$, $span(T_2) = \ker(\hat{E})$, *then* T *is pointwise orthogonal and* $\hat{E}\dot{x} = \hat{A}x + \hat{f}$ *is equivalent to*

$$\begin{bmatrix} \hat{E}_{11} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \end{bmatrix} = \begin{bmatrix} \hat{A}_{11} - \hat{E}_{11}T_1^T\dot{T}_1 & \hat{A}_{12} - \hat{E}_{11}T_1^T\dot{T}_2 \\ \hat{A}_{21} & \hat{A}_{22} \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} + \begin{bmatrix} \hat{f}_1 \\ \hat{f}_2 \end{bmatrix}. \tag{101}$$

2. *If* $span(T_1) = \ker(\hat{A})$, $span(T_2) = coker(\hat{A})$, *then* T *is pointwise orthogonal and* $\hat{E}\dot{x} = \hat{A}x + \hat{f}$ *is equivalent to*

$$\begin{bmatrix} \hat{E}_{11} & \hat{E}_{11} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \end{bmatrix} = \begin{bmatrix} \hat{A}_{11} - \hat{E}_{11}T_1^T\dot{T}_1 & \hat{A}_{12} - \hat{E}_{11}T_1^T\dot{T}_2 \\ 0 & \hat{A}_{22} \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} + \begin{bmatrix} \hat{f}_1 \\ \hat{f}_2 \end{bmatrix}. \tag{102}$$

3. *If* $span(T_1) = \ker(\hat{A})$, $span(T_2) = \ker(\hat{E})$, *then* $\hat{E}\dot{x} = \hat{A}x + \hat{f}$ *is equivalent to*

$$\begin{bmatrix} \hat{E}_{11} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \end{bmatrix} = \begin{bmatrix} \hat{A}_{11} - \hat{E}_{11}T_1^T\dot{T}_1 & \hat{A}_{12} - \hat{E}_{11}T_1^T\dot{T}_2 \\ 0 & \hat{A}_{22} \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} + \begin{bmatrix} \hat{f}_1 \\ \hat{f}_2 \end{bmatrix}. \tag{103}$$

Proof. Let $T \in C^1(\mathcal{I}, \mathbb{R}^{n\times n})$ is pointwise nonsingular and $x \in C^1(\mathcal{I}, \mathbb{R}^n)$. Setting $\tilde{x} = T^{-1}x$, then $\dot{\tilde{x}} = \dot{T}^{-1}x + T^{-1}\dot{x}$ and $\hat{E}\dot{x} = \hat{A}x + \hat{f}$ is equivalent to $\hat{E}T\dot{\tilde{x}} = (\hat{A}T - \hat{E}\dot{T})\tilde{x} + \hat{f}$. Partitioning T as proposed in the assertions, then we verify the given basis representations (101), (102), (103). □

Decomposition (101) yields an explicit differential equation for $\tilde{x}_1$ as $\hat{E}_{11}$ is pointwise nonsingular by the choice of T. Also, $\tilde{A}_{22}$ is pointwise nonsingular since $\hat{E}\dot{x} = \hat{A}x + \hat{f}$ is s-free, i.e., $\ker(\hat{E}) \cap \ker(\hat{A}) = \{0\}$, and we get a uniquely solvable algebraic equation for $\tilde{x}_2$. Using block Gaussian elimination, then we can eliminate the upper left entry $\tilde{A}_{12}$ and decouple the algebraic equation from the differential one, i.e., we obtain that

$$\begin{bmatrix} I_d & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \hat{E}_{11}^{-1}\hat{A}_{\mathcal{S}} - \tilde{A}_{\mathcal{C}} & 0 \\ \tilde{A}_{22}^{-1}\tilde{A}_{21} & I_a \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \hat{E}_{11}^{-1}\hat{f}_{\mathcal{S}} \\ \hat{A}_{22}^{-1}\hat{f}_2 \end{bmatrix}, \tag{104}$$

where $\tilde{A}_{\mathcal{S}} := \tilde{A}_{11} - \tilde{A}_{12}\tilde{A}_{22}^{-1}\tilde{A}_{21}$, $\tilde{A}_{\mathcal{C}} := T_1^T(\dot{T}_1 - \dot{T}_2\tilde{A}_{22}^{-1}\tilde{A}_{21})$, $\tilde{f}_{\mathcal{S}} := \hat{f}_1 - (A_{12} - \hat{E}_{11}T_1^T\dot{T}_2)\hat{A}_{22}^{-1}\hat{f}_2$. The decomposition (104) allows to treat the differential components in the standard ODE setting, and concepts like stability, see e.g. [40], are straightforwardly generalized under suitable conditions on the algebraic components. Moreover, this decomposition can be realized by orthogonal transformations, which is numerically more stable and simplifies the analysis. Noting that the projection $P_d = \hat{E}^+\hat{E}$ can be diagonalized for this choice of T, then we find that (104) is the basis representation of (58).

Likewise, decomposition (102) can be realized by orthogonal transformations. However, $\mathrm{coker}(\hat{A}_2)$ is not $\hat{E}$ invariant in general, such that the system is still implicit in the differential variables and involves derivatives of the algebraic components.

Contrarily, decomposition (103) yields an explicit differential and algebraic equation for $\tilde{x}_1$ and $\tilde{x}_2$, respectively, both of which are uniquely solvable by the choice of T. Using block Gaussian elimination from the right, these equations can be completely decoupled. However, the corresponding transformations are not orthogonal in general.

Remark 4.3. *Regarding the triangular structure of* (101), (102), (103), *we find that for a s-free pair of variable matrices* $\hat{E}, \hat{A} \in (\mathcal{I}, \mathbb{R}^{n\times n})$, *the space* $\mathbb{R}^n$ *does not decompose into a direct sum* $\mathbb{K}\oplus\mathbb{K}'$ *of complementary subspaces* $\mathbb{K}, \mathbb{K} \subset \mathbb{R}^n$ *that* both *are* $\hat{E}$ *and* $\hat{A}$ *invariant. Consequently, in the associated DAE, the differential and algebraic components cannot be completely decoupled without additional block Gaussian elimination from the left. For constant,* regular *matrix pairs* $E, A \in \mathbb{R}^{n\times n}$, *such a decomposition exists and substantially simplifies the analysis of the associated DAE, cp. [37].*

5 Examples

Example 1 Consider $E\dot{x} = Ax + f$ with

$$E = \begin{bmatrix} \frac{t+1}{t+2} & \frac{\sqrt{t+1}}{t+2} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} \frac{2(t+1)}{(t+2)^2} & \frac{2(t+1)}{2(t+2)^2\sqrt{t+1}} & 0 \\ -\frac{1}{\sqrt{t+2}} & \frac{\sqrt{t+1}}{\sqrt{t+2}} & 0 \\ -1 & -\frac{1}{\sqrt{t+1}} & 1 \end{bmatrix}$$

and

$$f = \begin{bmatrix} \frac{3t+4}{(t+2)^2} + t & -2\sqrt{t+2} & -t^2 \end{bmatrix}.$$

The matrices E, A satisfy Theorem 4.2 with $\mu = 0$, $d = 1$ and $a = n - d = 2$, i.e., $E\dot{x} = Ax + f$ is strangeness-free and already given in strangeness-free form (50). Noting that $\mathrm{rank}(E) = 1$ on $(-1, \infty)$, we apply Theorem 2.1 and obtain that

$$E = \begin{bmatrix} \frac{t+1}{t+2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{t+1}}{\sqrt{t+2}} & \frac{1}{\sqrt{t+2}} & 0 \\ -\frac{1}{\sqrt{t+2}} & \frac{\sqrt{t+1}}{\sqrt{t+2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

on $(-1, \infty)$. The Moore-Penrose is given by

$$E^+ = \begin{bmatrix} \frac{\sqrt{t+1}}{\sqrt{t+2}} & -\frac{1}{\sqrt{t+2}} & 0 \\ \frac{1}{\sqrt{t+2}} & \frac{\sqrt{t+1}}{\sqrt{t+2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{t+2}{t+1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{\sqrt{t+1}} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

cp. Lemma 2.4, and we obtain the projections

$$P_d = \begin{bmatrix} \frac{t+1}{t+2} & \frac{\sqrt{t+1}}{t+2} & 0 \\ \frac{\sqrt{t+1}}{t+2} & \frac{1}{t+2} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad P_d^{'} = \begin{bmatrix} \frac{1}{t+2} & -\frac{\sqrt{t+1}}{t+2} & 0 \\ -\frac{\sqrt{t+1}}{t+2} & \frac{t+1}{t+2} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{105a}$$

and

$$Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad Q^{'} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{105b}$$

To verify Lemma 4.2, we note that

$$Q^{'}AP_d^{'} = \begin{bmatrix} 0 & 0 & 0 \\ -\frac{1}{\sqrt{t+2}}A_{22} & \frac{\sqrt{t+1}}{\sqrt{t+2}}A_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & A_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{t+1}}{\sqrt{t+2}} & \frac{1}{\sqrt{t+2}} & 0 \\ -\frac{1}{\sqrt{t+2}} & \frac{\sqrt{t+1}}{\sqrt{t+2}} & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

such that we get the Moore-Penrose inverse

$$(Q^{'}AP_d^{'})^{+} = \begin{bmatrix} \frac{\sqrt{t+1}}{\sqrt{t+2}} & -\frac{1}{\sqrt{t+2}} & 0 \\ \frac{1}{\sqrt{t+2}} & \frac{\sqrt{t+1}}{\sqrt{t+2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{A_{22}} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{\sqrt{t+2}}\frac{1}{A_{22}} & 0 \\ 0 & \frac{\sqrt{t+1}}{\sqrt{t+2}}\frac{1}{A_{22}} & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

and verify that $(Q^{'}AP_d^{'})^{+}Q^{'}AP_d^{'} = P_d^{'}$ and $Q^{'}AP_d^{'}(Q^{'}AP_d^{'})^{+} = Q^{'}$. Now, we compute $D_{\mathcal{C}}$ and f_a, i.e.,

$$D_{\mathcal{C}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & -\frac{1}{\sqrt{t+1}} & 0 \end{bmatrix}$$

and

$$f_a = (Q^{'}AP_d^{'})^{+}f = \begin{bmatrix} 2 & -2\sqrt{t+1} & -t^2 \end{bmatrix}^T,$$

and obtain the projections

$$P = P_d - D_{\mathcal{C}} = \begin{bmatrix} \frac{t+1}{t+2} & \frac{\sqrt{t+1}}{t+2} & 0 \\ \frac{\sqrt{t+1}}{t+2} & \frac{1}{t+2} & 0 \\ 1 & \frac{1}{\sqrt{t+1}} & 0 \end{bmatrix}, \quad P^{'} = P_d^{'} + D_{\mathcal{C}} = \begin{bmatrix} \frac{1}{t+2} & -\frac{\sqrt{t+1}}{t+2} & 0 \\ -\frac{\sqrt{t+1}}{t+2} & \frac{t+1}{t+2} & 0 \\ -1 & -\frac{1}{\sqrt{t+1}} & 1 \end{bmatrix}.$$

For the inherent ODE, we compute the derivative

$$\dot{P}_d = \begin{bmatrix} \frac{1}{(t+2)^2} & -\frac{t}{2(t+2)^2\sqrt{t+1}} & 0 \\ -\frac{t}{2(t+2)^2\sqrt{t+1}} & -\frac{1}{(t+2)^2} & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

such that

$$D = E^+A + \dot{P}_d = \begin{bmatrix} \frac{2t+3}{(t+2)^2} & \frac{t+2}{2(t+2)^2\sqrt{t+1}} & 0 \\ \frac{3t+4}{2(t+2)^2\sqrt{t+1}} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

and we get the system matrix

$$D_{\mathcal{S}} = DP = \begin{bmatrix} \frac{2t+3}{(t+2)^2} & \frac{t+2}{2(t+2)^2\sqrt{t+1}} & 0 \\ \frac{3t+4}{2(t+2)^2\sqrt{t+1}} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

For the dynamic inhomogeneities, we note that

$$E^+f = \begin{bmatrix} \frac{3t+4}{(t+2)^2} + t & \frac{3t+4}{(t+2)^2\sqrt{t+1}} + \frac{t}{\sqrt{t+1}} & 0 \end{bmatrix}^T,$$

$$Df_a = \begin{bmatrix} 2\frac{2t+3}{(t+2)^2} - \frac{2(t+2)\sqrt{t+1}}{2(t+2)^2\sqrt{t+1}} & \frac{3t+4}{(t+2)^2\sqrt{t+1}} & 0 \end{bmatrix}^T,$$

such that

$$f_d = E^+f - Df_a = \begin{bmatrix} t & \frac{t}{\sqrt{t+1}} & 0 \end{bmatrix}^T.$$

In conclusion, the remodeling (58) is given by

$$\begin{bmatrix} \dot{x}_{d,1} \\ \dot{x}_{d,2} \\ \dot{x}_{d,3} \end{bmatrix} = \begin{bmatrix} \frac{2t+3}{(t+2)^2} & \frac{t+2}{2(t+2)^2\sqrt{t+1}} & 0 \\ \frac{3t+4}{2(t+2)^2\sqrt{t+1}} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_{d,1} \\ x_{d,2} \\ x_{d,3} \end{bmatrix} + \begin{bmatrix} t \\ \frac{t}{\sqrt{t+1}} \\ 0 \end{bmatrix}, \tag{106a}$$

$$\begin{bmatrix} x_{a,1} \\ x_{a,2} \\ x_{a,3} \end{bmatrix} = -\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & -\frac{1}{\sqrt{t+1}} & 0 \end{bmatrix} \begin{bmatrix} x_{d,1} \\ x_{d,2} \\ x_{d,3} \end{bmatrix} - \begin{bmatrix} 2 \\ -2\sqrt{t+1} \\ -t^2 \end{bmatrix} \tag{106b}$$

with

$$x_d = \begin{bmatrix} \frac{\sqrt{t+1}(\sqrt{t+1}x_1+x_2)}{t+2} \\ \frac{\sqrt{t+1}x_1+x_2}{t+2} \\ 0 \end{bmatrix}, \quad x_a = \begin{bmatrix} \frac{x_1-\sqrt{t+1}x_2}{t+2} \\ -\frac{\sqrt{t+1}(x_1-\sqrt{t+1}x_2)}{t+2} \\ x_3 \end{bmatrix}. \tag{107}$$

The set $\mathcal{C}_{E,A,f}$ is given by

$$\mathcal{C}_{E,A,f} = \left\{ x_0 \in \mathbb{R}^3 \middle| \begin{bmatrix} \frac{x_{0,1}-\sqrt{t_0+1}x_{0,2}}{t_0+2} \\ -\frac{\sqrt{t_0+1}(x_{0,1}-\sqrt{t_0+1}x_{0,2})}{t_0+2} \\ -\frac{\sqrt{t_0+1}x_{0,1}+x_{0,2}}{\sqrt{t_0+1}} + x_{0,3} \end{bmatrix} = \begin{bmatrix} 2 \\ -2\sqrt{t+1} \\ -t^2 \end{bmatrix} \right\}.$$

with associated projection

$$P = P - D_{\mathcal{C}} = \begin{bmatrix} \frac{t+1}{t+2} & \frac{\sqrt{t+1}}{t+2} & 0 \\ \frac{\sqrt{t+1}}{t+2} & \frac{1}{t+2} & 0 \\ 1 & \frac{1}{\sqrt{t+1}} & 0 \end{bmatrix}, \quad P' = P' + D_{\mathcal{C}} = \begin{bmatrix} \frac{1}{t+2} & -\frac{\sqrt{t+1}}{t+2} & 0 \\ -\frac{\sqrt{t+1}}{t+2} & \frac{t+1}{t+2} & 0 \\ -1 & -\frac{1}{\sqrt{t+1}} & 1 \end{bmatrix}.$$

The flow $\Phi_{D_{\mathcal{S}}}$ associated with (106a) is given by

$$\Phi^t_{D_{\mathcal{S}}}(t_0) = \begin{bmatrix} \exp(\int_{t_0}^t \frac{2s+3}{(s+2)^2}\,ds) & \int_{t_0}^t \exp(\int_{t_0}^s \frac{2\hat{s}+3}{(\hat{s}+2)^2}\,d\hat{s})\frac{s+2}{2(s+2)^2\sqrt{s+1}}\,ds & 0 \\ \exp(\int_{t_0}^t \frac{3s+4}{2(s+2)^2\sqrt{s+1}}\,ds) & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

implying that

$$\begin{aligned} &\Phi^t_{E,A}(t_0) = (P(t) - D_{\mathcal{C}}(t))\Phi^t_{D_{\mathcal{S}}}(t_0)P(t_0)x_0 \\ &= \begin{bmatrix} \frac{t+1}{t+2} & \frac{\sqrt{t+1}}{t+2} & 0 \\ \frac{\sqrt{t+1}}{t+2} & \frac{1}{t+2} & 0 \\ 1 & \frac{1}{\sqrt{t+1}} & 0 \end{bmatrix} \begin{bmatrix} \exp(\int_{t_0}^t \frac{2s+3}{(s+2)^2}\,ds) & \int_{t_0}^t \exp(\int_{t_0}^s \frac{2\hat{s}+3}{(\hat{s}+2)^2}\,d\hat{s})\frac{s+2}{2(s+2)^2\sqrt{s+1}}\,ds & 0 \\ \exp(\int_{t_0}^t \frac{3s+4}{2(s+2)^2\sqrt{s+1}}\,ds) & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{t_0+1}{t_0+2} & \frac{\sqrt{t_0+1}}{t_0+2} & 0 \\ \frac{\sqrt{t_0+1}}{t_0+2} & \frac{1}{t_0+2} & 0 \\ 1 & \frac{1}{\sqrt{t_0+1}} & 0 \end{bmatrix}. \end{aligned}$$

The inhomogeneous flow then is given by

$$\Phi^t_{E,A,f}(t_0, x_0) = \Phi^t_{E,A}(t_0)x_0 + \int_{t_0}^t \Phi^t_{E,A}(s)f_d\,ds - f_a(t).$$

Example 1 illustrates the decomposition of a linear DAE using projections and verifies the flow formula.

Example 2 Consider $F(t,x,\dot{x})=0$ with

$$F(t,x,\dot{x})=\begin{bmatrix}\frac{x_1}{2(t+2)\sqrt{t+1}}+\frac{\sqrt{t+1}\dot{x}_1+\dot{x}_2}{t+2}-\frac{\sqrt{t+1}x_1+x_2}{(t+2)^2}-\frac{(\sqrt{t+1}x_1+x_2)^2}{(t+2)^2}\\ \frac{(x_1-\sqrt{t+1}x_2)^2}{(t+2)^2}-2\\ x_3^2-\frac{\sqrt{t+1}x_1+x_2}{t+2}-1\end{bmatrix}$$

The Jacobians are given by

$$F_{\dot{x}}(t,x,\dot{x})=\begin{bmatrix}\frac{\sqrt{t+1}}{t+2} & \frac{1}{t+2} & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

and

$$F_x(t,x,\dot{x})=\begin{bmatrix}\frac{1}{2(t+2)\sqrt{t+1}}-\frac{\sqrt{t+1}}{(t+2)^2}-\frac{2(\sqrt{t+1}x_1+x_2)\sqrt{t+1}}{(t+2)^2} & -\frac{1}{(t+2)^2}-\frac{2(\sqrt{t+1}x_1+x_2)}{(t+2)^2} & 0\\ \frac{x_1-\sqrt{t+1}x_2}{(t+2)^2} & -\frac{(x_1-\sqrt{t+1}x_2)\sqrt{t+1}}{(t+2)^2} & 0\\ -\frac{\sqrt{t+1}}{t+2} & -\frac{1}{t+2} & 2x_3\end{bmatrix}$$

Noting that

$$F_{\dot{x}}^{+}(t,x,\dot{x})=\begin{bmatrix}\sqrt{t+1} & 0 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{bmatrix},$$

we obtain the same projections as in (105). We verify Lemma 4.6, i.e., with

$$\begin{aligned}&Q^{'}F_x(t,x,\dot{x})P_d^{'}\\ &=\begin{bmatrix}0 & 0 & 0\\ \frac{x_1-\sqrt{t+1}x_2}{(t+2)^2} & -\frac{(x_1-\sqrt{t+1}x_2)\sqrt{t+1}}{(t+2)^2} & 0\\ 0 & 0 & 2x_3\end{bmatrix}=\begin{bmatrix}0 & 0 & 0\\ 0 & \frac{x_1-\sqrt{t+1}x_2}{(t+2)^{3/2}} & 0\\ 0 & 0 & 2x_3\end{bmatrix}\begin{bmatrix}\frac{\sqrt{t+1}}{\sqrt{t+2}} & \frac{1}{\sqrt{t+2}} & 0\\ \frac{1}{\sqrt{t+2}} & -\frac{\sqrt{t+1}}{\sqrt{t+2}} & 0\\ 0 & 0 & 1\end{bmatrix}\end{aligned}$$

we have that

$$(Q^{'}F_x(t,x,\dot{x})P^{'}(t))^{+}=\begin{bmatrix}\frac{\sqrt{t+1}}{\sqrt{t+2}} & \frac{1}{\sqrt{t+2}} & 0\\ \frac{1}{\sqrt{t+2}} & -\frac{\sqrt{t+1}}{\sqrt{t+2}} & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}0 & 0 & 0\\ 0 & \frac{(t+2)^{3/2}}{x_1-\sqrt{t+1}x_2} & 0\\ 0 & 0 & \frac{1}{2x_3}\end{bmatrix},$$

such that

$$\begin{aligned}&(Q^{'}F_x(t,x,\dot{x})P^{'}(t))^{+}Q^{'}F_x(t,x,\dot{x})P^{'}(t)\\ &=\begin{bmatrix}0 & \frac{1}{\sqrt{t+2}} & 0\\ 0 & -\frac{\sqrt{t+1}}{\sqrt{t+2}} & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}0 & 0 & 0\\ \frac{1}{\sqrt{t+2}} & -\frac{\sqrt{t+1}}{\sqrt{t+2}} & 0\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}\frac{1}{t+2} & -\frac{\sqrt{t+1}}{t+2} & 0\\ -\frac{\sqrt{t+1}}{t+2} & \frac{t+1}{t+2} & 0\\ 0 & 0 & 1\end{bmatrix}=P_d^{'}.\end{aligned}$$

To compute the local remodeling (89), we formulate $F(t,x,\dot{x})=0$ in terms of $x_d = P_d x$, $x_a = P_d^{'} x$. With x_d, x_a given by (107), we obtain that

$$F(t, x_d + x_a, \dot{x}_d + \dot{x}_a) = \begin{bmatrix} \dot{x}_{d,2} - x_{d,2}^2 \\ x_{a1}^2 - 2 \\ x_{a3}^2 - x_{d,2} - 1 \end{bmatrix} = 0.$$

With $Q, Q^{'}$ given by (105) and noting that $x_{a2} = -\sqrt{t+1}x_{a1}$ and $x_{d1} = \sqrt{t+1}x_{d2}$, we successively solve $Q^{'}F(t, x_d + x_a, \dot{x}_d + \dot{x}_a) = 0$ and $QF(t, x_d + x_a, \dot{x}_d + \dot{x}_a) = 0$ for x_a, x_d and obtain that

$$x_a = g(t, x_d) = \begin{bmatrix} \sqrt{2}, & -\sqrt{2(t+1)} & \sqrt{1 + x_{d,2}} \end{bmatrix}^T, \tag{108a}$$

$$\dot{x}_d = h(t, x_d) = \begin{bmatrix} x_{d,2}^2 \sqrt{t+1} + \frac{x_{d,2}}{2\sqrt{t+1}} & x_{d,2}^2 & 0 \end{bmatrix}^T. \tag{108b}$$

The flow associated with (108b) is given by

$$\Phi_h^t(t_0, x_{d0}) = \begin{bmatrix} -\frac{\sqrt{t+1}}{t-t_0-x_{d,2,0}} & -\frac{1}{t-t_0-x_{d,2,0}} & 0 \end{bmatrix}^T,$$

defined for $(t_0, x_{d0}) \in \mathcal{I} \times \Omega$ with $t_0 \neq x_{d,0,2}$ and maximal interval of existence $\mathcal{J}_{max}(t_0, x_0) = [t_0, t_0 + x_{d,2,0})$ if $x_{d,2,0} \geq 0$ and $\mathcal{J}_{max}(t_0, x_0) = [t_0, \infty)$ if $x_{d,2,0} < 0$. Then,

$$g(t, x_d) = \begin{bmatrix} \sqrt{2} & -\sqrt{2(t+1)} & \sqrt{1 - \frac{1}{t-t_0-x_{d,2,0}}} \end{bmatrix}^T$$

is defined for $(t_0, x_{d0}) \in \mathcal{I} \times \Omega$ with $t_0 \neq x_{d,0,2}$ and $t \in [t_0, t_0 + x_{d,2,0})$ if $x_{d,2,0} \geq 0$ and $t \in [t_0, \infty)$ if $x_{d,2,0} < 0$. Noting that $x_{d,2,0} = \frac{\sqrt{t_0+1}x_{1,0}+x_{2,0}}{t_0+2}$, then we get that

$$\Phi_F^t(t_0, x_0) = \begin{bmatrix} -\frac{(t_0+2)\sqrt{t+1}}{(t-t_0)(t_0+2)-(\sqrt{t_0+1}x_{1,0}+x_{2,0})} + \sqrt{2} \\ -\frac{t_0+2}{(t-t_0)(t_0+2)-(\sqrt{t_0+1}x_{1,0}+x_{2,0})} - \sqrt{2(t+1)} \\ \sqrt{1 - \frac{t_0+2}{(t-t_0)(t_0+2)-(\sqrt{t_0+1}x_{1,0}+x_{2,0})}} \end{bmatrix}.$$

The set of consistent initial values is given by

$$\mathcal{C}_F = \left\{ x_0 \in \mathbb{R}^3 \middle| \; \exists\, v_1, v_2 \in \mathbb{R} : \begin{bmatrix} x_{0,1} - \frac{(t_0+2)\sqrt{t+1}}{\sqrt{t_0+1}x_{1,0}+x_{2,0}} + \sqrt{2} \\ x_{0,2} - \frac{t_0+2}{\sqrt{t_0+1}x_{1,0}+x_{2,0}} - \sqrt{2(t_0+1)} \\ x_{0,3} - \sqrt{1 + \frac{t_0+2}{\sqrt{t_0+1}x_{1,0}+x_{2,0}}} \end{bmatrix} = 0 \right\}$$

and the maximal interval of existence is given by $\mathcal{J}_{max}(t_0, x_0) = [t_0, t_0 + \frac{\sqrt{t_0+1}x_{1,0}+x_{2,0}}{t_0+2})$ if $t_0 \neq \sqrt{t_0+1}x_{1,0} + x_{2,0} > 0$ and $\mathcal{J}_{max}(t_0, x_0) = [t_0, \infty)$ if $t_0 \neq \sqrt{t_0+1}x_{1,0} + x_{2,0} \leq 0$.

6 Conclusion

We have generalized the concept of a flow to linear and nonlinear DAEs of arbitrary index. Based on the concept of the strangeness-index [37], we have used projections provided by the Moore-Penrose to remodel a given DAE as a semi explicit system. For linear systems $E\dot{x} = Ax + f$, this remodeling is explicitly given in terms of the coefficients E, A, f and globally defined on every interval $\mathcal{I}$ on which E, A satisfy certain rank assumptions. For nonlinear systems $F(t, x, \dot{x}) = 0$, the remodeling is stated terms of implicitly defined functions and locally defined in the neighborhood of every consistent initial value.

Exploiting the solution formulas for algebraic and differential equations in a subspace derived in [2], we have solved the semi-explicit remodeling and computed a closed solution formula for linear and nonlinear systems. Verifying that this solution formula is independent of the the strangeness-free remodeling and uniquely related with a given, consistent initial values, then we have shown the existence of a unique function that generalized the concept of a flow to constrained problems. For linear systems, the flow is globally defined on the set of consistent initial values, for nonlinear systems the flow is a local quantity defined in the neighborhood of every consistent value. For linear problems, the flow possesses a semi-inverse allowing to recover the initial value and the set of consistent initial values is flow invariant, in particular. For both linear and nonlinear problems, the algebraic and differential components can be obtained by projecting the flow onto the respective subspace, respectively. Stated in the original coordinate system due to the projection approach, the flow allows to study coordinate depending properties like positivity.

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