Propagation of Singularities in the Semi-Fractional Brownian Sheet *

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Abstract

Let X be a semi-fractional Brownian sheet, that is a centred and continuous Gaussian random field with $\mathbb{E}[X(s,t)X(\hat{s},\hat{t})] = (t \wedge \hat{t})(s^\alpha + \hat{s}^\alpha - |s-\hat{s}|^\alpha)/2$. We provide, for $\alpha \in (0,2)$, an analysis of the propagation of singularities into the fractional direction of X. Here, singularities are times where the law of the iterated logarithm fails, such as fast points.

1 Introduction

Let B be a standard Brownian motion on $\mathbb{R}$ and denote by

$$F(\lambda) = \left\{ t \geq 0 : \limsup_{h \to 0+} \frac{|B(t+h) - B(t)|}{\sqrt{2h \ln(1/h)}} \geq \lambda \right\}$$

the random set of λ-*fast* or λ-*rapid points*. Orey and Taylor [8] proved that for $0 < \lambda \leq 1$, the set $F(\lambda)$ has almost surely Hausdorff dimension $1 - \lambda^2$. Fast points give rise to a notion of singularities, since for such $t \in F(\lambda)$, one has that the law of the iterated logarithm (LIL) fails, i.e.

$$\limsup_{h \to 0+} \frac{|B(t+h) - B(t)|}{\sqrt{2h \ln \ln(1/h)}} = \infty.$$

We call such a time where the LIL fails a *singularity* (see [9]), and denote the random collection of times which are singularities by $Si(\mathbb{R}_+)$. From Orey and Taylor's dimension formula follows immediately that $Si(\mathbb{R}_+)$ has Hausdorff dimension one, almost surely.

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In the two-dimensional setting, there are singularities which have a striking property. Fix $\alpha \in (0,2)$ and define X to be a *semi-fractional Brownian sheet*, that is, a centred and continuous Gaussian random field with

$$\mathbb{E}\big[X(s,t)X(\hat{s},\hat{t}\,)\big] = (t \wedge \hat{t}\,)(s^\alpha + \hat{s}^\alpha - |s-\hat{s}|^\alpha)/2,$$

for $(s,t),(\hat{s},\hat{t}) \in (\mathbb{R}_+)^2$. In the s-coordinate X is a *fractional Brownian motion* with Hurst-parameter $\alpha/2$, and in the t-coordinate X is a standard Brownian motion. For $\alpha = 1$, the random field X is often called a *Brownian sheet*. Walsh [9] showed that for the Brownian sheet there are singularities which propagate along characteristic directions, i.e. there exists a positive and finite random time T associated properly to some s^*, such that

$$\limsup_{h\to 0+} \frac{|X(s^*,T+h)-X(s^*,T)|}{\sqrt{2h\ln\ln(1/h)}} = \infty \tag{1}$$

with probability one, and, moreover, equation (1) implies

$$\limsup_{h\to 0+} \frac{|X(s,T+h)-X(s,T)|}{\sqrt{2h\ln\ln(1/h)}} = \infty$$

simultaneously for all s in some continuous set I, almost surely. The size of I depends on a measurability condition on T and can, for example, be the positive real half line or an interval. Propagating singularities are like wrinkles in a sheet.

However, the analysis of the propagation of singularities for a Brownian sheet relies heavily on special properties of this process. The independence of its increments is of particular importance. Passing to more general Gaussian fields, one is forced to find proofs which rely on more general principles. We propose the random field X as a prototype to do that. The increments in the fractional coordinate of the field are no longer independent if $\alpha \neq 1$, but positively correlated for $\alpha > 1$ and negatively correlated for $\alpha < 1$.

Our aim in this article is the analysis of the propagation of singularities for X into the direction of the fractional coordinate. The paper is organised as follows.

In Section 2, we show that a *law of the iterated logarithm* for the t-coordinate holds simultaneously for all $s \geq 0$, i.e. we prove in Theorem 2.8 that

$$\mathbb{P}\Big\{\limsup_{h\to 0+} \frac{|X(s,t+h)-X(s,t)|}{\sqrt{2h\ln\ln(1/h)}} = \sqrt{s^\alpha} \quad \text{for all} \quad s \geq 0\Big\} = 1$$

for each $t \geq 0$. This is one of the principle tools to prove propagation.

In Section 3, we first construct, by means of a Section Theorem (see Thm 37, p18, [2]), a positive and finite random time T which is a singularity almost surely; see Proposition 3.2. Then we prove in Theorem 3.8 that the singularity at T propagates along the fractional coordinate of X. It is interesting to see that *fast points* either speed up or slow down, according to the value of $\alpha \in (0,2)$, while propagating along the fractional direction. Finally, it even turns out to be possible to determine the *origin* of a singularity.

We conclude with the remark that propagation *into the fractional direction* is the interesting case for the semi-fractional Brownian sheet, since propagation *into the standard-Brownian direction* can be checked easily. Indeed, Marcus [6] showed that fractional Brownian motion also obeys a law of the iterated logarithm. Moreover, Khoshnevisan and Shi [3] proved the existence of fast points for fractional Brownian motion. As with Brownian motion, these points are singularities in the sense that the LIL fails. Therefore the random field X has singular times in the fractional direction, and one can show that they propagate into the non-fractional direction; see [9] and [7] for details.

2 The Law of the Iterated Logarithm

The main result of this section is a law of the iterated logarithm (LIL), Theorem 2.8, for the semi-fractional Brownian sheet $X = \{X(s,t)\}$ in the t-coordinate, which holds simultaneously for all points s in the fractional coordinate, and which essentially follows from Proposition 2.6.

Definition 2.1 *Fix $\alpha \in (0,2)$ and define X to be the centred and continuous Gaussian random field with*

$$\mathbb{E}\big[X(s,t)X(\hat{s},\hat{t}\,)\big] = (t \wedge \hat{t}\,)(s^\alpha + \hat{s}^\alpha - |s-\hat{s}|^\alpha)/2,$$

for $(s,t), (\hat{s},\hat{t}) \in (\mathbb{R}_+)^2$.

We begin with a useful standard result of Gaussian theory, which is based on Theorem 12.1 in Lifschitz (1992) [4], see Equation (11) on p147.

Proposition 2.2 *Let $U \subset \mathbb{R}^n$ and let $Z = \{Z(u) : u \in U\}$ be a continuous centred Gaussian random field with variance*

$$\sigma^2 := \sup_{u \in U} \mathbb{V}[Z(u)] > 0.$$

Then,

$$\lim_{\lambda\to\infty} \frac{1}{\lambda^2} \ln \mathbb{P}\Big\{ \sup_{u\in U} Z(u) > \lambda \Big\} = -\frac{1}{2\sigma^2}.$$

For our purposes, we need to consider two important special cases.

Corollary 2.3 *For $r > 0$, $U = [0,r]$ and $\alpha \in (0,2)$, let $B^\alpha = \{B^\alpha(s) : s \in U\}$ be a standard* α-fractional Brownian motion *with values in $\mathbb{R}$. Then,*

$$\lim_{\lambda\to\infty} \frac{1}{\lambda^2} \ln \mathbb{P}\Big\{ \sup_{s\in[0,r]} B^\alpha(s) > \lambda \Big\} = -\frac{1}{2r^\alpha}.$$

Proof. B^α is a continuous centred Gaussian process on $\mathbb{R}$ and Variance $\sigma^2 = \sup_{s\in[0,r]} s^\alpha = r^\alpha$. Hence, Proposition 2.2 immediately gives the result. □

Corollary 2.4 *For $s_0, t_0 > 0$, let $U = [0,s_0] \times [0,t_0]$. For $\alpha \in (0,2)$, let X be the semi-fractional Brownian sheet from Definition 2.1. Then, we have*

$$\lim_{\lambda\to\infty} \frac{1}{\lambda^2} \ln \mathbb{P}\Big\{ \sup_{[0,s_0]\times[0,t_0]} X(s,t) > \lambda \Big\} = -\frac{1}{2s_0^\alpha t_0}.$$

Proof. Note that X is a continuous centred Gaussian random field on $\mathbb{R}^2$ and that

$$\sigma^2 := \sup_{(s,t)\in[0,s_0]\times[0,t_0]} \mathbb{V}[X(s,t)] = \sup_{(s,t)\in[0,s_0]\times[0,t_0]} s^\alpha t = s_0^\alpha t_0.$$

□

Remark 2.5 Observe that in the case $\alpha \in [1,2)$, we could have used Slepian's inequality (see e.g. [1], Corollary 2.4, p49) to compare X to the classical Brownian sheet in order to obtain the required estimates. However, this obviously does not work in the case $\alpha \in (0,1)$.

Note that for fixed $s_0 > 0$, the stochastic process $\{s_0^{-\alpha/2} X(s_0,t);\, t \geq 0\}$ is a standard Brownian motion and hence the law of the iterated logarithm (LIL) for Brownian motion states that

$$\mathbb{P}\left\{ \limsup_{h\to 0+} \frac{|X(s_0,t+h) - X(s_0,t)|}{\sqrt{2h \ln\ln(1/h)}} = \sqrt{s_0^\alpha} \right\} = 1$$

for each $t \geq 0$. However, we need a much stronger result, that is, we need the above LIL to hold *simultaneously* in the fractional coordinate s. This will, for each $t \geq 0$, be established in Theorem 2.8.

Nearly all of the work needed for this result is hidden in the following proposition. The proof follows the lines of the proof of Theorem 3, p1237, in [10]. However, Zimmerman considers only the classical Brownian sheet, which is is a special case of our proposition.

Proposition 2.6 *Let $0 < a \leq b < \infty$. Then,*

$$\mathbb{P}\left\{ \limsup_{t\to\infty} \left\{ \frac{s^{-\alpha/2} X(s,t)}{\sqrt{2t \ln\ln t}} \right\} = 1 \quad \text{for all} \quad s \in [a,b] \right\} = 1.$$

Proof. We proceed in three steps.

Step 1. Fix $0 < \varepsilon < 1$. The first partial result we establish is

$$\mathbb{P}\left\{ \limsup_{t\to\infty} \left\{ \sup_{a\leq s\leq b} \frac{s^{-\alpha/2} X(s,t)}{\sqrt{2t \ln\ln t}} \right\} \leq 1 + \varepsilon \right\} = 1, \tag{2}$$

by means of a Borel-Cantelli argument. Let $\delta > 0$, $u \geq a$ and set $v = u + \delta$. Moreover, let $q > 1$ and define for each $k \in \mathbb{N}$ the set

$$A_{k,\delta} = \left\{ \sup_{(s,t)\in[0,v]\times[0,q^k]} X(s,t) > (1+\varepsilon)\sqrt{u^\alpha 2 q^{k-1} \ln\ln q^{k-1}} \right\}.$$

For

$$\lambda(k) := (1+\varepsilon)\sqrt{u^\alpha 2 q^{k-1} \ln\ln q^{k-1}}$$

we have $\lim_{k\to\infty} \lambda(k) = \infty$ since $q > 1$, and by Corollary 2.4, it follows that

$$\lim_{k\to\infty} \frac{1}{\lambda(k)^2} \ln \mathbb{P}\big(A_{k,\delta}\big) = -\frac{1}{2 v^\alpha q^k}.$$

In other words, for each $\tilde{\varepsilon} \in (0, 1/2)$, there exists a $K(\tilde{\varepsilon}) \in \mathbb{N}$, so that for every $k > K(\tilde{\varepsilon})$

$$\frac{1}{\lambda(k)^2} \ln \mathbb{P}\big(A_{k,\delta}\big) \leq -\Big(\frac{1}{2} - \tilde{\varepsilon}\Big) \frac{1}{v^\alpha q^k}. \tag{3}$$

A simple computation then shows that for all $k > K(\tilde{\varepsilon})$, using $v = u + \delta$,

$$\begin{aligned} \mathbb{P}\big(A_{k,\delta}\big) &\leq \exp\Big(-(1-2\tilde{\varepsilon})(1+\varepsilon)^2 \frac{u^\alpha}{qv^\alpha} \ln\ln q^{k-1}\Big) \\ &= (k-1)^{-\frac{(1-2\tilde{\varepsilon})(1+\varepsilon)^2}{q(1+\delta/u)^\alpha}} (\ln q)^{-\frac{(1-2\tilde{\varepsilon})(1+\varepsilon)^2}{q(1+\delta/u)^\alpha}}. \end{aligned}$$

Now, it is easily checked that if $0 < \delta \leq a\varepsilon$ and $\tilde{\varepsilon}$ and q are chosen so that $1 < q < (1-2\tilde{\varepsilon})(1+\varepsilon)^{2-\alpha}$, then

$$\gamma := \frac{(1-2\tilde{\varepsilon})(1+\varepsilon)^2}{q(1+\delta/u)^\alpha} > 1,$$

so that

$$\sum_{k=1}^{\infty} \mathbb{P}\big(A_{k,\delta}\big) \leq (\ln q)^{-\gamma} \sum_{k=1}^{\infty} (k-1)^{-\gamma} < \infty.$$

Hence the Lemma of Borel-Cantelli applies and gives, for our choice of constants,

$$\mathbb{P}\Big(\bigcup_{k\in\mathbb{N}} \bigcap_{j\geq k} A_{j,\delta}^c\Big) = 1.$$

Thus there exists some k (depending on ω), such that, for almost all ω,

$$\sup_{0\leq t\leq q^j} \sup_{u\leq s\leq v} \frac{X(s,t)[\omega]}{\sqrt{u^\alpha 2q^{j-1} \ln\ln q^{j-1}}} \leq 1 + \varepsilon$$

for all $j \geq k$, which implies

$$\limsup_{t\to\infty} \Big\{ \sup_{u\leq s\leq v} \frac{s^{-\alpha/2} X(s,t)}{\sqrt{2t \ln\ln t}} \Big\} \leq 1 + \varepsilon$$

with probability one, respectively. Finally, the interval $[a, b]$ can be finitely covered by closed intervals of the form $[u, v]$ of length $0 < \delta \leq a\varepsilon$ and (2) follows.

Step 2. Next, observe that the counterpart of (2), i.e., for $\varepsilon > 0$,

$$\mathbb{P}\Big\{ \limsup_{t\to\infty} \Big\{ \sup_{a\leq s\leq b} \frac{s^{-\alpha/2} X(s,t)}{\sqrt{2t \ln\ln t}} \Big\} \geq 1 - \varepsilon \Big\} = 1 \tag{4}$$

results directly from the law of the iterated logarithm for a standard Brownian motion; see [5], p249. Moreover, equations (2) and (4) hold for any choice of ε. Thus,

$$\mathbb{P}\Big\{ \limsup_{t\to\infty} \Big\{ \sup_{a\leq s\leq b} \frac{s^{-\alpha/2} X(s,t)}{\sqrt{2t \ln\ln t}} \Big\} = 1 \Big\} = 1. \tag{5}$$

Step 3. It remains to get rid of the innermost "sup" in (5). Here, the choice of parameters is slightly more involved. By the scaling property of X we can assume, without loss of generality, that $0 < a \le b \le 1$. The Lemma of Borel-Cantelli is again our main tool. Fix $\varepsilon, \tilde\varepsilon \in (0, 1/2)$, let $u \ge a$, and set

$$v = \frac{u}{1 - \left(\frac{\epsilon^4(1-2\tilde\varepsilon)^\alpha}{4}\right)^{1/\alpha}}, \qquad \text{so that} \qquad \delta := v - u \ge a\left[\frac{1}{1 - \left(\frac{\epsilon^4(1-2\tilde\varepsilon)^\alpha}{4}\right)^{1/\alpha}} - 1\right] > 0. \tag{6}$$

This choice of parameters will prove to be useful soon. Moreover, let, for $x, y \in \mathbb{R}_+$

$$\Box X\left((x,y),(u,v)\right] = X(u,v) - X(u,y) - X(x,v) + X(x,y),$$

and define for $q > 1$ and $s \in [u, v]$ the decreasing function

$$\gamma(s) = \left(1 - \frac{\varepsilon}{4}\right)\sqrt{v^\alpha}\sqrt{\frac{q-1}{q}} - 2\sqrt{\frac{s^\alpha}{q}} - (1-\varepsilon)\sqrt{s^\alpha}.$$

It is easy to show that there exists a constant $q^* = q^*(\varepsilon) > 1$, independent of u and v, so that for all $q > q^*$, $\gamma(\cdot)$ is always positive. Choose, e.g.

$$q^* = \max\left\{\frac{16}{\varepsilon^2}, \frac{1}{1 - \left(\frac{1-\varepsilon/2}{1-\varepsilon/4}\right)^2}\right\} > 1.$$

For $q > q^*$, $s \in [u, v]$ and $n \in \mathbb{N}$, we may then define the sets

$$A_n(s) = \left\{\Box X\left((s, q^{n-1}), (v, q^n)\right] \ge \gamma(s)\sqrt{2q^n \ln\ln q^n}\right\}$$

and

$$A_n = \left\{\sup_{u \le s \le v} \Box X\left((s, q^{n-1}), (v, q^n)\right] \ge \gamma(v)\sqrt{2q^n \ln\ln q^n}\right\}.$$

One has for each $s \in [u, v]$ that $\gamma(v) \le \gamma(s)$, thus $A_n(s)$ is a subset of A_n and consequently $\bigcup_{s\in[u,v]} A_n(s) \subset A_n$. Again, we wish to apply the Lemma of Borel-Cantelli and hence need to find a summable upper bound for the $\mathbb{P}(A_n)$. To this end, we consider the continuous and centred Gaussian process

$$\left\{Y(s) = \Box X((s, q^{n-1}), (v, q^n)];\ s \in [u, v]\right\}.$$

For $s, \hat s \in [0, v-u]$, it is easy to verify that

$$\begin{aligned}\mathbb{E}\left[Y(v-s)Y(v-\hat s)\right] &= \mathbb{E}\left[\Box X((v-s, q^{n-1}), (v, q^n)]\Box X((v-\hat s, q^{n-1}), (v, q^n)]\right] \\ &= (q^n - q^{n-1})(s^\alpha + \hat s^\alpha - |s - \hat s|^\alpha)/2,\end{aligned}$$

and therefore, the process B^α, defined by

$$\left\{B^\alpha(s) = (q^n - q^{n-1})^{1/2} Y(v-s);\ s \in [0, v-u]\right\},$$

is an α-fractional Brownian motion. Hence, for each $\lambda \in \mathbb{R}_+$, we have

$$\begin{aligned}\mathbb{P}\Big\{\sup_{s\in[u,v]}(q^n-q^{n-1})^{-1/2}Y(s)>\lambda\Big\} &= \mathbb{P}\Big\{\sup_{s\in[0,v-u]}(q^n-q^{n-1})^{-1/2}Y(v-s)>\lambda\Big\}\\ &= \mathbb{P}\Big\{\sup_{s\in[0,v-u]}B^\alpha(s)>\lambda\Big\}.\end{aligned}$$

In particular, Corollary 2.3 implies the existence of some constant $\lambda(\tilde{\varepsilon})>0$, so that for all $\lambda>\lambda(\tilde{\varepsilon})$, we have

$$\mathbb{P}\Big\{\sup_{s\in[u,v]}(q^n-q^{n-1})^{-1/2}Y(s)>\lambda\Big\}\le \exp\Big(-\Big(\frac{1}{2}-\tilde{\varepsilon}\Big)\frac{\lambda^2}{(v-u)^\alpha}\Big).$$

This result applied to A_n gives

$$\begin{aligned}\mathbb{P}(A_n) &= \mathbb{P}\Big\{\sup_{s\in[u,v]}(q^n-q^{n-1})^{-1/2}Y_n(s)\ge\sqrt{\frac{2q^n\ln\ln q^n}{q^n-q^{n-1}}}\gamma(v)\Big\}\\ &\le \exp\Big(-\frac{(1-2\tilde{\varepsilon})\gamma(v)^2q^n\ln\ln q^n}{(q^n-q^{n-1})(v-u)^\alpha}\Big)\\ &= n^{-\frac{(1-2\tilde{\varepsilon})q\gamma(v)^2}{(q-1)(v-u)^\alpha}}(\ln q)^{-\frac{(1-2\tilde{\varepsilon})q\gamma(v)^2}{(q-1)(v-u)^\alpha}}\end{aligned}$$

for all n large enough, since $\sqrt{\frac{2q^n\ln\ln q^n}{q^n-q^{n-1}}}\gamma(v)$ goes off to ∞ with $n\to\infty$ (recall $q>q^*>1$). Next, we check that with our choice of parameters v,u in (6), and with q greater than the maximum of $q^*, 1+64/\varepsilon^2$ and $(1-4/(2+\varepsilon)^2)^{-1}$, the exponent of n and $\ln q$ is smaller than -1 and hence

$$\sum_{n=1}^{\infty}\mathbb{P}(A_n)<\infty.$$

Indeed, it is easy to verify that

$$\begin{aligned}\frac{(1-2\tilde{\varepsilon})q\gamma(v)^2}{(q-1)(v-u)^\alpha} &= \frac{(1-2\tilde{\varepsilon})v^\alpha}{(v-u)^\alpha}\Big[\Big(1-\frac{\varepsilon}{4}\Big)-\frac{2}{\sqrt{q-1}}-(1-\varepsilon)\sqrt{\frac{q}{q-1}}\Big]^2\\ &> \frac{\varepsilon^4(1-2\tilde{\varepsilon})}{4}\Big(\frac{v}{v-u}\Big)^\alpha=1.\end{aligned}$$

The last equality follows from our special choice of u,v in (6). Thus we may deduce from the Lemma of Borel-Cantelli and continuity, that

$$\mathbb{P}\Big\{\bigcup_{s\in[u,v]\cap\mathbb{Q}}A_n(s),\ \text{infinitely often}\Big\}\le\mathbb{P}\{A_n,\ \text{infinitely often}\}=0.$$

Taking the complement yields that for almost every ω there is an integer $n_1(\omega)$, such that for all $n\ge n_1(\omega)$ and all $s\in[u,v]$,

$$\square X\left((s,q^{n-1}),(v,q^n)\right][\omega]<\gamma(s)\sqrt{2q^n\ln\ln q^n}. \tag{7}$$

In addition to this inequality we need two other inequalities. Recall that the stochastic process $\{v^{-\alpha/2}X(v,t);\, t \geq 0\}$ is a Brownian motion. From [5], p560, 19b follows that for every $q > 1$,

$$v^{-\alpha/2}\Big(X(v,q^n) - X(v,q^{n-1})\Big) > \Big(1 - \frac{\varepsilon}{4}\Big)\sqrt{\frac{q-1}{q}}\sqrt{2q^n \ln\ln q^n} \tag{8}$$

n-infinitely often, with probability one. Moreover, from (2), together with the symmetry of the law of X, we get that for every $q > 1$,

$$|v^{-\alpha/2}X(s,q^{n-1})| \leq 2\sqrt{2q^{n-1}\ln\ln q^n} = \frac{2}{\sqrt{q}}\sqrt{2q^n \ln\ln q^n} \tag{9}$$

for every $n \geq n_2(q,\omega)$ and simultaneously for all $a \leq s \leq b$, almost surely. Putting things together, from the three inequalities (7), (8) and (9) follows that for infinitely many $n \geq (n_1(\omega) \vee n_2(q,\omega))$ and for all $s \in [u,v]$,

$$\begin{aligned}
X(s,q^n) &> X(s,q^n) + \Box X\left((s,q^{n-1}),(v,q^n)\right] - \gamma(s)\sqrt{2q^n\ln\ln q^n} \\
&= X(v,q^n) - X(v,q^{n-1}) + X(s,q^{n-1}) - \gamma(s)\sqrt{2q^n \ln\ln q^n} \\
&\geq (1-\varepsilon)s^{\alpha/2}\sqrt{2q^n \ln\ln q^n},
\end{aligned}$$

with probability one. This gives immediately

$$\mathbb{P}\bigg\{\limsup_{t\to\infty}\bigg\{\frac{s^{-\alpha/2}X(s,t)}{\sqrt{2t\ln\ln t}}\bigg\} > (1-\varepsilon) \quad \text{for all} \quad s \in [u,v]\bigg\} = 1.$$

The interval $[a,b]$ can be covered by finitely many closed intervals $[u,v]$ of length $\delta > 0$ and ε can be chosen arbitrarily small, thus the assertion follows. □

Corollary 2.7

$$\mathbb{P}\bigg\{\limsup_{t\to\infty}\bigg\{\frac{|X(s,t)|}{\sqrt{2t\ln\ln t}}\bigg\} = \sqrt{s^\alpha} \quad \text{for all} \quad s \in [0,\infty)\bigg\} = 1.$$

Proof. Proposition 2.6 says that

$$\mathbb{P}\bigg\{\limsup_{t\to\infty}\bigg\{\frac{s^{-\alpha/2}X(s,t)}{\sqrt{2t\ln\ln t}}\bigg\} = 1 \quad \text{for all} \quad s \in [a,b]\bigg\} = 1$$

for any $0 < a \leq b < \infty$. However, the law of X is symmetric and we can rewrite the modulus into $|x| = \max(x,0) + \min(-x,0)$. Thus the assertion follows. □

Although being a result about the behaviour of "regular points", the following Theorem will be our main tool to establish the propagation of singularities along the fractional coordinate.

Theorem 2.8 (Law of the Iterated Logarithm) *For each $t_0 \geq 0$,*

$$\mathbb{P}\bigg\{\limsup_{h\to 0+}\bigg\{\frac{|X(s,t_0+h) - X(s,t_0)|}{\sqrt{2h\ln\ln(1/h)}}\bigg\} = \sqrt{s^\alpha} \quad \text{for all} \quad s \in [0,\infty)\bigg\} = 1.$$

Proof. The law of $\{tX(s,1/t);\,(s,t)\in\mathbb{R}_+\times\mathbb{R}_+\}$ is equal to the law of X. Thus Corollary 2.7 yields

$$\mathbb{P}\Big\{\limsup_{h\to 0+}\Big\{\frac{|X(s,h)|}{\sqrt{2h\ln\ln(1/h)}}\Big\}=\sqrt{s^\alpha}\quad\text{for all}\quad s\in[0,\infty)\Big\}=1,$$

where $h=1/t$. Finally, use that $\hat{X}$ with $\hat{X}(s,t)=X(s,t_0+t)-X(s,t_0)$ is equal in law to X and the proof is done. □

Corollary 2.9 *Let ψ be a continuous increasing function, such that $\psi(0)=0$ and $\lim_{h\to 0+}(2h\ln\ln(1/h))^{1/2}/\psi(h)=0$. Then for each $t\geq 0$,*

$$\mathbb{P}\Big\{\limsup_{h\to 0+}\Big\{\frac{|X(s,t+h)-X(s,t)|}{\psi(h)}\Big\}=0\quad\text{for all}\quad s\in[0,\infty)\Big\}=1.$$

In particular, we may choose $\psi(h)=(2h\ln(1/h))^{1/2}$.

3 Propagation of Singularities

It is a consequence of Theorem 2.8, our LIL for X, that the time at which a singularity appears depends on the choice of ω. However, we may construct, by means of the Section Theorem, a random variable T, which identifies a singularity for almost all given ω.

Definition 3.1 *Let $s,t\in\mathbb{R}_+$ and define the random variables $R(s,t)$ and $RR(s,t)$ by*

$$R(s,t)=\limsup_{h\to 0+}\frac{|X(s,t+h)-X(s,t)|}{\sqrt{2h\ln\ln(1/h)}}$$

and

$$RR(s,t)=\limsup_{h\to 0+}\frac{|X(s,t+h)-X(s,t)|}{\sqrt{2h\ln(1/h)}}.$$

The function RR is similar to R up to a different scaling function in the denominator, involving a single logarithm only, so that $RR(s,t)\leq R(s,t)$ for all $s,t>0$. Now, fix $s_0>0$, set $X_{s_0}=\{X(s_0,t)\,:\,t\geq 0\}$ and denote by $\{\mathcal{F}_{s_0,t}\,:\,t\geq 0\}$ its natural filtration. Define $\mathcal{F}_{s_0,\infty}:=\sigma\{X(s_0,t)\,:\,t\geq 0\}$.

Proposition 3.2 *There exists a positive, finite and $\mathcal{F}_{s_0,\infty}$-measurable random variable T, such that with probability one, $R(s_0,T)=\infty$.*

Proof. By definition, X_{s_0} has almost surely continuous paths and is therefore measurable with respect to $(\Omega\times\mathbb{R}_+,\mathcal{F}_{s_0,\infty}\otimes\mathcal{B}(\mathbb{R}_+))$. In addition, continuity allows us to rewrite $R(s_0,\cdot)$ as

$$R(s_0,t)\;=\;\lim_{h\to 0+}\;\sup_{0<k\leq h,\,k\in\mathbb{Q}_+}\frac{|X(s_0,t+k)-X(s_0,t)|}{\sqrt{2k\ln\ln(1/k)}}$$

which shows that $R(s_0, \cdot)$ is measurable, too, and that

$$Si(\Omega \times \mathbb{R}_+) := \{(\omega, s) \in \Omega \times \mathbb{R}_+ : R(s_0, s)[\omega] = \infty\} \quad \in \quad \mathcal{F}_{s_0,\infty} \otimes \mathcal{B}(\mathbb{R}_+).$$

Thus, by the Section Theorem (Thm 37, p18, [2]), there is a positive $\mathcal{F}_{s_0,\infty}$-measurable random variable T with

a) $\text{graph}[T] = \{(\omega, t) \in \Omega \times \mathbb{R}_+ : T(\omega) = t\} \subset Si(\Omega \times \mathbb{R}_+)$,

b) $\{T < \infty\}$ is equal to the projection of $Si(\Omega \times \mathbb{R}_+)$ onto Ω.

Moreover, the set $Si(\mathbb{R}_+)$ has almost surely Hausdorff dimension one; see [8]. Hence the projection of $Si(\Omega \times \mathbb{R}_+)$ on Ω is equal to Ω up to a set of measure zero and T is almost surely finite. □

In fact, for the proof it is enough to know that $Si(\mathbb{R}_+)$ is non-empty almost surely, but, remarkably, it is even rather large. We now give a refinement of this result.

Corollary 3.3 *There exists a positive, finite and $\mathcal{F}_{s_0,\infty}$-measurable random variable TR, such that with probability one, $RR(s_0, TR) \in (0, s_0^{\alpha/2}]$.*

Proof. Orey and Taylor [8] proved the existence of such *fast points* and the corresponding random time TR can be constructed similar to the above. □

After having established the *existence* of singularities and suitably measurable random variables T, TR that *find* such singularities, we now turn our attention to the *propagation* of singularities in the fractional direction. In the sequel we need to consider the shift of our Gaussian random field X by such random times in the t-direction. Here is the necessary notation.

Definition 3.4 *Let τ be a positive and finite random variable and let U be a random field on $\mathbb{R}_+^2$. Then the* shifted random field *U^τ is defined by*

$$U^\tau(s, t) = U(s, \tau + t) - U(s, \tau).$$

The next aim is to present a Gaussian random field which is invariant under a given random shift. To this end, we need to collect some properties of the "fractional part" of the covariance function of X.

Definition 3.5 *For $\alpha \in (0, 2)$, let the function a_α on $(0, \infty) \times [0, \infty)$ be given by*

$$a_\alpha(s_0, s) = (s_0^\alpha + s^\alpha - |s_0 - s|^\alpha)/(2s_0^\alpha).$$

Proposition 3.6 *Let $\alpha \in (0, 2)$ and $s_0 > 0$.*

i) The function $a_\alpha(s_0,\cdot)$ *is non-negative and continuous on* $[0,\infty)$ *with* $a_\alpha(s_0,0)=0$ *and* $a_\alpha(s_0,s_0)=1$. *Furthermore, for* $\alpha\in[1,2)$,

$$\frac{\alpha}{2s_0}\le\frac{\partial}{\partial s}a_\alpha(s_0,s)<\infty \quad on \quad [0,s_0), \qquad and \qquad 0\le\frac{\partial}{\partial s}a_\alpha(s_0,s)<\frac{\alpha}{2s_0} \quad on \quad (s_0,\infty).$$

In particular, $a_\alpha(s_0,\cdot)$ *is monotone increasing on* $[0,\infty)$. *For* $\alpha=1$, *we have that*

$$\frac{\partial}{\partial s}a_1(s_0,s)=\frac{1}{s_0} \quad on \quad [0,s_0), \qquad and \qquad \frac{\partial}{\partial s}a_1(s_0,s)=0 \quad on \quad (s_0,\infty).$$

In particular, $a_1(s_0,\cdot)$ *is monotone increasing on* $[0,s_0)$ *and constant on* (s_0,∞). *Finally, for* $\alpha\in(0,1)$, *we have that*

$$\frac{\alpha}{2s_0}\le\frac{\partial}{\partial s}a_\alpha(s_0,s)<\infty \quad on \quad [0,s_0), \qquad and \qquad -\infty<\frac{\partial}{\partial s}a_\alpha(s_0,s)<0 \quad on \quad (s_0,\infty).$$

In particular, $a_\alpha(s_0,\cdot)$ *is monotone increasing on* $[0,s_0)$ *and monotone decreasing on* (s_0,∞).

ii) There exist $s_1,\ s_2\in(0,\infty)$ *with* $s_1<s_0<s_2/2$, *such that for* $s\in(s_1,s_2/2)$,

$$a_\alpha(s,s_1)a_\alpha(s_0,s_2)=a_\alpha(s,s_2)a_\alpha(s_0,s_1) \tag{10}$$

if and only if $s=s_0$.

Proof. Recall that, for $0\le r<\hat r<\infty$, one has $(\hat r-r)^\beta\le\hat r^\beta-r^\beta$ for $\beta\in[1,2)$, and $(\hat r-r)^\beta\ge\hat r^\beta-r^\beta$ for $\beta\in(-\infty,1)$.

i) For the partial derivative of $a_\alpha(s_0,s)$ with respect to s, we have

$$\frac{\partial}{\partial s}a_\alpha(s_0,s)=\begin{cases}\frac{\alpha(s^{\alpha-1}+(s_0-s)^{\alpha-1})}{2s_0^\alpha} & \text{for} \quad s<s_0,\\ \frac{\alpha(s^{\alpha-1}-(s-s_0)^{\alpha-1})}{2s_0^\alpha} & \text{for} \quad s_0<s.\end{cases}$$

The result follows easily from the second inequality above.

ii) Fix some $s_1,\ s_2\in(0,\infty)$ with $s_1<s_0<s_2/2$. If $s=s_0$ then (10) is clearly satisfied. Define a differentiable function $f:(s_1,s_2/2)\to\mathbb{R}_+$ by

$$f(s)=\frac{\left(\frac{s}{s_1}\right)^\alpha+1-\left(\frac{s}{s_1}-1\right)^\alpha}{\left(\frac{s_0}{s_1}\right)^\alpha+1-\left(\frac{s_0}{s_1}-1\right)^\alpha}-\frac{\left(\frac{s}{s_2}\right)^\alpha+1-\left(1-\frac{s}{s_2}\right)^\alpha}{\left(\frac{s_0}{s_2}\right)^\alpha+1-\left(1-\frac{s_0}{s_2}\right)^\alpha}.$$

and observe that Equation (10) is equivalent to $f(s)=0$. Our goal is to show that f is strictly decreasing for $s\in(s_1,s_2/2)$ if s_1 is sufficiently small and s_2 is sufficiently large. This is an easy, but somewhat tedious exercise. Note that the first derivative of f is

$$f'(s)=\frac{\alpha\left(\left(\frac{s}{s_1}\right)^{\alpha-1}-\left(\frac{s}{s_1}-1\right)^{\alpha-1}\right)}{s_1\left(\left(\frac{s_0}{s_1}\right)^\alpha+1-\left(\frac{s_0}{s_1}-1\right)^\alpha\right)}-\frac{\alpha\left(\left(\frac{s}{s_2}\right)^{\alpha-1}+\left(1-\frac{s}{s_2}\right)^{\alpha-1}\right)}{s_2\left(\left(\frac{s_0}{s_2}\right)^\alpha+1-\left(1-\frac{s_0}{s_2}\right)^\alpha\right)}. \tag{11}$$

Now, for $\alpha \in (1,2)$ and $s_1 < s < s_2/2$, we have, from the inequalities at the beginning of the proof,

$$f'(s) \leq \frac{\alpha}{s_1\left(\left(\frac{s_0}{s_1}\right)^\alpha + 1 - \left(\frac{s_0}{s_1} - 1\right)^\alpha\right)} - \frac{\alpha}{s_2\left(\left(\frac{s_0}{s_2}\right)^\alpha + 1 - \left(1 - \frac{s_0}{s_2}\right)^\alpha\right)},$$

Hence L'Hôpital's rule implies the assertion for $\alpha \in (1,2)$. The case $\alpha = 1$ is readily checked. In the remaining case $\alpha \in (0,1)$, observe that the first numerator on the right-hand side in (11) is negative, whereas the denominators and the numerator in the second term are all positive. □

Definition 3.7 *Let, for $(s,t) \in \mathbb{R}_+ \times \mathbb{R}_+$, the random field Y_{s_0} be defined by*

$$Y_{s_0}(s,t) = X(s,t) - a_\alpha(s_0,s)X(s_0,t).$$

It is straightforward that Y_{s_0} is a continuous and centred Gaussian random field. Moreover, it is independent of the sigma field $\mathcal{F}_{s_0,\infty}$ introduced above, since, for $t \in \mathbb{R}_+$,

$$\mathbb{E}\big[X(s_0,t)Y_{s_0}(s,\hat{t})\big] = (t \wedge \hat{t})\big((s_0^\alpha + s^\alpha - |s_0 - s|^\alpha)/2 - a_\alpha(s_0,s)s_0^\alpha\big) = 0,$$

and we can decompose X into

$$\begin{aligned} X(s,t) &= a_\alpha(s_0,s)X(s_0,t) + \big(X(s,t) - a_\alpha(s_0,s)X(s_0,t)\big) \\ &= \mathbb{E}\big[X(s,t)|\mathcal{F}_{s_0,\infty}\big] + Y_{s_0}(s,t). \end{aligned}$$

Observe that if we let τ be a positive, finite and $\mathcal{F}_{s_0,\infty}$-measurable random variable, then we have for the shifted random field $Y_{s_0}^\tau$, defined in the sense of Definition 3.4, that

$$\text{law}(Y_{s_0}^\tau) = \text{law}(Y_{s_0}),$$

since τ is independent of Y_{s_0}. Now we are ready to present the main result of this section.

Theorem 3.8 *Let $s_0 > 0$ and let τ be a positive, finite and $\mathcal{F}_{s_0,\infty}$-measurable random variable. Then with probability one, we have*

(A) *for all $s \geq 0$,*

$$a_\alpha(s_0,s)R(s_0,\tau) - \sqrt{s^\alpha} - a_\alpha(s_0,s)\sqrt{s_0^\alpha} \;\leq\; R(s,\tau) \;\leq\; a_\alpha(s_0,s)R(s_0,\tau) + \sqrt{s^\alpha} + a_\alpha(s_0,s)\sqrt{s_0^\alpha},$$

(B) *for all $s \geq 0$,*

$$a_\alpha(s_0,s)RR(s_0,\tau) = RR(s,\tau).$$

In the proof and in the sequel we need the following property of the lim sup: suppose that $\limsup |g| < \infty$, then

$$\limsup |f| - \limsup |g| \leq \limsup |f+g| \leq \limsup |f| + \limsup |g|. \tag{12}$$

Proof of Theorem 3.8. Property (12) and Theorem 2.8 imply that with probability one

$$\limsup_{h\to 0+}\left\{\frac{|Y_{s_0}(s,t+h)-Y_{s_0}(s,t)|}{\sqrt{2h\ln\ln(1/h)}}\right\} \le \sqrt{s^\alpha}+a_\alpha(s_0,s)\sqrt{s_0^\alpha}<\infty \tag{13}$$

simultaneously for all $s\ge 0$. Moreover, the random fields $Y_{s_0}^\tau$ and Y_{s_0} are equal in law, thus (13) applies to $Y_{s_0}^\tau$, too. We rewrite X to

$$X(s,\tau+h)-X(s,\tau)=a(s_0,s)\big(X(s_0,\tau+h)-X(s_0,\tau)\big)+Y_{s_0}^\tau(s,h),$$

use property (12) again and deduce part (A). The proof of part (B) is similar and uses Corollary 2.9 instead of Theorem 2.8. □

Remark 3.9 (Propagation of singularities) From Proposition 3.2 we know that there exists an almost surely finite, positive and $\mathcal{F}_{s_0,\infty}$-measurable random variable T with $\mathbb{P}\{R(s_0,T)=\infty\}=1$. Thus there are singularities and Theorem 3.8 applied with $\tau=T$ shows that they propagate along the s-coordinate.

Remark 3.10 (Propagation and speed of fast points) From Corollary 3.3 we know that there is an almost surely finite, positive and $\mathcal{F}_{s_0,\infty}$-measurable random variable TR with $\mathbb{P}\big\{RR(s_0,TR)\in(0,\sqrt{s_0^\alpha}]\big\}=1$. Thus we can find by TR a λ-fast point for $X(s_0,\cdot)$, and this point is then an $a_\alpha(s_0,s)\lambda$-fast point for $X(s,\cdot)$. Recall from Proposition 3.6 that $a_\alpha(s_0,\cdot)$ behaves qualitatively different in each of the cases $\alpha\in(0,1)$, $\alpha=1$ and $\alpha=(1,2)$. For example, for $s>s_0$, the positive correlation in the case $\alpha\in(1,2)$ seems to *"speed up"* the singularity, whereas in the case $\alpha\in(0,1)$, the singularity seems to *"slow down"*, compared to the classical case $\alpha=1$. See Figure 1.

Is it possible to find the origin of a propagating singularity? Assume τ to be an almost surely finite and positive random variable which is measurable with respect to a sigma field of the form $\mathcal{F}_{s_0,\infty}$ for some (unique) $s_0>0$, but s_0 is not given explicitly. If we have $\mathbb{P}\{R(s,\tau)=\infty \text{ for all } s>0\}=1$, can we find the parameter s_0?

Definition 3.11 *For* $0<s_1<s_2$ *and* $s,t\ge 0$, *let*

$$N(s_1,s_2,s,t)=\limsup_{h\to 0+}\frac{|a(s,s_1)(X(s_2,t+h)-X(s_2,t))-a(s,s_2)(X(s_1,t+h)-X(s_1,t))|}{\sqrt{2h\ln\ln(1/h)}}$$

Proposition 3.12 *Let* $s_0>0$ *and let* τ *be a positive, finite and* $\mathcal{F}_{s_0,\infty}$*-measurable random variable. Then with probability one, for all* $0<s_1<s_2$ *and* $s\ge 0$ *holds*

$$\begin{aligned}
&\big(a_\alpha(s,s_1)a_\alpha(s_0,s_2)-a_\alpha(s,s_2)a_\alpha(s_0,s_1)\big)R(s_0,\tau)\\
&\qquad -a_\alpha(s,s_1)\big(\sqrt{s_2^\alpha}+a_\alpha(s_0,s_2)\sqrt{s_0^\alpha}\big)-a_\alpha(s,s_2)\big(\sqrt{s_1^\alpha}+a_\alpha(s_0,s_1)\sqrt{s_0^\alpha}\big)\\
&\le N(s_1,s_2,s,\tau)\\
&\le \big(a_\alpha(s,s_1)a_\alpha(s_0,s_2)-a_\alpha(s,s_2)a_\alpha(s_0,s_1)\big)R(s_0,\tau)\\
&\qquad +a_\alpha(s,s_1)\big(\sqrt{s_2^\alpha}+a_\alpha(s_0,s_2)\sqrt{s_0^\alpha}\big)+a_\alpha(s,s_2)\big(\sqrt{s_1^\alpha}+a_\alpha(s_0,s_1)\sqrt{s_0^\alpha}\big).
\end{aligned}$$

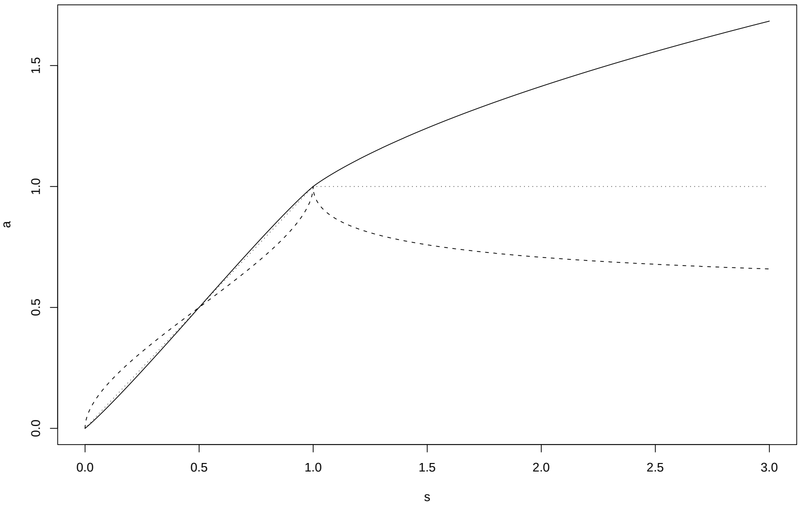

Figure 1: *Values of* a_α *for* $s \in [0,3]$, $s_0 = 1$ *and* $\alpha \in \{1/2, 1, 3/2\}$. *Solid lines:* $\alpha = 1$, *dashed lines:* $\alpha = 1/2$, *dotted lines:* $\alpha = 3/2$.

Remark 3.13 (Origin of a singularity) Suppose we are given a positive and finite random variable τ with the properties that it is measurable with respect to a sigma field of the form $\mathcal{F}_{s_0,\infty}$ for some $s_0 > 0$, which may not be given explicitly, and

$$\mathbb{P}\{R(s,\tau) = \infty \text{ for all } s > 0\} = 1.$$

Then, (10) says that for s_1 sufficiently small and s_2 sufficiently large

$$a_\alpha(s, s_1)a_\alpha(s_0, s_2) = a_\alpha(s, s_2)a_\alpha(s_0, s_1) \quad \text{if and only if s} = \text{s}_0.$$

Thus from Proposition 3.12 we get that $N(s_1, s_2, s, \tau)$ is infinite unless $s = s_0$ and we found s_0.

Proof of Proposition 3.12. Similar as in the proof of part (A) of Theorem 3.8 we obtain that with probability one for all $0 < s_1 < s_2$ and $s \geq 0$

$$\limsup_{h\to 0+} \left\{ \frac{|a_\alpha(s, s_1)Y^\tau_{s_0}(s_2, h)|}{\sqrt{2h \ln\ln(1/h)}} \right\} \leq a_\alpha(s, s_1)\big(\sqrt{s_2^\alpha} + a_\alpha(s_0, s_2)\sqrt{s_0^\alpha}\big),$$

$$\limsup_{h\to 0+} \left\{ \frac{|a_\alpha(s, s_2)Y^\tau_{s_0}(s_1, h)|}{\sqrt{2h \ln\ln(1/h)}} \right\} \leq a_\alpha(s, s_2)\big(\sqrt{s_1^\alpha} + a_\alpha(s_0, s_1)\sqrt{s_0^\alpha}\big).$$

Moreover, we have the decomposition

$$\begin{aligned} &a_\alpha(s, s_1)\big(X(s_2, \tau + h) - X(s_2, \tau)\big) - a_\alpha(s, s_2)\big(X(s_1, \tau + h) - X(s_1, \tau)\big) \\ &\quad = \big(a_\alpha(s, s_1)a_\alpha(s_0, s_2) - a_\alpha(s, s_2)a_\alpha(s_0, s_1)\big)\big(X(s_0, \tau + h) - X(s_0, \tau)\big) \\ &\qquad + a_\alpha(s, s_1)Y^\tau_{s_0}(s_2, h) - a_\alpha(s, s_2)Y^\tau_{s_0}(s_1, h). \end{aligned}$$

This gives with property (12) of the lim sup the assertion. □

Remark 3.14 Define the random set of singularities by

$$\mathcal{T} = \{t \geq 0\,;\, R(s,t) = \infty \quad \text{for all} \quad s > 0\}$$

and for each $s > 0$ define the random set

$$\mathcal{T}(s) = \big\{t \in \mathcal{T}\,;\, \inf\{\sigma > 0\,;\, \liminf_{n\to\infty} N(1/n, n, \sigma, t) < \infty\} = s\big\}.$$

The sets $\mathcal{T}(s)$ are disjoint and $\mathcal{T} = \bigcup_{s>0} \mathcal{T}(s)$. Let T_{s_0} be a positive, finite and $\mathcal{F}_{s_0,\infty}$-measurable random variable with $\mathbb{P}\{R(s_0, T_{s_0}) = \infty\} = 1$. From the considerations above follows

$$\liminf_{n\to\infty} N(1/n, n, \sigma, T_{s_0}) < \infty \Leftrightarrow \sigma = s_0$$

and $T_{s_0} \in \mathcal{T}(s_0)$ almost surely. This fact is true for each T_{s_0} with $s_0 > 0$. If we believe that T_{s_0} was chosen 'at random' and there are many more random variables with the same properties measurable for $\mathcal{F}_{s_0,\infty}$, then we can think of $\mathcal{T}$ as an uncountable union of large, maybe uncountable sets.

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