Technische Universität Berlin
Institut für Mathematik

Linear Discrete-Time Descriptor Systems

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Preprint 30/2007

Preprint-Reihe des Instituts für Mathematik
Technische Universität Berlin

Report 30/2007 **January 2008**

Linear Discrete-Time Descriptor Systems

Diplomarbeit von

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Berlin, den 21. August 2007

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Chapter 1

Introduction

The purpose of this diploma thesis is to transfer the concepts and results presented in [8] into the discrete-time case. In [8] continuous-time differential-algebraic equations of the form

$$\begin{gathered} F(t, x, \dot{x}) = 0, \\ \text{with } F : \mathbb{I} \times \mathbb{D}_x \times \mathbb{D}_{\dot{x}} \to \mathbb{C}^m, \\ \text{where } \mathbb{I} \subset \mathbb{R} \text{ and } \mathbb{D}_x, \mathbb{D}_{\dot{x}} \subset \mathbb{C}^n \quad , \end{gathered} \tag{1.1}$$

are examined. Given such an equation one is looking for a differentiable function $x : \mathbb{I} \to \mathbb{C}^n$ which solves (1.1) in the sense that $F(t, x(t), \dot{x}(t)) = 0$ for all $t \in \mathbb{I}$, where $\dot{x}$ denotes the derivative of x. The analogous general form of a discrete-time descriptor system is

$$\begin{gathered} F(k, x^k, x^{k+1}) = 0, \\ \text{with } F : \mathbb{K} \times \mathbb{D}_{x^k} \times \mathbb{D}_{x^{k+1}} \to \mathbb{C}^m, \\ \text{where } \mathbb{K} := \{k \in \mathbb{Z} : k_b \leq k \leq k_f\},\ k_b \in \mathbb{Z} \cup \{-\infty\},\ k_f \in \mathbb{Z} \cup \{\infty\} \text{ and} \\ \mathbb{D}_{x^k}, \mathbb{D}_{x^{k+1}} \subset \mathbb{C}^n. \end{gathered} \tag{1.2}$$

Such systems could also be called discrete-time systems of differential-algebraic equations, since systems of the form (1.2) naturally arise by discretizing systems of the form (1.1) through a difference quotient, e.g., $\dot{x}(t_i) \approx \frac{x(t_{i+1}) - x(t_i)}{t_{i+1} - t_i}$. Other more common names are discrete-time singular systems (e.g., [16]), discrete-time semi-state systems and discrete-time generalized state-space systems. One is looking for a sequence $\{y^k\}$ which solves (1.2) in the sense that $F(k, y^k, y^{k+1}) = 0$ for all $k \in \mathbb{K}$. Such a sequence is called a solution of (1.2).
Although not all concepts in [8] are reasonable in the discrete-time case (e.g., generalized solutions) there are still many concepts which can be carried over into the discrete-time case. To study all these concepts is a major task. For this reason, we only consider some of the concepts in [8] for linear discrete-time systems, despite the importance of non-linear discrete-time descriptor systems of the form (1.2).
We distinguish two types of linear discrete-time descriptor systems, namely systems with constant and systems with variable coefficients. To introduce these two types we first define the discrete interval

$$\mathbb{K} := \{k \in \mathbb{Z} : k_b \leq k \leq k_f\},\ k_b \in \mathbb{Z} \cup \{-\infty\},\ k_f \in \mathbb{Z} \cup \{\infty\}. \tag{1.3}$$

With this definition we call

$$Ex^{k+1} = Ax^k + f^k,\ k \in \mathbb{K}, \tag{1.4}$$
$$\text{where } E, A \in \mathbb{C}^{m,n}, f^k \in \mathbb{C}^m,$$

a *linear discrete-time descriptor system with constant coefficients*. Such systems represent a special case of the second type

$$E_k x^{k+1} = A_k x^k + f^k,\ k \in \mathbb{K}, \tag{1.5}$$
$$\text{where } E_k, A_k \in \mathbb{C}^{m,n}, f^k \in \mathbb{C}^m,$$

which is called a *linear discrete-time descriptor system with variable coefficients*. Together with one of the systems (1.4) or (1.5) we also often require a solution to satisfy an initial condition

$$x^{k_0} = \hat{x}, \text{ where } k_0 \in \mathbb{K}. \tag{1.6}$$

Other notation is listed in the following table.

$\operatorname{diag}(A_1, \ldots, A_k)$	Denotes the block diagonal matrix with $A_1, \ldots, A_k$ on the block diagonal.
$\mathbb{N}$	$\{1, 2, 3, \ldots\}$
$\mathbb{N}_0$	$\mathbb{N} \cup \{0\} = \{0, 1, 2, 3, \ldots\}$
$\mathbb{R}^+$	$\mathbb{R} \cap]0; \infty] =]0; \infty]$
Z is a basis of the vector space $\mathfrak{V}$	When talking of a matrix Z being a basis of some vector space $\mathfrak{V} \subset \mathbb{R}^l$ then this means that the columns of Z form a basis of $\mathfrak{V}$.
x_i^k	For $x^k \in \mathbb{C}^n$, x_i^k means the i-th component of x^k.
$\{A_k\}_{k=k_b}^{k_f}$	A sequence of $A_k \in \mathbb{C}^{m,n}$ for $k = k_b, \ldots, k_f$ where $k_b, k_f \in \mathbb{Z}$.
$\{A_k\}_{k \geq k_b}$	A sequence of $A_k \in \mathbb{C}^{m,n}$ for $k \geq k_b \in \mathbb{Z}$.
$\{A_k\}_{k \leq k_f}$	A sequence of $A_k \in \mathbb{C}^{m,n}$ for $k \leq k_f \in \mathbb{Z}$.
$\{A_k\}_{k \in \mathbb{K}}$	For a subset $\mathbb{K} \subset \mathbb{Z}$, this expression means the sequence of the $A_k \in \mathbb{C}^{m,n}$ for $k \in \mathbb{K}$.

Table 1.1: Notation used in this text

1.1 Applications

In this section we will discuss some applications where linear discrete-time descriptor systems are used.

1.1.1 Discretization of linear differential algebraic equations

Consider the general linear continuous-time differential algebraic equation

$$E(t)\dot{x}(t) = A(t)x(t) + f(t),\ t \in [t_0, t_f]. \tag{1.7}$$

We define a grid $t_0 < t_1 < \ldots < t_N = t_f$ and introduce $x^i := x(t_i)$, $f^i := f(t_i)$, $E_i := \frac{1}{t_{i+1}-t_i}E(t_i)$ and $A_i := \frac{1}{t_{i+1}-t_i}E(t_i) + A(t_i)$. Approximating

$$\dot{x}(t_i) \approx \frac{x^{i+1} - x^i}{t_{i+1} - t_i},$$

in (1.7) yields

$$E(t_i)\frac{x^{i+1} - x^i}{t_{i+1} - t_i} = A(t_i)x(t_i) + f(t_i), \text{ for all } i \in \{0, \ldots, N-1\}.$$

This system is equivalent to

$$E_i x^{i+1} = A_i x^i + f^i, \text{ for all } i \in \{0, \ldots, N-1\},$$

which is a linear discrete-time descriptor system with variable coefficients.

1.1.2 Singular Leontief Systems

Leontief systems [3, 14] have the form

$$\begin{aligned} x^k &= Ax^k + \underbrace{B}_{:=\tilde{E}}(x^{k+1} - x^k) + \underbrace{d^k}_{:=-f^k} \\ \Leftrightarrow \tilde{E}x^{k+1} &= \underbrace{(I - A + B)}_{:=\tilde{A}}x^k + f^k \\ \Leftrightarrow \tilde{E}x^{k+1} &= \tilde{A}x^k + f^k \end{aligned} \tag{1.8}$$

where

$$A,\ B \in \mathbb{R}^{n,n},\ x^k,\ d^k \in \mathbb{R}^n, n \in \mathbb{N}.$$

Such systems describe the production of an economy with n distinct sectors. A widely used example is $n = 3$ with the sectors agriculture, manufacturing, and service, i.e., the primary, secondary, and tertiary sector of industry. Here x_i^k is the (monetary) output of the sector i in the time period k, whereas d_i^k is the (monetary) customer demand for products of the sector i in the time period k and is prescribed.
The term Ax^k is there to consider inter-sector relations in the economy, i.e., it takes into account that any sector may need output from all the other sectors to produce its output. For example, it seems reasonable that any produced good of the secondary sector requires

some service (like telecommunications) from the tertiary sector. In this case the entry $a_{3,2}$ of the matrix A would have to be positive. In this way the (also often considered) equation

$$x^k = Ax^k + d^k$$

arises, which describes the output of the sectors in dependence of the customer demand. This is a linear equation and the matrix A is called *consumption matrix*.
Finally, the term $B(x^{k+1} - x^k)$ may describe investments. When the output level in sector i increases from period k to $k+1$ there may be investments necessary to accomplish this increase. For example, a growth in the primary sector demands for an increase of the agricultural machinery. So when the primary sector grows from time period k to $k+1$, it is necessary to produce this machinery a priori in period k. In this case the entry $b_{2,1}$ of the matrix B would have to be positive. In this way equation (1.8) arises. This is a linear discrete-time descriptor system. The matrix B is called the *capital coefficients matrix*.
The matrix B (and thus $\tilde{E}$) may be singular, as stated in [3, 14], since some rows of B may only contain zero elements. For example, a growth in any of the three sectors of industry does not really demand for an increase in the production of the primary sector (i.e., of food). At least the increase is (monetarily) inferior which means that the first row of B is zero.

1.1.3 Backward Leslie Model

A Leslie Model [2] has the form

$$x^{k+1} = Tx^k,$$

where

$$T \in \mathbb{R}^{n,n},\ x^k \in \mathbb{R}^n,\ n \in \mathbb{N}.$$

The model describes the age distribution of a given population in time. The population is divided into n distinct age classes. Here x_i^k is the number of individuals in age class i in time period k. Considering the individual birth and death rates of the age classes, the matrix T is constructed. As shown in [2] this matrix may be singular. If one wants to determine an age distribution in the past, given a present age distribution, one has to solve the Leslie Model backwards, i.e., one has to solve a system of the form

$$Tx^{l+1} = x^l.$$

Obviously the same situation occurs every time one wants to determine a state in the past given a present state if the behavior of the system is given by a difference equation with singular T.

1.1.4 A self-excogitated example: The Bullwhip Effect

Consider a chain of four warehouses where each warehouse may send and receive goods to and from the preceding warehouse. The last warehouse accepts its orders from a customer. The first warehouse issues orders to a manufacturer and may also send goods back to the

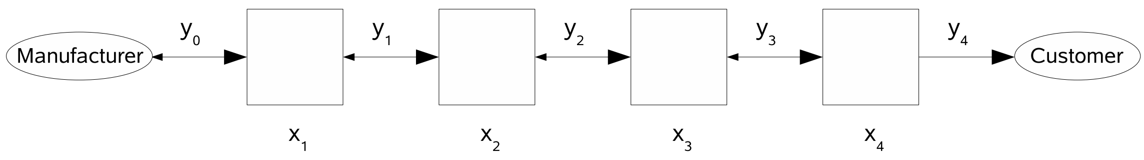

Figure 1.1: Sketch of the supply chain

manufacturer. Let x_i^k, $i \in \{1,2,3,4\}$ be the stock in warehouse i in time period k. Further, let y_i^k, $i \in \{0,1,2,3\}$ be the amount of goods moved from warehouse i to warehouse $i+1$ (where warehouse 0 is the manufacturer) after the time period k (and thus before time period $k+1$). A negative value of y_i^k means that goods were actually sent back from warehouse $i+1$ to warehouse i. Finally, let y_4^k be the demand of the customer which is satisfied after time period k (and thus before time period $k+1$). Assume that this demand is given externally by a (non-negative) sequence $f^k \in \mathbb{R}$. From this setup one obviously gets the five equations

$$\begin{aligned} x_i^{k+1} &= x_i^k + y_{i-1}^k - y_i^k \text{ for } i \in \{1,2,3,4\}, \\ y_4^k &= f^k. \end{aligned}$$

Since there are nine variables to be described there are four equations missing. Therefore suppose that the manager of every warehouse i in every time period k tries to have a stock in his warehouse that is equal to the amount of goods moved from warehouse i to warehouse $i+1$ ($=y_i^k$) after the time period k (and thus before time period $k+1$) plus a safety stock, which he chooses proportional to y_i^k. This assumption then leads to the four additional equations

$$x_i^k = a y_i^k \text{ for } i \in \{1,2,3,4\},$$

with $a > 1$.
Using the state vector $x^k = \begin{bmatrix} y_0^k \; x_1^k \; y_1^k \; x_2^k \; y_2^k \; x_3^k \; y_3^k \; x_4^k \; y_4^k \end{bmatrix}^T$ and the matrices

$$E := \begin{bmatrix} 0 &&&&&&&& \\ & 1 &&&&&&& \\ && 0 &&&&&& \\ &&& 1 &&&&& \\ &&&& 0 &&&& \\ &&&&& 1 &&& \\ &&&&&& 0 && \\ &&&&&&& 1 & \\ &&&&&&&& 0 \end{bmatrix} \quad A := \begin{bmatrix} 0 & -1 & a & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & a & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & a & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & a \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix},$$

one can write the system as

$$Ex^{k+1} = Ax^k + \begin{bmatrix} 0 \; 0 \; 0 \; 0 \; 0 \; 0 \; 0 \; 0 \; f^k \end{bmatrix}^T.$$

Since the matrix A is regular (which can be seen by exchanging adjacent rows), the pencil (E, A) is regular and one can compute the unique solution, given a sequence of f^k. In Figure

1.2 the sequence f^k and the corresponding x_i^k for $i \in \{1, 2, 3, 4\}$ of such a solution are plotted. One sees that the further the warehouses are away from the customer the bigger the stock fluctuations are. This is called the "Bullwhip Effect" by economists.
Note that this example makes the assumption of complete information. (It has a high index (I calculated an index of 5) and only infinite eigenvalues. Also it is possible for stocks to become negative. All this shows that the example is of no practical use.)

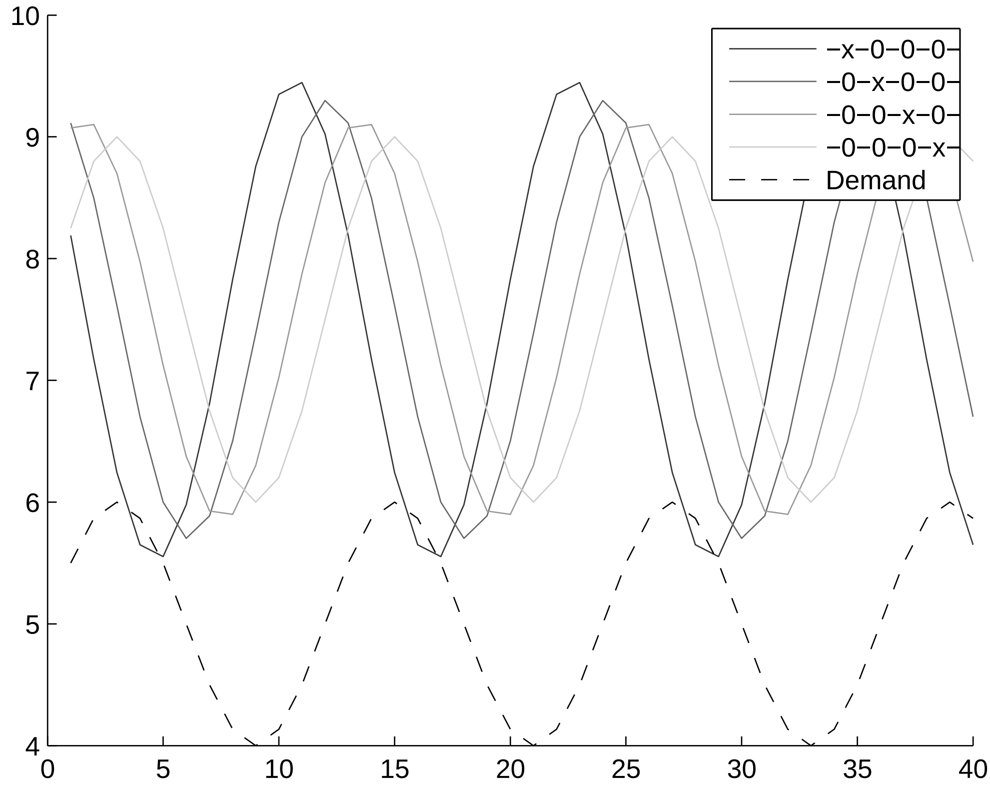

Figure 1.2: Stock development and demand

Chapter 2

Linear discrete-time descriptor systems with constant coefficients

It would be ideal to develop a theory for the most general type of discrete-time descriptor systems (1.2). Nevertheless, such a theory is unlikely to exist. Thus, restrictions have to be imposed on (1.2) in order to get a nice theory. Obviously, the theory gets nicer the more restrictions one imposes on the problem.
It seems reasonable to start by imposing very strong restrictions on (1.2) and then loosen the restrictions as one proceeds. The advantage in doing so is that one may first derive some basic results, which than can be generalized.
Here we start with the very simple case of linear discrete-time descriptor systems with constant coefficients (i.e., systems of the type (1.4)), analogous to [8].

2.1 Solution with the Kronecker canonical form

We begin by recapitulating some basic results of linear algebra, that are needed afterwards. First let us review the Kronecker canonical form.

Theorem 2.1. *[8] Let $E, A \in \mathbb{C}^{m,n}$. Then there exist nonsingular matrices $P \in \mathbb{C}^{m,m}$ and $Q \in \mathbb{C}^{n,n}$ such that for all $\lambda \in \mathbb{C}$*

$$P\left(\lambda E - A\right) Q = \operatorname{diag}\left(\mathfrak{L}_{\epsilon_1}, \dots, \mathfrak{L}_{\epsilon_p}, \mathfrak{M}_{\eta_1}, \dots, \mathfrak{M}_{\eta_q}, \mathfrak{J}_{\rho_1}, \dots, \mathfrak{J}_{\rho_r}, \mathfrak{N}_{\sigma_1}, \dots, \mathfrak{N}_{\sigma_s}\right), \tag{2.1}$$

where the diagonal blocks have the following forms:

1. *Every $\mathfrak{L}_{\epsilon_i}$ block is of size $\epsilon_i \times (\epsilon_i + 1), \epsilon_i \in \mathbb{N}_0$ and has the form*

$$\mathfrak{L}_{\epsilon_j} = \lambda \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix}. \tag{2.2}$$

2. *Every $\mathfrak{M}_{\eta_j}$ block is of size $(\eta_j + 1) \times \eta_j, \eta_j \in \mathbb{N}_0$ and has the form*

$$\mathfrak{M}_{\eta_j} = \lambda \begin{bmatrix} 1 & & \\ 0 & \ddots & \\ & \ddots & 1 \\ & & 0 \end{bmatrix} - \begin{bmatrix} 0 & & \\ 1 & \ddots & \\ & \ddots & 0 \\ & & 1 \end{bmatrix}. \tag{2.3}$$

3. *Every $\mathfrak{J}_{\rho_k}$ block is of size $\rho_k \times \rho_k, \rho_k \in \mathbb{N}$ and has the form*

$$\mathfrak{J}_{\rho_k} = \lambda \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} - \begin{bmatrix} \lambda_k & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_k \end{bmatrix}. \tag{2.4}$$

4. *Every $\mathfrak{N}_{\sigma_l}$ block is of size $\sigma_l \times \sigma_l, \sigma_l \in \mathbb{N}$ and has the form*

$$\mathfrak{N}_{\sigma_l} = \lambda \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix} - \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}. \tag{2.5}$$

The Kronecker canonical form is unique up to permutation of the blocks, i.e., the kind, size and number of the blocks are invariant for the matrix pair (E, A).

Definition 2.2. Let $E, A \in \mathbb{C}^{n,n}$. Then the matrix pair (E, A) is called *regular* if and only if $\det \lambda E - A$ does not vanish identically.

Theorem 2.3. *[8] Let $E, A \in \mathbb{C}^{n,n}$ and (E, A) be regular. Then there exist nonsingular matrices $P, Q \in \mathbb{C}^{n,n}$ such that for all $\lambda \in \mathbb{C}$ we have*

$$P\,(\lambda E - A)\,Q = \lambda \begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix} - \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix}, \tag{2.6}$$

where J is a matrix in Jordan canonical form and N is a nilpotent matrix also in Jordan canonical form. Moreover, it is allowed that one or the other block is not present.
Thus, the Kronecker canonical form of a regular matrix pencil only has blocks of type (2.4) and (2.5).

Definition 2.4. [8] Let $E, A \in \mathbb{C}^{n,n}$, let the matrix pair (E, A) be regular and let the Kronecker canonical form of (E, A) be given by (2.6). Then the quantity ν defined by $N^\nu = 0$, $N^{\nu-1} \neq 0$, i.e., by the index of nilpotency of N in (2.6), if the nilpotent block in (2.6) is present and by $\nu = 0$ if it is absent, is called the *index* of the matrix pair (E, A), denoted by $\nu = \operatorname{ind}(E, A)$.

Definition 2.5. Let $E \in \mathbb{C}^{n,n}$. Further, let ν be the index of the matrix pair (E, I). Then ν is also called the *index* of E and denoted by $\operatorname{ind}(E) = \nu$.

Consider an arbitrary matrix pencil $\lambda E - A$ with the Kronecker canonical from (2.1). When we are interested in the solution of the associated discrete-time descriptor system consisting of (1.4) and (1.6) with $k_b = k_0 \in \mathbb{Z}$ we can study the problem in the coordinates of the Kronecker canonical form, i.e., we can look at the equivalent problem

$$\begin{aligned} PEQ\tilde{x}^{k+1} &= PAQ\tilde{x}^k + Pf^k, \text{ for } k \in \mathbb{K}, \\ \tilde{x}^{k_0} &= Q^{-1}\hat{x}, \end{aligned}$$

with $\tilde{x}^k = Q^{-1}x^k$. Since the pencil (PEQ, PAQ) is block diagonal one can compute the solution for each block separately. This is done in the following.

1. Consider a block of type (2.2), i.e., let

$$\lambda E_{\mathfrak{L}} - A_{\mathfrak{L}} = \lambda \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix} \in \mathbb{C}^{\epsilon,\epsilon+1}.$$

   Then the system (1.4) corresponding to this pencil, together with (1.6), i.e.,

$$\begin{cases} E_{\mathfrak{L}}x^{k+1} = A_{\mathfrak{L}}x^k + f^k, \; k \geq k_0 \\ x^{k_0} = \hat{x}, \end{cases} \tag{2.7}$$

   is equivalent to

$$\begin{cases} x_{i+1}^{k+1} = x_i^k + f_i^k, \quad i = 1, \ldots, \epsilon, \\ x^{k_0} = \hat{x}. \end{cases} \tag{2.8}$$

   If we have $\epsilon = 0$ we have no equations but one variable. Thus, in this case every sequence satisfying the initial condition is a solution. Therefore let us assume in the following that $\epsilon > 0$. In this case recursion of (2.8) yields

$$\begin{aligned} x_i^k &= x_{i-1}^{k-1} + f_{i-1}^{k-1} \\ &= x_{i-2}^{k-2} + f_{i-2}^{k-2} + f_{i-1}^{k-1} \\ &= \ldots = \left(\sum_{j=1}^{\min(i-1,k-k_0)} f_{i-j}^{k-j} \right) + x_{i-\min(i-1,k-k_0)}^{k-\min(i-1,k-k_0)} \\ &= \begin{cases} \left(\sum_{j=1}^{i-1} f_{i-j}^{k-j} \right) + x_1^{k-i+1} & \text{, if } k - k_0 > i - 1, \\ \\ \left(\sum_{j=1}^{k-k_0} f_{i-j}^{k-j} \right) + \hat{x}_{i-k+k_0} & \text{, if } k - k_0 \leq i - 1, \end{cases} \end{aligned}$$

   for $k = k_0, \ldots, k_f$. Thus, once the initial condition $\hat{x}$, values for $x_1^{k_0+1}, x_1^{k_0+2}, \ldots$ and the right hand sides $f^{k_0}, f^{k_0+1}, \ldots$ are given one can uniquely determine the solution. For

a given initial condition and right hand side, there are still $k_f - k_0$ degrees of freedom, i.e., the solution space has dimension $k_f - k_0$.
Written with matrices the following result is obtained. Multiplying (2.7) with $E_{\mathfrak{L}}^T$ from the left results in

$$E_{\mathfrak{L}}^T E_{\mathfrak{L}} x^{k+1} = E_{\mathfrak{L}}^T A_{\mathfrak{L}} x^k + E_{\mathfrak{L}}^T f^k \ , \tag{2.9}$$

where

$$E_{\mathfrak{L}}^T E_{\mathfrak{L}} = \begin{array}{cc} \begin{bmatrix} 0 & 0 \\ 0 & I_\epsilon \end{bmatrix} & \begin{array}{c} 1 \\ \epsilon \end{array} \end{array} \text{ is a projector and}$$

$$E_{\mathfrak{L}}^T A_{\mathfrak{L}} = \begin{array}{cc} \begin{bmatrix} 0 & 0 \\ I_\epsilon & 0 \end{bmatrix} & \begin{array}{c} 1 \\ \epsilon \end{array} \end{array} \text{ is nilpotent with nilpotency index } \epsilon + 1.$$

Partitioning the vector $x^k = \begin{array}{cc} \begin{bmatrix} x_1^{k+1} \\ x_2^{k+1} \end{bmatrix} & \begin{array}{c} 1 \\ \epsilon \end{array} \end{array}$ in (2.9) results in

$$\begin{bmatrix} 0 \\ x_2^{k+1} \end{bmatrix} = E_{\mathfrak{L}}^T A_{\mathfrak{L}} x^k + E_{\mathfrak{L}}^T f^k$$

$$\Leftrightarrow \quad x^{k+1} = E_{\mathfrak{L}}^T A_{\mathfrak{L}} x^k + E_{\mathfrak{L}}^T f^k + \begin{bmatrix} x_1^{k+1} \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Recursion of this equation leads to

$$x^k = \left(E_{\mathfrak{L}}^T A_{\mathfrak{L}}\right)^{k-k_0} \hat{x} + \sum_{j=0}^{k-k_0-1} \left(E_{\mathfrak{L}}^T A_{\mathfrak{L}}\right)^j \left(E_{\mathfrak{L}}^T f^{k-1-j} + \begin{bmatrix} x_1^{k-j} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right).$$

2. Consider a block of type (2.3), i.e., let

$$\lambda E_{\mathfrak{M}} - A_{\mathfrak{M}} = \lambda \begin{bmatrix} 1 & & \\ 0 & \ddots & \\ & \ddots & 1 \\ & & 0 \end{bmatrix} - \begin{bmatrix} 0 & & \\ 1 & \ddots & \\ & \ddots & 0 \\ & & 1 \end{bmatrix} \in \mathbb{C}^{\eta+1,\eta}.$$

In the case $\eta = 0$ this means that we are looking at sequences in which each iterate is a vector of length 0. Such sequences are of no interest. Therefore let us assume in the following that $\eta > 0$. In this case the system (1.4) corresponding to this pencil, together with (1.6), i.e.,

$$\begin{cases} E_{\mathfrak{M}} x^{k+1} = A_{\mathfrak{M}} x^k + f^k, \ k \geq k_0 \\ x^{k_0} = \hat{x}, \end{cases} \tag{2.10}$$

is equivalent to

$$\begin{cases} x_1^{k+1} = f_1^k, \\ x_i^{k+1} = x_{i-1}^k + f_i^k, \quad i = 2, \dots, \eta, \\ 0 = x_\eta^k + f_{\eta+1}^k, \\ x^{k_0} = \hat{x}. \end{cases}$$

Recursion of the second equation and the first equation yields

$$\begin{aligned} x_i^k &= x_{i-1}^{k-1} + f_i^{k-1} \\ &= x_{i-2}^{k-2} + f_{i-1}^{k-2} + f_i^{k-1} \\ &= \dots = \left(\sum_{j=1}^{\min(i-1,k-k_0)} f_{i+1-j}^{k-j} \right) + x_{i-\min(i-1,k-k_0)}^{k-\min(i-1,k-k_0)} \\ &= \begin{cases} \left(\sum_{j=1}^{i-1} f_{i+1-j}^{k-j} \right) + \underbrace{x_1^{k-i+1}}_{=f_1^{k-i}} & \text{, if } k - k_0 > i - 1, \\ \left(\sum_{j=1}^{k-k_0} f_{i+1-j}^{k-j} \right) + x_{i-k+k_0}^{k_0} & \text{, if } k - k_0 \le i - 1, \end{cases} \\ &= \begin{cases} \sum_{j=1}^{i} f_{i+1-j}^{k-j} & \text{, if } k - k_0 > i - 1, \\ \left(\sum_{j=1}^{k-k_0} f_{i+1-j}^{k-j} \right) + \hat{x}_{i-k+k_0}^{k_0} & \text{, if } k - k_0 \le i - 1. \end{cases} \end{aligned}$$

Because of the additional equation $x_\eta^k = -f_{\eta+1}^k$ the previous equation implies that

$$-f_{\eta+1}^k = x_\eta^k = \begin{cases} \sum_{j=1}^{\eta} f_{\eta+1-j}^{k-j} & \text{, if } k - k_0 > \eta - 1, \\ \left(\sum_{j=1}^{k-k_0} f_{\eta+1-j}^{k-j} \right) + \hat{x}_{\eta-k+k_0}^{k_0} & \text{, if } k - k_0 \le \eta - 1. \end{cases}$$

This provides the consistency condition for the inhomogeneity

$$0 = \sum_{j=0}^{\eta} f_{\eta+1-j}^{k-j} \quad \text{for all } k > k_0 + \eta - 1,$$

and the consistency condition for the initial condition

$$\hat{x}_{\eta-k+k_0}^{k_0} = -\sum_{j=0}^{k-k_0} f_{\eta+1-j}^{k-j} \quad \text{for all } k_0 \le k \le k_0 + \eta - 1.$$

Replacing $i = \eta - k + k_0$ in the last equation yields the nicer form

$$\hat{x}_i^{k_0} = -\sum_{j=0}^{\eta-i} f_{\eta+1-j}^{\eta-i+k_0-j} \quad \text{for all } 1 \le i \le \eta.$$

Again one can get the same result in terms of matrices. Multiplying (2.10) with $E_{\mathfrak{M}}^T$ from the left leads to

$$\begin{aligned}\underbrace{E_{\mathfrak{M}}^T E_{\mathfrak{M}}}_{=I} x^{k+1} &= E_{\mathfrak{M}}^T A_{\mathfrak{M}} x^k + E_{\mathfrak{M}}^T f^k \\ \Rightarrow x^{k+1} &= E_{\mathfrak{M}}^T A_{\mathfrak{M}} x^k + E_{\mathfrak{M}}^T f^k \\ &= (E_{\mathfrak{M}}^T A_{\mathfrak{M}})^2 x^{k-1} + (E_{\mathfrak{M}}^T A_{\mathfrak{M}}) E_{\mathfrak{M}}^T f^{k-1} + E_{\mathfrak{M}}^T f^k \\ &= \ldots = (E_{\mathfrak{M}}^T A_{\mathfrak{M}})^{k-k_0+1} \hat{x} + \sum_{j=0}^{k-k_0} (E_{\mathfrak{M}}^T A_{\mathfrak{M}})^j E_{\mathfrak{M}}^T f^{k-j}. \qquad (2.11)\end{aligned}$$

One should notice, that

$$(E_{\mathfrak{M}}^T A_{\mathfrak{M}}) = \begin{bmatrix} 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & & \\ 1 & \ddots & \\ & \ddots & 0 \\ & & 1 \end{bmatrix} = \begin{bmatrix} 0 & & & \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix} \in \mathbb{C}^{\eta,\eta},$$

is nilpotent with nilpotency index η. Thus, (2.11) can also be written as

$$x^{k+1} = (E_{\mathfrak{M}}^T A_{\mathfrak{M}})^{k-k_0+1} \hat{x} + \sum_{j=0}^{\min(k-k_0,\eta-1)} (E_{\mathfrak{M}}^T A_{\mathfrak{M}})^j E_{\mathfrak{M}}^T f^{k-j}.$$

3. Consider a block of type (2.4), i.e., let

$$\lambda E_{\mathfrak{J}} - A_{\mathfrak{J}} = \lambda \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} - \begin{bmatrix} \lambda_k & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_k \end{bmatrix} \in \mathbb{C}^{\rho,\rho}.$$

Then the system (1.4) corresponding to this pencil, together with (1.6) is

$$\begin{cases} x^{k+1} = A_{\mathfrak{J}} x^k + f^k, \; k \geq k_0 \\ x^{k_0} = \hat{x}. \end{cases} \qquad (2.12)$$

Simple recursion yields

$$\begin{aligned} x^k &= A_{\mathfrak{J}} x^{k-1} + f^{k-1} \\ &= A_{\mathfrak{J}}^2 x^{k-2} + A_{\mathfrak{J}} f^{k-2} + f^{k-1} \\ &= \ldots = A_{\mathfrak{J}}^{k-k_0} \hat{x} + \sum_{i=1}^{k-k_0} A_{\mathfrak{J}}^{i-1} f^{k-i}. \end{aligned}$$

4. Consider a block of type (2.5), i.e., let

$$\lambda E_{\mathfrak{N}} - A_{\mathfrak{N}} = \lambda \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix} - \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} \in \mathbb{C}^{\sigma,\sigma}.$$

Then the system (1.4) corresponding to this pencil, together with (1.6) is

$$\begin{cases} E_{\mathfrak{N}} x^{k+1} = x^k + f^k, \; k \geq k_0 \\ x^{k_0} = \hat{x}. \end{cases} \tag{2.13}$$

It follows that

$$\begin{aligned} x^k &= E_{\mathfrak{N}} x^{k+1} - f^k \\ &= E_{\mathfrak{N}}^2 x^{k+2} - E_{\mathfrak{N}} f^{k+1} - f^k \\ &= \ldots = -\sum_{i=0}^{\sigma-1} E_{\mathfrak{N}}^i f^{k+i}, \end{aligned}$$

because the matrix $E_{\mathfrak{N}}$ is nilpotent with index σ. Thus, in this case we see that the solution only depends on present and future right hand sides.

2.2 Explicit representation of the solution

In order to determine an explicit solution of (1.4) one can employ the Drazin inverse.

Definition 2.6. Let $E \in \mathbb{C}^{n,n}$ have the index ν. A matrix $X \in \mathbb{C}^{n,n}$ satisfying

$$EX = XE, \tag{2.14}$$
$$XEX = X, \tag{2.15}$$
$$XE^{\nu+1} = E^{\nu}, \tag{2.16}$$

is called a *Drazin inverse* of E and denoted by E^D.

From this definition some basic results can be derived.

Lemma 2.7. *Consider matrices $E, A \in \mathbb{C}^{n,n}$ with $EA = AE$. Then*

$$\begin{aligned} EA^D &= A^D E, \\ E^D A &= AE^D, \\ E^D A^D &= A^D E^D \end{aligned} \tag{2.17}$$

where E^D denotes the Drazin inverse of E.

Proof. [8], Theorem (2.21), p. 25. □

Also, recall the following Theorem.

Theorem 2.8. *Let $E \in \mathbb{C}^{n,n}$ with $\nu = \operatorname{ind}(E)$. There is one and only one decomposition*

$$E = \tilde{C} + \tilde{N} \tag{2.18}$$

with the properties

$$\tilde{C}\tilde{N} = \tilde{N}\tilde{C} = 0, \quad \tilde{N}^{\nu} = 0, \quad \tilde{N}^{\nu-1} \neq 0, \quad \operatorname{ind}(\tilde{C}) \leq 1. \tag{2.19}$$

For this decomposition the following statements hold:

$$\tilde{C}^D\tilde{N} = \tilde{N}\tilde{C}^D = 0, \tag{2.20a}$$
$$E^D = \tilde{C}^D, \tag{2.20b}$$
$$\tilde{C}\tilde{C}^D\tilde{C} = \tilde{C}, \tag{2.20c}$$
$$\tilde{C}^D\tilde{C} = E^DE, \tag{2.20d}$$
$$\tilde{C} = EE^DE,\ \tilde{N} = E\left(I - E^DE\right). \tag{2.20e}$$

Proof. [8], Theorem (2.22), p. 25. □

With these preliminaries one can start like in [8] to find solutions in the special case that the matrices E and A commute.

Lemma 2.9. *Let $E, A \in \mathbb{C}^{n,n}$ with $EA = AE$ and $E = \tilde{C} + \tilde{N}$ be the decomposition (2.18). Then the following propositions hold.*

1. *Let $\{x^k\}_{k=k_0}^{k_f+1}$ be a solution of (1.4). Set*

$$\begin{aligned} x_1^k &:= E^DEx^k, \quad x_2^k := \left(I - E^DE\right)x^k, \\ f_1^k &:= E^DEf^k, \quad f_2^k := \left(I - E^DE\right)f^k. \end{aligned} \tag{2.21}$$

   Then, we have

$$\tilde{C}x_1^{k+1} = Ax_1^k + f_1^k, \tag{2.22}$$
$$\tilde{N}x_2^{k+1} = Ax_2^k + f_2^k, \tag{2.23}$$

   for $k = k_0, \ldots, k_f$.

2. *Let $\{x_1^k\}_{k=k_0}^{k_f+1}$ and $\{x_2^k\}_{k=k_0}^{k_f+1}$ be solutions of*

$$\begin{aligned} \tilde{C}x_1^{k+1} &= Ax_1^k + E^DEf^k, \\ \tilde{N}x_2^{k+1} &= Ax_2^k + (I - E^DE)f^k, \end{aligned} \tag{2.24}$$

   for $k = k_0, \ldots, k_f$, respectively. Then $\{x^k\}_{k=k_0}^{k_f+1}$ given by

$$x^k := E^DEx_1^k + (I - E^DE)x_2^k$$

   is a solution of (1.4).

3. Let $\{x_1^k\}_{k=k_0}^{k_f+1}$ have the form

$$x_1^k = E^D E x^k, \tag{2.25}$$

for some $\{x^k\}_{k=k_0}^{k_f+1}$ and let $\{f_1^k\}_{k=k_0}^{k_f+1}$ have the form

$$f_1^k = E^D E f^k, \tag{2.26}$$

for some $\{f^k\}_{k=k_0}^{k_f+1}$. Then $\{x_1^k\}_{k=k_0}^{k_f+1}$ is a solution of

$$\tilde{C} x_1^{k+1} = A x_1^k + f_1^k \tag{2.27}$$

if and only if $\{x_1^k\}_{k=k_0}^{k_f+1}$ is a solution of

$$x_1^{k+1} = E^D A x_1^k + E^D f_1^k. \tag{2.28}$$

Proof. 1. From (2.20) and (2.17) it follows that

$$\tilde{C}^D \tilde{C} A \overset{(2.20d)}{=} E^D E A \overset{(2.17)}{=} A E^D E = A \tilde{C}^D \tilde{C} \tag{2.29}$$

and

$$\tilde{N} x_1^k = \tilde{N} E^D E x^k \overset{(2.20d)}{=} \tilde{N} \tilde{C}^D \tilde{C} x^k \overset{(2.20a)}{=} 0, \tag{2.30}$$

$$\tilde{C} x_2^k = \left(\tilde{C} - \tilde{C}\tilde{C}^D\tilde{C}\right) x^k \overset{(2.20c)}{=} 0, \tag{2.31}$$

$$\begin{aligned} \tilde{N} f_1^k &= \tilde{N} E^D E f^k \overset{(2.20d)}{=} \tilde{N} \tilde{C}^D \tilde{C} f^k \overset{(2.20a)}{=} 0, \\ \tilde{C} f_2^k &= \left(\tilde{C} - \tilde{C}\tilde{C}^D\tilde{C}\right) f^k \overset{(2.20c)}{=} 0. \end{aligned} \tag{2.32}$$

for all $k = k_0, \ldots, k_f + 1$. Further, from (1.4) we obtain

$$\begin{aligned} &\left(\tilde{C} + \tilde{N}\right)\left(x_1^{k+1} + x_2^{k+1}\right) = A\left(x_1^k + x_2^k\right) + f_1^k + f_2^k \qquad (2.33) \\ \overset{(2.29)}{\Rightarrow}\ &(\tilde{C}^D\tilde{C}\tilde{C} + \tilde{C}^D \underbrace{\tilde{C}\tilde{N}}_{\overset{(2.19)}{=}0})\left(x_1^{k+1} + x_2^{k+1}\right) = A\tilde{C}^D\tilde{C}\left(x_1^k + x_2^k\right) + \tilde{C}^D\tilde{C}\left(f_1^k + f_2^k\right) \\ \overset{(2.31),(2.32)}{\Rightarrow}\ &\tilde{C}^D\tilde{C}\tilde{C}x_1^{k+1} \overset{(2.21)}{=} A\tilde{C}^D\tilde{C}x_1^k + f_1^k = AE^DEx_1^k + f_1^k \overset{(2.21)}{=} Ax_1^k + f_1^k, \end{aligned}$$

for all $k = k_0, \ldots, k_f$, which shows (2.22), since

$$\tilde{C}^D\tilde{C}\tilde{C} \overset{(2.14)}{=} \tilde{C}\tilde{C}^D\tilde{C} \overset{(2.20c)}{=} \tilde{C}.$$

Subtracting (2.22) from (2.33) yields

$$\underbrace{\tilde{C}x_2^{k+1}}_{\overset{(2.31)}{=}0} + \underbrace{\tilde{N}x_1^{k+1}}_{\overset{(2.30)}{=}0} + \tilde{N}x_2^{k+1} = Ax_2^k + f_2^k,$$

for all $k = k_0, \ldots, k_f$, which shows (2.23).

2. Applying the Definition of x^{k+1} leads to

$$\begin{aligned} Ex^{k+1} &= EE^DEx_1^{k+1} + E(I - E^DE)x_2^{k+1} \\ &\overset{(2.20e)}{=} \tilde{C}x_1^{k+1} + \tilde{N}x_2^{k+1} \\ &\overset{(2.24)}{=} A(x_1^k + x_2^k) + E^DEf^k + (I - E^DE)f^k \\ &= Ax^k + f^k. \end{aligned}$$

3. Multiplying (2.27) with $\tilde{C}^D \overset{(2.20b)}{=} E^D$ from the left leads to

$$\tilde{C}^D\tilde{C}x_1^{k+1} = E^DAx_1^k + E^Df_1^k. \tag{2.34}$$

From (2.25) one can also obtain

$$(I - \tilde{C}^D\tilde{C})x_1^{k+1} = 0. \tag{2.35}$$

Adding (2.34) and (2.35) then immediately shows (2.28). Conversely, multiplying (2.28) by $\tilde{C}$ from the left gives

$$\begin{aligned} \tilde{C}x_1^{k+1} &= \tilde{C}\tilde{C}^DAx_1^k + \tilde{C}\tilde{C}^Df_1^k \\ &= AE^DEx_1^k + E^DEf_1^k \\ &\overset{(2.25),(2.26)}{=} Ax_1^k + f_1^k, \end{aligned}$$

since

$$\tilde{C}\tilde{C}^D \overset{(2.20e),(2.20b)}{=} EE^DEE^D \overset{(2.17)}{=} E^DEE^DE \overset{(2.15)}{=} E^DE.$$

□

Lemma 2.10. *Let $E, A \in \mathbb{C}^{n,n}$ with $EA = AE$, $k_0 \in \mathbb{Z}$ and $v \in \mathbb{C}^n$. Then the following statements hold.*

1. Let $\hat{v} = E^DEv$. Then

$$x^k := (E^DA)^{k-k_0}\hat{v}, \quad k = k_0, k_0 + 1, \ldots \tag{2.36}$$

solves the homogeneous linear discrete-time descriptor system

$$Ex^{k+1} = Ax^k, \quad k = k_0, k_0 + 1, \ldots \tag{2.37}$$

2. Let $\hat{v} = A^DAv$. Then

$$x^k := (A^DE)^{k_0-k}\hat{v}, \quad k = k_0, k_0 - 1, \ldots \tag{2.38}$$

solves the homogeneous linear discrete-time descriptor system

$$Ex^{k+1} = Ax^k, \quad k = k_0 - 1, k_0 - 2, \ldots \tag{2.39}$$

3. *Let* $\hat{v} \in \operatorname{range}\left(A^D A\right) \cap \operatorname{range}\left(E^D E\right)$. *Then*

$$x^k := \begin{cases} (E^D A)^{k-k_0}\hat{v}, & k = k_0, k_0 - 1, \dots \\ (A^D E)^{k_0-k}\hat{v}, & k = k_0 - 1, k_0 - 2, \dots \end{cases} \tag{2.40}$$

solves the homogeneous linear discrete-time descriptor system

$$Ex^{k+1} = Ax^k, \quad k \in \mathbb{Z}. \tag{2.41}$$

Proof. 1. We have

$$\begin{aligned} Ex^{k+1} &= E(E^D A)(E^D A)^{k-k_0} E^D E v \\ &\overset{(2.17)}{=} A(E^D A)^{k-k_0} E^D E E^D E v \\ &= A(E^D A)^{k-k_0} E^D E v \\ &= Ax^k \text{ for all } k = k_0, k_0 + 1, \dots . \end{aligned}$$

2. In this case we obtain

$$\begin{aligned} Ax^k &= A(A^D E)^{k_0-k} A^D A v \\ &= A(A^D E)(A^D E)^{k_0-k-1} A^D A v \\ &\overset{(2.17)}{=} E(A^D E)^{k_0-k-1} A^D A A^D A v \\ &= E(A^D E)^{k_0-k-1} A^D A v \\ &= Ex^{k+1} \text{ for all } k = k_0 - 1, k_0 - 2, \dots . \end{aligned}$$

3. This follows from 1. and 2., since the definitions of x^{k_0} from 1. and 2. coincide.

□

Since

$$(E^D A)^{k-k_0} E^D E v = E^D E (E^D A)^{k-k_0} v,$$

it is clear, that the solution x^k stays in the subspace $\operatorname{range}\left(E^D E\right)$ for all $k \geq k_0$. An analogous conclusion is possible for the case 2. in Lemma 2.10. In case 3. of Lemma 2.10 the solution even stays in $\operatorname{range}\left(A^D A\right) \cap \operatorname{range}\left(E^D E\right)$ all the time.

Theorem 2.11. *Let* $E, A \in \mathbb{C}^{n,n}$ *with* $EA = AE$ *and suppose that*

$$\operatorname{kernel}(E) \cap \operatorname{kernel}(A) = \{0\}. \tag{2.42}$$

Then,

$$(I - E^D E)A^D A = (I - E^D E). \tag{2.43}$$

Proof. [8], Theorem (2.28), p. 30. □

Theorem 2.12. *Let $E, A \in \mathbb{C}^{n,n}$ with $EA = AE$ satisfy (2.42). Also, let $k_0 \in \mathbb{Z}$. Then the following statements hold.*

1. *Let $\{x^k\}_{k \geq k_0}$ be any solution of (2.37). Then $\{x^k\}_{k \geq k_0}$ has the form (2.36) for some $\hat{v} \in \operatorname{range}\left(E^D E\right)$.*

2. *Let $\{x^k\}_{k \leq k_0}$ be any solution of (2.39). Then $\{x^k\}_{k \leq k_0}$ has the form (2.38) for some $\hat{v} \in \operatorname{range}\left(A^D A\right)$.*

3. *Let $\{x^k\}_{k \in \mathbb{Z}}$ be any solution of (2.41). Then $\{x^k\}_{k \in \mathbb{Z}}$ has the form (2.40) for some $\hat{v} \in \operatorname{range}\left(A^D A\right) \cap \operatorname{range}\left(E^D E\right)$.*

Proof. Using the decomposition (2.18) we get the following results.

1. We have
$$A\tilde{N} \overset{(2.20e)}{=} AE(I - E^D E) \overset{(2.17)}{=} E(I - E^D E)A \overset{(2.20e)}{=} \tilde{N}A. \tag{2.44}$$
Furthermore for any $x \in \mathbb{C}^n$ one has
$$\begin{aligned} A\tilde{N}x = 0 \quad &\Rightarrow \quad A^D A\tilde{N}x = 0 \\ &\Rightarrow \quad (I - E^D E)A^D A\tilde{N}x = 0 \\ &\overset{(2.43)}{\Rightarrow} \quad (I - E^D E)\tilde{N}x = 0 \\ &\overset{(2.20e)}{\Rightarrow} \quad \tilde{N}x = 0. \end{aligned} \tag{2.45}$$
Let $\{x^k\}_{k \in \mathbb{Z}}$ be any solution of (2.37). From Lemma 2.9 part 1. we get $\{x_1^k\}_{k \geq k_0}$, $\{x_2^k\}_{k \geq k_0}$ with $x^k = x_1^k + x_2^k$ which solve (2.22) and (2.23), respectively. With $\nu = \operatorname{ind}(E)$ one then obtains
$$0 \overset{(2.19)}{=} \tilde{N}^\nu x_2^{k+1} \overset{(2.23)}{=} \tilde{N}^{\nu-1} A x_2^k \overset{(2.44)}{=} A\tilde{N}^{\nu-1} x_2^k, \quad k \geq k_0$$
$$\overset{(2.45)}{\Rightarrow} \tilde{N}^{\nu-1} x_2^k = 0, \quad k \geq k_0.$$
Discarding the identity for $k = k_0$ then yields
$$\tilde{N}^{\nu-1} x_2^k = 0, \quad k \geq k_0 + 1.$$
$$\begin{aligned} &\overset{\text{Shift}}{\Rightarrow} \tilde{N}^{\nu-1} x_2^{k+1} = 0, \quad k + 1 \geq k_0 + 1 \\ &\Rightarrow \tilde{N}^{\nu-1} x_2^{k+1} = 0, \quad k \geq k_0 \\ &\Rightarrow \ldots \Rightarrow \tilde{N} x_2^k = 0, \quad k \geq k_0 \\ &\overset{(2.23)}{\Rightarrow} A x_2^k = 0, \quad k \geq k_0 \end{aligned} \tag{2.46}$$
$$\Rightarrow x_2^k \overset{(2.21)}{=} (I - E^D E)x_2^k \overset{(2.43)}{=} (I - E^D E)A^D \underbrace{A x_2^k}_{\overset{(2.46)}{=} 0} = 0, \quad k \geq k_0$$

$$\Rightarrow x^k = x_1^k, \quad k \geq k_0$$
$$\overset{\text{Lemma 2.9 3.}}{\Rightarrow} x_1^k \text{ solves } x_1^{k+1} = (E^D A)x_1^k, \quad k \geq k_0$$
$$\overset{\text{Recursion}}{\Rightarrow} x_1^k = (E^D A)^{k-k_0} x_1^{k_0}, \quad k \geq k_0$$
$$\Rightarrow x^k = x_1^k = (E^D A)^{k-k_0} x_1^{k_0} \overset{(2.21)}{=} (E^D A)^{k-k_0} E^D E x^{k_0}. \tag{2.47}$$

2. Let $\{x^k\}_{k \leq k_0}$ be any solution of (2.39). Set

$$l_0 := -k_0 \text{ and } y^l := x^{-l}, \quad l \geq l_0.$$

By replacing $k = -l$ in (2.39) one obtains

$$Ex^{-l+1} = Ax^{-l}, \quad -l = -l_0 - 1, -l_0 - 2, \ldots$$
$$\Rightarrow Ex^{-(l-1)} = Ax^{-l}, \quad l = l_0 + 1, l_0 + 2, \ldots$$
$$\Rightarrow y^l,\, l \geq l_0 \text{ is a solution of } Ey^{l-1} = Ay^l, \quad l \geq l_0 + 1$$
$$\Rightarrow y^l,\, l \geq l_0 \text{ is a solution of } Ay^{l+1} = Ey^l, \quad l \geq l_0$$
$$\overset{(2.47)}{\Rightarrow} y^l = (A^D E)^{l-l_0} A^D A y^{l_0}, \quad l \geq l_0.$$

Undoing the replacements then yields

$$x^{-l} = (A^D E)^{l-l_0} A^D A x^{-l_0}, \quad l \geq l_0,$$
$$x^k = (A^D E)^{-k+k_0} A^D A x^{k_0}, \quad -k \geq -k_0,$$
$$x^k = (A^D E)^{k_0-k} A^D A x^{k_0}, \quad k \leq k_0. \tag{2.48}$$

3. Let $\{x^k\}_{k \in \mathbb{Z}}$ be any solution of (2.41). Then from (2.47) we have

$$x^k = (E^D A)^{k-k_0} E^D E x^{k_0}, \quad k \geq k_0$$
$$\overset{k=k_0}{\Rightarrow} x^{k_0} = E^D E x^{k_0} \Rightarrow x^{k_0} \in \operatorname{range}\left(E^D E\right).$$

Also we know from (2.48) that

$$x^k = (A^D E)^{k_0-k} A^D A x^{k_0}, \quad k \leq k_0$$
$$\overset{k=k_0}{\Rightarrow} x^{k_0} = A^D A x^{k_0} \Rightarrow x^{k_0} \in \operatorname{range}\left(A^D A\right).$$

Thus, the claim of the Theorem follows with $\hat{v} = x^{k_0}$. $\square$

Remark 2.13. One may think that it is not meaningful to look at case 3. of the previous Theorem, since in most cases one starts at some time point and then calculates into the future. But as shown by the following Lemma, also those solutions (where one starts at $k_0 \in \mathbb{Z}$ and calculates a solution for $k \geq k_0$) are almost completely in the subspace $\operatorname{range}\left(A^D A\right) \cap \operatorname{range}\left(E^D E\right)$.

Lemma 2.14. *Let $E, A \in \mathbb{C}^{n,n}$ with $EA = AE$ satisfy (2.42). Also, let $k_0 \in \mathbb{Z}$ and let $\nu_E = \operatorname{ind}(E)$, $\nu_A = \operatorname{ind}(A)$. Then the following statements hold.*

1. *Let $\{x^k\}_{k\geq k_0}$ be any solution of (2.37). Then for all $k \geq k_0 + \nu_A$ it holds that $x^k \in \operatorname{range}\left(A^D A\right) \cap \operatorname{range}\left(E^D E\right)$.*

2. *Let $\{x^k\}_{k\leq k_0}$ be any solution of (2.39). Then for all $k \leq k_0 - \nu_E$ it holds that $x^k \in \operatorname{range}\left(E^D E\right) \cap \operatorname{range}\left(A^D A\right)$.*

Proof. 1. Since $k \geq k_0 + \nu_A$ it follows that $k = \hat{k} + k_0 + \nu_A$ with $\hat{k} \geq 0$. From Theorem 2.12 we then know that for some $v \in \mathbb{C}^n$ we have

$$\begin{aligned} A^D A x^k &\overset{(2.36)}{=} A^D A \underbrace{(E^D A)^{k-k_0}}_{=(E^D)^{k-k_0} A^{k-k_0}} E^D E v \\ &= A^D A (E^D)^{k-k_0} A^{\nu_A} A^{\hat{k}} E^D E v \\ &\overset{(2.16)}{=} A^D A (E^D)^{k-k_0} A^D A^{\nu_A+1} A^{\hat{k}} E^D E v \\ &= (E^D)^{k-k_0} \underbrace{A^D A A^D}_{\overset{(2.15)}{=} A^D} A^{\nu_A+1} A^{\hat{k}} E^D E v \\ &= (E^D)^{k-k_0} A^D A^{\nu_A+1} A^{\hat{k}} E^D E v \\ &\overset{(2.16)}{=} (E^D A)^{k-k_0} E^D E v \\ &= x^k. \end{aligned}$$

Also, we naturally get

$$\begin{aligned} E^D E x^k &\overset{(2.36)}{=} E^D E (E^D A)^{k-k_0} E^D E v \\ &= (E^D A)^{k-k_0} \underbrace{E^D E E^D}_{\overset{(2.15)}{=} E^D} E v \\ &= x^k, \end{aligned} \tag{2.49}$$

and thus the assertion follows.

2. As in (2.49) one gets that $A^D A x^k = x^k$. Let $k = -\hat{k} + k_0 - \nu_E$ with $\hat{k} \geq 0$. Then again for some $v \in \mathbb{C}^n$ it follows that

$$\begin{aligned} E^D E x^k &\overset{(2.38)}{=} E^D E \underbrace{(A^D E)^{k_0-k}}_{=(A^D)^{k_0-k} E^{k_0-k}} A^D A v \\ &= E^D E (A^D)^{k_0-k} E^{\nu_E} E^{\hat{k}} A^D A v \\ &\overset{(2.16)}{=} E^D E (A^D)^{k_0-k} E^D E^{\nu_E+1} E^{\hat{k}} A^D A v \\ &= (A^D)^{k_0-k} \underbrace{E^D E E^D}_{\overset{(2.15)}{=} E^D} E^{\nu_E+1} E^{\hat{k}} A^D A v \end{aligned}$$

$$
\begin{aligned}
&= (A^D)^{k_0-k} E^D E^{\nu_E+1} E^{\hat{k}} A^D A v \\
&\overset{(2.16)}{=} (A^D E)^{k_0-k} A^D A v \\
&= x^k.
\end{aligned}
$$

□

Remark 2.15. From Lemma 2.14 one might presume that it is meaningful to require that the initial condition satisfies

$$x^{k_0} \in \operatorname{range}\left(A^D A\right) \cap \operatorname{range}\left(E^D E\right), \tag{2.50}$$

since only in this case it is possible to calculate the solution into the future (i.e., calculate x^k for $k \geq k_0$) and into the past (i.e., calculate x^k for $k \leq k_0$).
Also, only in case that (2.50) holds, we get something like an invertibility of the operator that calculates x^{k+1} from x^k. To understand this, imagine that a fixed x^{k_0} is given. From this we calculate a finite number of steps κ into the future. Thus, we have $x^{k_0+\kappa}$. From this state we then calculate κ steps back into the past to obtain $\tilde{x}^{k_0}$. We then have $x^{k_0} = \tilde{x}^{k_0}$ if condition (2.50) holds. Otherwise we cannot be sure that $x^{k_0} = \tilde{x}^{k_0}$ holds, as shown in the following example.

Example 2.16. Consider the homogeneous linear discrete-time descriptor system defined by

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{:=E} x^{k+1} = \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{:=A} x^k,\ k \geq 0, \qquad x^0 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}. \tag{2.51}$$

Clearly, we have $EA = AE$, $E^D = E$, $A^D = A$ and condition (2.42) holds. Thus, the pencil (E, A), corresponding to system (2.51), satisfies all assumptions from Lemma 2.14 which means that the iterate x^1 has to be in $\operatorname{range}\left(A^D A\right)$. Indeed,

$$Ax^0 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \Rightarrow x^1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \in \operatorname{range}\left(A^D A\right). \tag{2.52}$$

Now let us calculate back one step from (2.52), i.e., let us consider the reversed system

$$A\tilde{x}^{l+1} = E\tilde{x}^l,\ l \leq 0, \qquad \tilde{x}^{-1} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

We see that

$$E\tilde{x}^{-1} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \Rightarrow \tilde{x}^0 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

and thus

$$\tilde{x}^0 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \neq x^0.$$

Theorem 2.17. *Let $E, A \in \mathbb{C}^{n,n}$ with $EA = AE$ satisfy (2.42). Also, let $\nu_E = \operatorname{ind}(E)$, $\nu_A = \operatorname{ind}(A)$, $\{f^k\}_{k\in\mathbb{Z}}$ with $f^k \in \mathbb{C}^n$ and $k_0 \in \mathbb{Z}$. Then the following statements hold.*

1. *The linear discrete-time descriptor system*

$$Ex^{k+1} = Ax^k + f^k, \quad k \geq k_0$$

has the particular solution $\{x^k\}_{k\geq k_0}$ with

$$x^k := \underbrace{\sum_{j=k_0}^{k-1} (E^D A)^{k-j-1} E^D f^j}_{:=x_1^k} \underbrace{- (I - E^D E) \sum_{i=0}^{\nu_E - 1} (A^D E)^i A^D f^{k+i}}_{:=x_2^k} \quad \text{for } k \geq k_0. \tag{2.53}$$

For the construction of the iterate x^k only the f_k with $k \geq k_0$ have to be employed.

2. *The linear discrete-time descriptor system*

$$Ex^{k+1} = Ax^k + f^k, \quad k \leq k_0 - 1 \tag{2.54}$$

has the particular solution $\{x^k\}_{k\leq k_0}$ with

$$x^k := (I - A^D A) \sum_{i=0}^{\nu_A - 1} (E^D A)^i E^D f^{k-i-1} - \sum_{j=k+1}^{k_0} (A^D E)^{j-k-1} A^D f^{j-1} \quad \text{for } k \leq k_0. \tag{2.55}$$

For the construction of the iterate x^k only the f_k with $k \leq k_0 - 1$ have to be employed.

Proof. 1. Let $E = \tilde{C} + \tilde{N}$ be the decomposition (2.18). Then

$$E^D E x_1^k \overset{(2.17)}{=} \sum_{j=k_0}^{k-1} (E^D A)^{k-j-1} \underbrace{E^D E E^D}_{=E^D} f^j = x_1^k,$$

$$(I - E^D E) x_2^k = -(I - E^D E)(I - E^D E) \sum_{i=0}^{\nu_E - 1} (A^D E)^i A^D f^{k+i} = x_2^k.$$

One can also conclude, that for all $k \geq k_0$ it follows that

$$\tilde{C} x_1^{k+1} \quad = \quad \tilde{C} \sum_{j=k_0}^{k} (E^D A)^{k+1-1-j} E^D f^j$$

$$
\begin{aligned}
&= && \tilde{C}\left(\sum_{j=k_0}^{k-1}(E^DA)^{k-j}E^Df^j + E^Df^k\right)\\
&= && \tilde{C}\left((E^DA)\sum_{j=k_0}^{k-1}(E^DA)^{k-j-1}E^Df^j + E^Df^k\right)\\
&\overset{(2.20e),(2.15)}{=} && AE^DEx_1^k + E^DEf^k\\
&= && Ax_1^k + E^DEf^k,
\end{aligned}
$$

and with

$$
(I-E^DE)E^{\nu_E} = \begin{cases}(I-E^DE)E^DE^{\nu_E-1} = 0 & \text{, if } \nu_E \geq 1,\\ (I-E^DE) = (I-I) = 0 & \text{, if } \nu_E = 0,\end{cases} \tag{2.56}
$$

we obtain

$$
\begin{aligned}
\tilde{N}x_2^{k+1} &\overset{(2.20e)}{=} && E(I-E^DE)x_2^{k+1}\\
&= && -(I-E^DE)\sum_{i=0}^{\nu_E-1}(A^DE)^{i+1}f^{k+i+1}\\
&\overset{(2.56)}{=} && -(I-E^DE)\sum_{i=0}^{\nu_E-2}(A^DE)^{i+1}f^{k+i+1}\\
&\overset{(2.43)}{=} && -(I-E^DE)A^DA\sum_{i=1}^{\nu_E-1}(A^DE)^{i}f^{k+i}\\
&= && -A(I-E^DE)\sum_{i=0}^{\nu_E-1}(A^DE)^{i}A^Df^{k+i} + (I-E^DE)f^k\\
&= && Ax_2^k + (I-E^DE)f^k.
\end{aligned}
$$

With these results and Lemma 2.9 part 2 one immediately gets that

$$
x^k = E^DEx_1^k + (I-E^DE)x_2^k = x_1^k + x_2^k
$$

is a solution and thus the assertion follows.

2. By replacing $l := -k$ and $l_0 := -k_0$ in (2.54) one gets the system

$$
\begin{aligned}
&Ex^{-l+1} = Ax^{-l} + f^{-l}, \quad -l \leq -l_0 - 1\\
&\Rightarrow Ex^{-(l-1)} = Ax^{-l} + f^{-l}, \quad l \geq l_0 + 1.
\end{aligned}
$$

By further replacing $y^l := x^{-l}, g^l := -f^{-l-1}$ for $l \geq l_0$ one gets

$$
\begin{aligned}
Ey^{l-1} &= Ay^l + f^{-l}, \quad l \geq l_0 + 1\\
\Rightarrow Ey^l &= Ay^{l+1} + f^{-l-1}, \quad l+1 \geq l_0 + 1\\
\Rightarrow Ay^{l+1} &= Ey^l - f^{-l-1}, \quad l \geq l_0\\
\Rightarrow Ay^{l+1} &= Ey^l + g^l, \quad l \geq l_0.
\end{aligned}
$$

We then get a solution of this last system as

$$
\begin{aligned}
\overset{1.}{\Rightarrow} y^l &= \sum_{j=l_0}^{l-1}(A^DE)^{l-j-1}A^Dg^j - (I-A^DA)\sum_{i=0}^{\nu_A-1}(E^DA)^iE^Dg^{l+i} \\
\Rightarrow x^{-l} &= -\sum_{j=l_0}^{l-1}(A^DE)^{l-j-1}A^Df^{-j-1} + (I-A^DA)\sum_{i=0}^{\nu_A-1}(E^DA)^iE^Df^{-(l+i)-1} \\
\Rightarrow x^k &= (I-A^DA)\sum_{i=0}^{\nu_A-1}(E^DA)^iE^Df^{k-i-1} - \sum_{j=-k_0}^{-k-1}(A^DE)^{-k-j-1}A^Df^{-j-1} \\
&= (I-A^DA)\sum_{i=0}^{\nu_A-1}(E^DA)^iE^Df^{k-i-1} - \sum_{j=k+1}^{k_0}(A^DE)^{j-k-1}A^Df^{j-1}.
\end{aligned}
$$

□

Finding a particular solution for the general case (i.e., (1.4) with $\mathbb{K} = \mathbb{Z}$) is more complicated. Similar to Lemma 2.9 we obtain the following result.

Lemma 2.18. *Let $E, A \in \mathbb{C}^{n,n}$ with $EA = AE$ satisfy (2.42). Further, let $E = \tilde{C} + \tilde{N}$ and analogously $A = \tilde{D} + \tilde{M}$ be decompositions as in (2.18). Let $\{x_1^k\}_{k\in\mathbb{Z}}$, $\{x_2^k\}_{k\in\mathbb{Z}}$, $\{x_3^k\}_{k\in\mathbb{Z}}$ be solutions of*

$$
\begin{aligned}
\tilde{C}x_1^{k+1} &= \tilde{M}x_1^k + (I-A^DA)f^k, &(2.57)\\
\tilde{C}x_2^{k+1} &= \tilde{D}x_2^k + A^DAE^DEf^k, &(2.58)\\
\tilde{N}x_3^{k+1} &= \tilde{D}x_3^k + (I-E^DE)f^k, &(2.59)
\end{aligned}
$$

respectively. Then $\{x^k\}_{k\in\mathbb{Z}}$ with

$$x^k := (I-A^DA)x_1^k + A^DAE^DEx_2^k + (I-E^DE)x_3^k,$$

is a solution of

$$Ex^{k+1} = A^k + f^k.$$

Proof. First of all, we see that

$$
\begin{aligned}
&(I-A^DA) + (I-E^DE) + A^DAE^DE \\
=&I - A^DA + I - \underbrace{(I-A^DA)E^DE}_{\overset{(2.43)}{=}(I-A^DA)} \\
=&I - A^DA + I - (I-A^DA) = I.
\end{aligned}
\tag{2.60}
$$

Furthermore, we have

$$\tilde{D}(I-A^DA) = AA^DA(I-A^DA) = 0\ , \tag{2.61}$$

$$\tilde{M}(A^DAE^DE) = A(I-A^DA)(A^DAE^DE) = 0\ , \tag{2.62}$$

$$\tilde{M}(I-E^DE) = A(I-A^DA)(I-E^DE) = A\left((I-E^DE) - (I-E^DE)A^DA\right) = 0. \tag{2.63}$$

With these identities we get

$$
\begin{aligned}
Ex^{k+1} &= E\underbrace{(I-A^DA)}_{\overset{(2.43)}{=}(I-A^DA)E^DE}x_1^{k+1} + EA^DA\underbrace{E^DE}_{\overset{(2.15)}{=}E^DEE^DE}x_2^{k+1} + E\underbrace{(I-E^DE)}_{=(I-E^DE)(I-E^DE)}x_3^{k+1} \\
&= (I-A^DA)\tilde{C}x_1^{k+1} + A^DAE^DE\tilde{C}x_2^{k+1} + (I-E^DE)\tilde{N}x_3^{k+1} \\
&= (I-A^DA)\tilde{M}x_1^k + (I-A^DA)f^k + \\
&\quad A^DAE^DE\tilde{D}x_2^k + A^DAE^DEf^k + \\
&\quad (I-E^DE)\tilde{D}x_3^k + (I-E^DE)f^k \\
&\overset{(2.60)}{=} f^k + (I-A^DA)\tilde{M}x_1^k + A^DAE^DE\tilde{D}x_2^k + (I-E^DE)\tilde{D}x_3^k \\
&= f^k + \\
&\quad \tilde{M}(I-A^DA)x_1^k + \underbrace{\tilde{D}(I-A^DA)x_1^k}_{\overset{(2.61)}{=}0} \\
&\quad \underbrace{\tilde{M}A^DAE^DEx_2^k}_{\overset{(2.62)}{=}0} + \tilde{D}A^DAE^DEx_2^k \\
&\quad \underbrace{\tilde{M}(I-E^DE)x_3^k}_{\overset{(2.63)}{=}0} + \tilde{D}(I-E^DE)x_3^k \\
&= f^k + Ax^k.
\end{aligned}
$$

Here we have used, that $\tilde{D} = AA^DA$ and thus $\tilde{D}$ commutes with the matrices E and A. □

Using Lemma 2.18 we can construct a particular solution for the general case (i.e., (1.4) with $\mathbb{K} = \mathbb{Z}$).

Lemma 2.19. *Let $E, A \in \mathbb{C}^{n,n}$ with $EA = AE$ satisfying (2.42). Also, let $\nu_E = \operatorname{ind}(E)$, $\nu_A = \operatorname{ind}(A)$, $\{f^k\}_{k\in\mathbb{Z}} \subset \mathbb{C}^n$ and $k_0 \in \mathbb{Z}$. Then a solution $\{x^k\}_{k\in\mathbb{Z}}$ of*

$$Ex^{k+1} = Ax^k + f^k$$

is given by

$$
\begin{aligned}
x^k :=& \underbrace{(I-A^DA)\sum_{i=0}^{\nu_A-1}(E^DA)^iE^Df^{k-i-1}}_{=:x_1^k} \\
&\underbrace{+\begin{cases} A^DA\sum_{j=k_0}^{k-1}(E^DA)^{k-j-1}E^Df^j & \text{for } k \geq k_0 \\ -E^DE\sum_{j=k+1}^{k_0}(A^DE)^{j-k-1}A^Df^{j-1} & \text{for } k \leq k_0 \end{cases}}_{:=x_2^k} \\
&\underbrace{-(I-E^DE)\sum_{i=0}^{\nu_E-1}(A^DE)^iA^Df^{k+i}}_{:=x_3^k}, \text{ for } k \in \mathbb{Z}.
\end{aligned}
\tag{2.64}
$$

Proof. Considering the decompositions $E = \tilde{C} + \tilde{N}$ and $A = \tilde{D} + \tilde{M}$ as in (2.18) we have

$$\begin{aligned}
\tilde{M}x_1^k &= A(I - A^D A) \sum_{i=0}^{\nu_A - 1} (E^D A)^i E^D f^{k-i-1} \\
&\overset{(2.17)}{=} (I - A^D A) \sum_{i=0}^{\nu_A - 1} (E^D A)^{i+1} f^{k-i-1} \\
&\overset{\text{see (2.56)}}{=} (I - A^D A) \sum_{i=0}^{\nu_A - 2} (E^D A)^{i+1} f^{k-i-1} \\
&= (I - A^D A) \sum_{i=1}^{\nu_A - 1} (E^D A)^{i} f^{k-i} \\
&= (I - A^D A) \left(\sum_{i=0}^{\nu_A - 1} (E^D A)^{i} f^{k-i} - f^k \right) \\
&\overset{(2.43)}{=} -(I - A^D A) f^k + (I - A^D A) E^D E \sum_{i=0}^{\nu_A - 1} (E^D A)^{i} f^{k-i} \\
&\overset{(2.17)}{=} -(I - A^D A) f^k + E(I - A^D A) \sum_{i=0}^{\nu_A - 1} (E^D A)^{i} E^D f^{k-i} \\
&= -(I - A^D A) f^k + E x_1^{k+1} \\
&= -(I - A^D A) f^k + (\tilde{C} + \tilde{N}) x_1^{k+1} \\
&= -(I - A^D A) f^k + \tilde{C} x_1^{k+1},
\end{aligned}$$

where the last identity holds, since x_1^k has the form $x_1^k = (I - A^D A) y_1^k$ for some y_1^k and

$$\tilde{N}x_1^k = E(I - E^D E)(I - A^D A) y_1^k = E \left(I - A^D A - \underbrace{E^D E (I - A^D A)}_{\overset{(2.43)}{=} (I - A^D A)} \right) y_1^k = 0. \tag{2.65}$$

As in Theorem 2.17, part 1. one obtains

$$\tilde{N} x_3^{k+1} = A x_3^k + (I - E^D E) f^k = (\tilde{D} + \tilde{M}) x_3^k + (I - E^D E) f^k.$$

Again as in (2.65) it follows that

$$\tilde{M} x_3^k = 0,$$

and thus

$$\tilde{N} x_3^{k+1} = \tilde{D} x_3^k + (I - E^D E) f^k.$$

Finally, for $k \geq k_0$ one has

$$\begin{aligned}
\tilde{C}x_2^{k+1} &= \tilde{C}A^DA\sum_{j=k_0}^{k}(E^DA)^{k-j}E^Df^j \\
&= \underbrace{\tilde{C}}_{=EE^DE}E^DA^DA\left(\sum_{j=k_0}^{k-1}(E^DA)^{k-j}f^j + f^k\right) \\
&= E^DEA^DA\left(\sum_{j=k_0}^{k-1}(E^DA)^{k-j}f^j + f^k\right) \\
&= E^DEA^DA\left(E^DA\sum_{j=k_0}^{k-1}(E^DA)^{k-j-1}f^j + f^k\right) \\
&= AA^DA\sum_{j=k_0}^{k-1}(E^DA)^{k-j-1}E^Df^j + A^DAE^DEf^k \\
&= AA^DAA^DA\sum_{j=k_0}^{k-1}(E^DA)^{k-j-1}E^Df^j + A^DAE^DEf^k \\
&= \tilde{D}A^DA\sum_{j=k_0}^{k-1}(E^DA)^{k-j-1}E^Df^j + A^DAE^DEf^k \\
&= \tilde{D}x_2^k + A^DAE^DEf^k,
\end{aligned}$$

and for $k < k_0$ analogously,

$$\begin{aligned}
\tilde{D}x_2^k &= -\tilde{D}E^DE\sum_{j=k+1}^{k_0}(A^DE)^{j-k-1}A^Df^{j-1} \\
&= -AA^DAE^DEA^D\sum_{j=k+1}^{k_0}(A^DE)^{j-k-1}f^{j-1} \\
&= -AA^DE^DE\sum_{j=k+1}^{k_0}(A^DE)^{j-k-1}f^{j-1} \\
&= -AA^DE^DE\left(\sum_{j=k+2}^{k_0}(A^DE)^{j-k-1}f^{j-1} + f^k\right) \\
&= -AA^DE^DEf^k - A^DAA^DEE^DE\sum_{j=k+2}^{k_0}(A^DE)^{j-k-2}f^{j-1} \\
&= -AA^DE^DEf^k - A^DEE^DEE^DE\sum_{j=k+2}^{k_0}(A^DE)^{j-k-2}f^{j-1}
\end{aligned}$$

$$
\begin{aligned}
&= -AA^D E^D E f^k + EE^D E \left(-E^D E \sum_{j=k+2}^{k_0} (A^D E)^{j-k-2} A^D f^{j-1} \right) \\
&= -AA^D E^D E f^k + \tilde{C} x_2^{k+1}.
\end{aligned}
$$

Lemma 2.18 then implies the assertion. □

We finally combine Lemmas 2.10, 2.14, 2.19 and Theorem 2.17.

Theorem 2.20. *Let $E, A \in \mathbb{C}^{n,n}$ with $EA = AE$ satisfy (2.42). Also, let $\nu_E = \operatorname{ind}(E)$, $\nu_A = \operatorname{ind}(A)$, $\{f^k\}_{k\in\mathbb{Z}}$ with $f^k \in \mathbb{C}^n$ and $k_0 \in \mathbb{Z}$. Then the following statements hold.*

1. *Every solution $\{x^k\}_{k\geq k_0}$ of*

$$
Ex^{k+1} = Ax^k + f^k, \quad k \geq k_0 \tag{2.66}
$$

satisfies

$$
\begin{aligned}
x^k = \left(E^D A\right)^{k-k_0} E^D E v + \sum_{j=k_0}^{k-1} \left(E^D A\right)^{k-j-1} E^D f^j \\
- \left(I - E^D E\right) \sum_{i=0}^{\nu_E - 1} \left(A^D E\right)^i A^D f^{k+i}
\end{aligned} \tag{2.67}
$$

for $k \geq k_0$ and for some $v \in \mathbb{C}^n$.

2. *Every solution $\{x^k\}_{k\leq k_0}$ of*

$$
Ex^{k+1} = Ax^k + f^k, \quad k \leq k_0 - 1 \tag{2.68}
$$

satisfies

$$
\begin{aligned}
x^k = \left(A^D E\right)^{k_0-k} A^D A v + (I - A^D A) \sum_{i=0}^{\nu_A - 1} (E^D A)^i E^D f^{k-i-1} \\
- \sum_{j=k+1}^{k_0} (A^D E)^{j-k-1} A^D f^{j-1}
\end{aligned} \tag{2.69}
$$

for $k \leq k_0$ and for some $v \in \mathbb{C}^n$.

3. *Every solution $\{x^k\}_{k\in\mathbb{Z}}$ of*

$$
Ex^{k+1} = Ax^k + f^k, \quad k \in \mathbb{Z} \tag{2.70}
$$

satisfies

$$
\begin{aligned}
x^k =& (I - A^D A) \sum_{i=0}^{\nu_A - 1} (E^D A)^i E^D f^{k-i-1} \\
&+ \begin{cases} A^D A \left((E^D A)^{k-k_0} E^D E v + \sum_{j=k_0}^{k-1} (E^D A)^{k-j-1} E^D f^j \right) & \text{for } k \geq k_0 \\ E^D E \left((A^D E)^{k_0 - k} A^D A v - \sum_{j=k+1}^{k_0} (A^D E)^{j-k-1} A^D f^{j-1} \right) & \text{for } k \leq k_0 \end{cases} \\
&- (I - E^D E) \sum_{i=0}^{\nu_E - 1} (A^D E)^i A^D f^{k+i}
\end{aligned}
\tag{2.71}
$$

for $k \in \mathbb{Z}$ and for some $v \in \mathbb{C}^n$.

Proof. Since the problem is linear any solution may be written as a particular solution of the inhomogeneous problem plus a solution of the homogeneous problem. □

Corollary 2.21. *Let the assumptions of Theorem 2.20 hold. Then the following statements hold.*

1. *The initial value problem consisting of (2.66) and (1.6) possesses a solution if and only if there exists a $v \in \mathbb{C}^n$ with*

$$
\hat{x} = E^D E v - \left(I - E^D E\right) \sum_{i=0}^{\nu_E - 1} \left(A^D E\right)^i A^D f^{k_0 + i}. \tag{2.72}
$$

   If this is the case, then the solution is unique.

2. *The initial value problem consisting of (2.68) and (1.6) possesses a solution if and only if there exists a $v \in \mathbb{C}^n$ with*

$$
\hat{x} = A^D A v + (I - A^D A) \sum_{i=0}^{\nu_A - 1} (E^D A)^i E^D f^{k_0 - i - 1}. \tag{2.73}
$$

   If this is the case, then the solution is unique.

3. *The problem consisting of (2.70) and (1.6) possesses a solution if and only if there exists a $v \in \mathbb{C}^n$ with*

$$
\begin{aligned}
\hat{x} \quad = \quad & (I - A^D A) \sum_{i=0}^{\nu_A - 1} (E^D A)^i E^D f^{k_0 - i - 1} \\
& + A^D A E^D E v \\
& - (I - E^D E) \sum_{i=0}^{\nu_E - 1} (A^D E)^i A^D f^{k_0 + i}.
\end{aligned}
\tag{2.74}
$$

   If this is the case, then the solution is unique.

Remark 2.22. Analogously to Remark 2.15 one might presume that it is meaningful to require that the initial condition satisfies (2.74), even when only calculating into the future.

Following the approach in [8] we now consider the case that (E, A) is regular but E and A do not commute. In this case we need the following Lemma.

Lemma 2.23. *[1] Let $E, A \in \mathbb{C}^{n,n}$ with (E, A) regular. Let $\tilde{\lambda} \in \mathbb{C}$ be chosen such that $\tilde{\lambda}E - A$ is nonsingular. Then the matrices*

$$\tilde{E} = \left(\tilde{\lambda}E - A\right)^{-1} E, \qquad \tilde{A} = \left(\tilde{\lambda}E - A\right)^{-1} A$$

commute.

Proof. We have

$$\begin{aligned}\tilde{\lambda}\tilde{E} - \tilde{A} &= \tilde{\lambda}\left(\tilde{\lambda}E - A\right)^{-1} E - \left(\tilde{\lambda}E - A\right)^{-1} A \\ &= \left(\tilde{\lambda}E - A\right)^{-1}\left(\tilde{\lambda}E - A\right) \\ &= I\end{aligned}$$

and thus

$$\tilde{A} = \tilde{\lambda}\tilde{E} - I.$$

Therefore we finally obtain

$$\begin{aligned}\tilde{E}\tilde{A} &= \tilde{E}\left(\tilde{\lambda}\tilde{E} - I\right) \\ &= \left(\tilde{\lambda}\tilde{E} - I\right)\tilde{E} \\ &= \tilde{A}\tilde{E}.\end{aligned}$$

□

Remark 2.24. [8] Since the factor $\left(\tilde{\lambda}E - A\right)^{-1}$ represents a simple scaling of the descriptor system, results similar to Theorem 2.20 and Corollary 2.21 hold for the general case provided that the coefficient matrices form a regular matrix pair. We only need to perform the replacements

$$E \leftarrow \left(\tilde{\lambda}E - A\right)^{-1} E, \qquad A \leftarrow \left(\tilde{\lambda}E - A\right)^{-1} A, \qquad f \leftarrow \left(\tilde{\lambda}E - A\right)^{-1} f,$$

in Theorem 2.20 and Corollary 2.21.
Also note that condition (2.42) is equivalent to the regularity of a matrix pair (E, A), which satisfies $EA = AE$. Thus, the assumptions of Theorem 2.20 and Corollary 2.21 can essentially be reduced to the regularity of the original matrix pair (E, A).

2.3 Conclusion

In this chapter we have first examined the behavior of a general linear discrete-time descriptor system with constant coefficients, i.e., of a system of the form (1.4), with the help of the Kronecker canonical form. Here we have only considered the case where one has an initial condition at some point $k_0 \in \mathbb{Z}$ and wants to get a solution for all $k \geq k_0$.
Then we concentrated on regular systems, i.e., on systems of the form (1.4) where the matrix pair (E, A) is regular. For such systems we wrote down the explicit solution with the help of the Drazin inverse. In contrast to the continuous-time case one can distinguish between three different cases for such systems. The first case is where one has an initial condition given at point $k_0 \in \mathbb{Z}$ and only wants to get a solution for indices $k \geq k_0$. The second case is where one has an initial condition given at point $k_0 \in \mathbb{Z}$ and only wants to get a solution for indices $k \leq k_0$. These first two cases are closely related, since the first case can be transferred into the second one by a variable substitution. The third case is really different from the first two cases. Here also an initial condition is given at some point $k_0 \in \mathbb{Z}$ but one is looking for a solution for indices $k \geq k_0$ as well as for indices $k \leq k_0$. This puts stronger restrictions on the initial condition, i.e., the set of consistent initial conditions in the third case is smaller than in the first or second case.

Chapter 3

Linear discrete-time descriptor systems with variable coefficients

In this chapter we consider the more general case, where the matrices E and A are allowed to change with time, i.e., system (1.5) is considered. Thus, we give up one restriction and expect the theory to become more complicated. This analysis is done analogously as in [8]. Note that there are other approaches to systems of the form (1.5), for example [12].

3.1 Canonical forms

As in the continuous case (see [8]) the unique solvability of (1.5) and the regularity of all (E_k, A_k) for $k \in \mathbb{K}$ are completely independent in the discrete case. This can be seen by the following three examples.

Example 3.1. Let $\mathbb{K} = \mathbb{N}$ and let the matrix sequences $\{E_k\}_{k\in\mathbb{K}}$, $\{A_k\}_{k\in\mathbb{K}} \subset \mathbb{C}^{2,2}$ be given by

$$\begin{aligned} (E_1, A_1) &= \left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right), \\ (E_i, A_i) &= \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right) \text{ for } i \geq 2. \end{aligned}$$

Then (1.5) is equivalent to

$$\begin{aligned} x^1 &= -f^1, \\ x^{i+1} &= f^i \text{ for } i \geq 2. \end{aligned}$$

Thus, there is no equation for x^2 and it can be chosen arbitrarily.

Example 3.2. From [6] pp.24-25. Let $\mathbb{K} = \mathbb{Z}$ and let the matrix sequences $\{E_k\}_{k\in\mathbb{K}}$, $\{A_k\}_{k\in\mathbb{K}} \subset \mathbb{C}^{2,2}$ be given by

$$(E_k, A_k) = \left(\begin{bmatrix} 0 & 0 \\ -1 & k \end{bmatrix}, \begin{bmatrix} -1 & k-1 \\ 0 & 0 \end{bmatrix} \right) \text{ for } k \in \mathbb{K}.$$

Since

$$\det(\lambda E_k - A_k) = \begin{vmatrix} 1 & 1-k \\ -\lambda & \lambda k \end{vmatrix} = \lambda k - (-\lambda)(1-k) = \lambda,$$

the matrix pair (E_k, A_k) is regular for all $k \in \mathbb{K}$. By defining a sequence $\{x^k\}_{k\in\mathbb{K}}$ through

$$x^k := c^k \begin{bmatrix} k-1 \\ 1 \end{bmatrix},$$

where the scalar sequence $\{c^k\}$ may be chosen arbitrarily, one gets

$$\begin{aligned} E_k x^{k+1} &= \begin{bmatrix} 0 & 0 \\ -1 & k \end{bmatrix} \begin{bmatrix} (k+1)-1 \\ 1 \end{bmatrix} c^{k+1} = 0, \\ A_k x^k &= \begin{bmatrix} -1 & k-1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} k-1 \\ 1 \end{bmatrix} c^k = 0. \end{aligned}$$

Thus, the solution is not unique, even if one defines an initial condition like $x^0 = 0$.

Example 3.3. From [8] pp.56-57. Let $\mathbb{K} = \mathbb{Z}$ and let the matrix sequences $\{E_k\}_{k\in\mathbb{K}}$, $\{A_k\}_{k\in\mathbb{K}} \subset \mathbb{C}^{2,2}$ be given by

$$(E_k, A_k) = \left(\begin{bmatrix} 0 & 0 \\ 1 & -k \end{bmatrix}, \begin{bmatrix} -1 & k \\ 0 & 0 \end{bmatrix} \right) \text{ for } k \in \mathbb{K}.$$

Let $f^k = [f_1^k, f_2^k]^T$ be partitioned conformably with E_k and A_k and let x_1^k and x_2^k be defined according to Table 1.1. Then (1.5) implies

$$\begin{aligned} 0 &= -x_1^k + kx_2^k + f_1^k, \\ x_1^{k+1} - kx_2^{k+1} &= f_2^k, \end{aligned}$$

which implies that

$$\begin{aligned} -x_1^{k+1} + (k+1)x_2^{k+1} &= -f_1^{k+1} \\ \Rightarrow -x_1^{k+1} + kx_2^{k+1} &= -x_2^{k+1} - f_1^{k+1} \\ \Rightarrow f_2^k &= x_2^{k+1} + f_1^{k+1} \\ \Rightarrow x_2^{k+1} &= f_2^k - f_1^{k+1} \\ \Rightarrow x_1^{k+1} &= f_2^k + kf_2^k - kf_1^{k+1}. \end{aligned}$$

Thus, the solution is uniquely determined by the sequence $\{f^k\}_{k\in\mathbb{K}}$ although all (E_k, A_k) are singular.

Considering an arbitrary discrete-time descriptor system with variable coefficients and an initial condition we see that

$$\begin{aligned} E_k x^{k+1} = A_k x^k + f^k,\ k \in \mathbb{Z}, &\qquad x^0 = \hat{x} \\ \Leftrightarrow P_k E_k Q_{k+1} Q_{k+1}^{-1} x^{k+1} = P_k A_k Q_k \underbrace{Q_k^{-1} x^k}_{=:\tilde{x}^k} + P_k f^k,\ k \in \mathbb{Z}, &\qquad Q_0^{-1} x^0 = Q_0 \hat{x} \\ \Leftrightarrow P_k E_k Q_{k+1} \tilde{x}^{k+1} = P_k A_k Q_k \tilde{x}^k + P_k f^k,\ k \in \mathbb{Z}, &\qquad \tilde{x}^0 = Q_0 \hat{x}, \end{aligned}$$

as long as all P_k and Q_k are invertible. This leads to the following definition.

Definition 3.4. Two sequences of matrix pairs $\{(E_k, A_k)\}_{k\in\mathbb{K}}$ and $\{(\tilde{E}_k, \tilde{A}_k)\}_{k\in\mathbb{K}}$ with $E_k, A_k, \tilde{E}_k, \tilde{A}_k \in \mathbb{C}^{m,n}$ are called *globally equivalent* (on $\mathbb{K}$) if there exist two pointwise nonsingular matrix sequences

$$\begin{aligned} \{P_k\}_{k\in\mathbb{K}} \quad & \text{with } P_k \in \mathbb{C}^{m,m}, \\ \{Q_k\}_{k\in\mathbb{K}\cup\{k_f+1\}} \quad & \text{with } Q_k \in \mathbb{C}^{n,n}, \end{aligned}$$

such that

$$P_k E_k Q_{k+1} = \tilde{E}_k \quad \text{and} \quad P_k A_k Q_k = \tilde{A}_k, \tag{3.1}$$

for all $k \in \mathbb{K}$. We denote this by $\{(E_k, A_k)\}_{k\in\mathbb{K}} \sim \{(\tilde{E}_k, \tilde{A}_k)\}_{k\in\mathbb{K}}$.

Lemma 3.5. *The relation introduced in Definition 3.4 is an equivalence relation.*

Proof. Reflexivity: We have $\{(E_k, A_k)\}_{k\in\mathbb{K}} \sim \{(E_k, A_k)\}_{k\in\mathbb{K}}$ with $P_k = I_m$ and $Q_k = I_n$ for all $k \in \mathbb{K}$ and $Q_{k_f+1} = I_n$.
Symmetry: From $\{(E_k, A_k)\} \sim \{(\tilde{E}_k, \tilde{A}_k)\}$, it follows that

$$P_k E_k Q_{k+1} = \tilde{E}_k \quad \text{and} \quad P_k A_k Q_k = \tilde{A}_k,$$

with nonsingular P_k, Q_k. Hence,

$$E_k = P_k^{-1} \tilde{E}_k Q_{k+1}^{-1} \quad \text{and} \quad A_k = P_k^{-1} \tilde{A}_k Q_k^{-1}.$$

Transitivity: From $\{(E_k, A_k)\} \sim \{(\tilde{E}_k, \tilde{A}_k)\}$ and $\{(\tilde{E}_k, \tilde{A}_k)\} \sim \{(\hat{E}_k, \hat{A}_k)\}$ it follows that

$$\begin{aligned} P_k E_k Q_{k+1} = \tilde{E}_k \quad & \text{and} \quad P_k A_k Q_k = \tilde{A}_k, \\ \tilde{P}_k \tilde{E}_k \tilde{Q}_{k+1} = \hat{E}_k \quad & \text{and} \quad \tilde{P}_k \tilde{A}_k \tilde{Q}_k = \hat{A}_k, \end{aligned}$$

for all $k \in \mathbb{K}$, where all Q_k, $\tilde{Q}_k$, P_k, $\tilde{P}_k$, are nonsingular. From this one immediately sees that $\{(E_k, A_k)\} \sim \{(\hat{E}_k, \hat{A}_k)\}$ using the matrix sequences $\tilde{P}_k P_k$ and $Q_k \tilde{Q}_k$. ☐

Definition 3.6. Two pairs of matrices (E, A), $(\tilde{E}, \tilde{A}) \in \mathbb{C}^{m,n}$ are called *locally equivalent* if there exist matrices $P \in \mathbb{C}^{m,m}$ and $Q, R \in \mathbb{C}^{n,n}$ that are all nonsingular, such that

$$\tilde{E} = PEQ \quad \text{and} \quad \tilde{A} = PAR. \tag{3.2}$$

Again, we denote this by $(E, A) \sim (\tilde{E}, \tilde{A})$.

Lemma 3.7. *The relation introduced in Definition 3.6 is an equivalence relation.*

Proof. By using Lemma 3.5 in the special case $\mathbb{K} = \{1\}$ we immediately obtain the assumption. ☐

For the proof of Theorem 3.9 we recall the notion of the echelon form used here.

Lemma 3.8. *Let* $A \in \mathbb{C}^{m,n}$ *be a matrix with* $\operatorname{rank}(A) = r$. *Then there exist invertible matrices* $P \in \mathbb{C}^{m,m}$ *and* $Q \in \mathbb{C}^{n,n}$ *such that*

$$PAQ = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \tag{3.3}$$

is in echelon form.

For convenience, we say in the following that a matrix is a basis of a vector space if this is valid for its columns. For matrix pairs of block matrices we also use the convention that corresponding blocks (i.e., blocks in the same block row and block column) have the same number of rows and columns.

Theorem 3.9. *Let* $E, A \in \mathbb{C}^{m,n}$ *and introduce the following matrices:*

$$\begin{aligned} &Z \quad \textit{basis of} \quad \operatorname{corange}(E) = \operatorname{kernel}\left(E^H\right), && (3.4)\\ &Y \quad \textit{basis of} \quad \operatorname{corange}(A) = \operatorname{kernel}\left(A^H\right). && (3.5) \end{aligned}$$

Then, the quantities

$$\begin{aligned}
r_f &= \operatorname{rank}(E), && \textit{(rank of E; corresponds to forward direction)} && (3.6a)\\
r_b &= \operatorname{rank}(A), && \textit{(rank of A; corresponds to backward direction)} && (3.6b)\\
h_f &= \operatorname{rank}\left(Z^H A\right), && \textit{(rank of } Z^H A\textit{; forward direction)} && (3.6c)\\
h_b &= \operatorname{rank}\left(Y^H E\right) && && \\
&= r_f + h_f - r_b, && \textit{(rank of } Y^H E\textit{; backward direction)} && (3.6d)\\
c &= r_b - h_f, && \textit{(common part)} && (3.6e)\\
a &= \min(h_f, n - r_f), && \textit{(algebraic part)} && (3.6f)\\
s &= h_f - a, && \textit{(strangeness)} && (3.6g)\\
d &= r_f - c - s, && \textit{(differential part)} && (3.6h)\\
u &= n - r_f - a, && \textit{(undetermined variables)} && (3.6i)\\
v &= m - r_f - h_f, && \textit{(vanishing equations)} && (3.6j)
\end{aligned}$$

are invariant under (3.2), and (E, A) *is locally equivalent to the canonical form*

$$(\tilde{E}, \tilde{A}) = \left(\begin{bmatrix} I_s & 0 & 0 & 0 & 0 \\ 0 & I_d & 0 & 0 & 0 \\ 0 & 0 & I_c & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I_c & 0 & 0 \\ 0 & 0 & 0 & I_a & 0 \\ I_s & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right). \tag{3.7}$$

We have that either $s = 0$, $u = 0$ *or* $s = u = 0$. *The quantities (3.6) are called* local characteristics *or* local invariants *of the matrix pair* (E, A).

Proof. Let (E_i, A_i), $i = 1, 2$, be locally equivalent, i.e., let P, Q, R be invertible matrices of appropriate size such that

$$E_2 = PE_1Q \quad \text{and} \quad A_2 = PA_1R. \tag{3.8}$$

Since

$$\begin{aligned} \operatorname{rank}(E_2) &= \operatorname{rank}(PE_1Q) = & \operatorname{rank}(E_1), \\ \operatorname{rank}(A_2) &= \operatorname{rank}(PA_1R) = & \operatorname{rank}(A_1), \end{aligned}$$

it follows that r_f and r_b are invariant under local equivalence. First note that h_f is independent of the choice of the basis of Z. To see this let Z and $\tilde{Z}$ be two bases of $\operatorname{corange}(E)$. Then there exists a regular matrix M_Z with

$$\tilde{Z} = ZM_Z.$$

Then, from

$$\operatorname{rank}\left(\tilde{Z}^H A\right) = \operatorname{rank}\left({M_Z}^H Z^H A\right) = \operatorname{rank}\left(Z^H A\right)$$

the statement follows.
Let Z_2 be a basis of $\operatorname{corange}(E_2)$, i.e $\operatorname{kernel}\left({E_2}^H\right) = \operatorname{range}(Z_2)$. Then $Z_1 := P^H Z_2$ is a basis of $\operatorname{corange}(E_1)$, since Z_1 has full column rank and

$$\operatorname{kernel}\left({E_1}^H\right) = \operatorname{range}(Z_1).$$

To prove this, note that

$$\begin{aligned} x &\in \operatorname{kernel}\left({E_1}^H\right) \\ \Rightarrow 0 &= {E_1}^H x = Q^{-H}{E_2}^H P^{-H} x \\ \Rightarrow 0 &= {E_2}^H P^{-H} x \\ \Rightarrow P^{-H} x &\in \operatorname{kernel}\left({E_2}^H\right) = \operatorname{range}(Z_2) \\ \Rightarrow \text{there exists a } z \text{ such that } P^{-H} x &= Z_2 z \\ \Rightarrow x &= P^H Z_2 z = Z_1 z \in \operatorname{range}(Z_1). \end{aligned}$$

Conversely, we have

$$\begin{aligned} x &\in \operatorname{range}(Z_1) \\ \Rightarrow \text{there exists a } z \text{ such that } x &= Z_1 z = P^H Z_2 z \\ \Rightarrow P^{-H} x &= Z_2 z \in \operatorname{range}(Z_2) = \operatorname{kernel}\left({E_2}^H\right) \\ \Rightarrow 0 &= {E_2}^H P^{-H} x = Q^H {E_1}^H P^H P^{-H} x = Q^H {E_1}^H x \\ \Rightarrow 0 &= {E_1}^H x \\ \Rightarrow x &\in \operatorname{kernel}\left({E_1}^H\right). \end{aligned}$$

This implies

$$\operatorname{rank}\left({Z_2}^H A_2\right) = \operatorname{rank}\left({Z_2}^H P A_1 Q\right) = \operatorname{rank}\left({Z_1}^H A_1 Q\right) = \operatorname{rank}\left({Z_1}^H A_1\right),$$

which shows that also h_f is invariant under local equivalence. By exchanging the roles of A and E as well as the roles of Z and Y in the previous argument we also see that h_b (defined as $\operatorname{rank}\left(Y^H E\right)$) is invariant under local equivalence. Since all other quantities are functions of those first three invariant quantities all quantities are invariant under local equivalence.
Let Z' be a matrix, such that the composed matrix $\begin{bmatrix} Z' & Z \end{bmatrix}$ is invertible. Also let X, Y be matrices, such that $Y\left((Z')^H E\right) X = \begin{bmatrix} I_{r_f} & 0 \end{bmatrix}$ is in echelon form (3.3). Note that $\left((Z')^H E\right)$ has full row rank, since Z' spans the vector space $\operatorname{range}(E)$. Thus, the following equivalence transformations in the sense of Definition 3.6 can be applied to the matrix pair (E, A) (where all corresponding blocks of both matrices have the same size).

$$\begin{aligned}
(E, A) &\sim \left(\begin{bmatrix} (Z')^H E \\ 0 \end{bmatrix}, \begin{bmatrix} (Z')^H A \\ Z^H A \end{bmatrix} \right) \\
&\sim \left(\begin{bmatrix} \begin{bmatrix} I_{r_f} & 0 \end{bmatrix} \\ 0 \end{bmatrix}, \begin{bmatrix} Y (Z')^H A \\ Z^H A \end{bmatrix} \right) \\
&\sim \left(\begin{bmatrix} I_s & 0 & 0 & 0 \\ 0 & I_{d+c} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & I_a & 0 \\ I_s & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right).
\end{aligned} \tag{3.9}$$

Looking at matrix pair (3.9) one can (because of the equivalence relation (3.2)) permute the columns of the second matrix without permuting the columns of the first matrix. This shows that the size of the I_a block can be increased by decreasing the size of the I_s block (as long as the last block column does not vanish) and, conversely, that the size of the I_s block can be increased at the expense of the size of the I_a block (as long as the second block column does not vanish). To get a canonical form we choose the I_a block to be of maximum size. Hence, it is clear that either the last block column vanishes, i.e., $u = 0$, or that there is no I_s, i.e., $s = 0$ (or even $s = u = 0$), as stated in the assertion. These cases are considered separately in the following.
If $u = 0$, then (3.9) reads as

$$\begin{aligned}
(E, A) &\sim \left(\begin{bmatrix} I_s & 0 & 0 \\ 0 & I_{d+c} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} * & * & * \\ * & * & * \\ 0 & 0 & I_a \\ I_s & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) \\
&\sim \left(\begin{bmatrix} I_s & 0 & 0 \\ 0 & I_{d+c} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & * & 0 \\ 0 & * & 0 \\ 0 & 0 & I_a \\ I_s & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right)
\end{aligned}$$

$$\sim \left(\begin{bmatrix} I_s & 0 & 0 & 0 \\ 0 & I_d & 0 & 0 \\ 0 & 0 & I_c & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I_c & 0 \\ 0 & 0 & 0 & I_a \\ I_s & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right).$$

In this case we have $h_f = s + a \geq a = n - r_f$, which is consistent with (3.6f).
If $s = 0$, then (3.9) reads as

$$\begin{aligned} (E, A) &\sim \left(\begin{bmatrix} I_{d+c} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} * & * & * \\ 0 & I_a & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) \\ &\sim \left(\begin{bmatrix} I_{d+c} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} * & 0 & * \\ 0 & I_a & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) \\ &\sim \left(\begin{bmatrix} I_d & 0 & 0 & 0 \\ 0 & I_c & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & I_c & 0 & 0 \\ 0 & 0 & I_a & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right). \end{aligned}$$

In this case we have $h_f = a = n - r_f - u \leq n - r_f$ which is also consistent with (3.6f).
Finally, the identity (3.6d) can be derived from the canonical form (3.7). Therefore, let $\tilde{Z}$ and $\tilde{Y}$ be bases of corange$\left(\tilde{E}\right)$ and corange$\left(\tilde{A}\right)$, where $\tilde{E}$ and $\tilde{A}$ are the matrices in the canonical form (3.7), respectively. Then we have

$$\tilde{Y} = \begin{bmatrix} I_s & 0 & 0 \\ 0 & I_d & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_v \end{bmatrix}.$$

From this we see, that

$$\begin{aligned} \operatorname{rank}\left(\tilde{Y}^H \tilde{E}\right) &= \operatorname{rank} \left(\begin{bmatrix} I_s & 0 & 0 & 0 & 0 & 0 \\ 0 & I_d & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_v \end{bmatrix} \begin{bmatrix} I_s & 0 & 0 & 0 & 0 \\ 0 & I_d & 0 & 0 & 0 \\ 0 & 0 & I_c & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right) \\ &= s + d \overset{(3.6h)}{=} s + r_f - c - s \\ &= r_f - c \overset{(3.6e)}{=} r_f - r_b + h_f, \end{aligned}$$

which shows (3.6d). ☐

Note that the form (3.7) can also be obtained by first reducing (E, A) to Kronecker canonical form and then applying further local equivalence transformations. Doing so shows that if $s > 0$, then there has to be at least one real Kronecker block of the form (2.3), i.e., a Kronecker block with dimension greater or equal to 2×1. Anyway, s may be zero even when there is any number of Kronecker blocks of the form (2.3), i.e., the presence of Kronecker blocks of the form (2.3) is necessary for $s > 0$ but not sufficient.
Comparing this result to the analogous result from [8] (Theorem 3.7) one notices the additional "common" part. This part cannot be eliminated, since with the equivalence relation (3.2) the matrix A may not be changed by means of the matrix E.

3.2 Forward global canonical form

Like in the constant rank case we first study the case where one starts at some time point (here this time point is always $k = 0$) and calculates into the future, i.e., one tries to get a solution for $k \geq 0$. In order to derive a global canonical form, some constant rank assumptions are introduced. Milder assumptions are necessary in this case, than in the case where one wants to get a solution for all $k \in \mathbb{Z}$. Despite the issue that we only want to get a solution for $k \geq 0$ we may also consider linear descriptor systems with equations for all $k \in \mathbb{Z}$ (i.e., systems of the form $\{(E_k, A_k)\}_{k\in\mathbb{Z}}$), since this simplifies moving to the case where we want to get a solution for all $k \in \mathbb{Z}$. This is no restriction, since every linear descriptor system of the form $\{(E_k, A_k)\}_{k\in\mathbb{N}_0}$ can be extended to one of the form $\{(E_k, A_k)\}_{k\in\mathbb{Z}}$ by choosing $E_k = E_0$ and $A_k = A_0$ for all $k < 0$.
Note that we use here the term canonical form in a way that differs from the terminology of abstract algebra.

Lemma 3.10. *Consider system (1.5) and introduce the matrix sequence $\{Z_k\}_{k\in\mathbb{K}}$ where*

$$Z_k \quad \textit{is a basis of} \quad \operatorname{corange}(E_k) = \operatorname{kernel}\left(E_k^H\right) \textit{ for all } k \in \mathbb{K}. \tag{3.10}$$

Let

$$\begin{aligned} r_f^k &= \operatorname{rank}(E_k)\ ,\ k \in \mathbb{K}, & (3.11a)\\ r_b^k &= \operatorname{rank}(A_k)\ ,\ k \in \mathbb{K}, & (3.11b)\\ h_f^k &= \operatorname{rank}\left({Z_k}^H A_k\right)\ ,\ k \in \mathbb{K}, & (3.11c) \end{aligned}$$

be the local characteristics of each matrix pencil (E_k, A_k) with $k \in \mathbb{K}$. Then, these characteristic sequences are invariant under global equivalence (3.1).
Assume further that the two local characteristic sequences

$$r_f \equiv r_f^k \textit{ and } h_f \equiv h_f^k \tag{3.12}$$

are constant for all $k \in \mathbb{K}$. Then the sequence of matrix pairs $\{(E_k, A_k)\}_{k\in\mathbb{K}}$ is globally equivalent to the sequence

$$\left\{\left(\begin{bmatrix} E_k^{(1)} & E_k^{(2)} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A_k^{(1)} & 0 \\ 0 & I_{h_f} \\ 0 & 0 \end{bmatrix}\right)\right\}_{k\in\mathbb{K}}, \tag{3.13}$$

where all matrices $[\, E_k^{(1)}\; E_k^{(2)}\,]$ *have full row rank, i.e., they all are of rank* r_f.

Proof. The invariance of the local characteristics follows directly from Theorem 3.9. Let

$$Z'_k \text{ be a basis of range}\,(E_k) \text{ for all } k \in \mathbb{K}.$$

Then $[Z'_k\; Z_k]$ is invertible for all $k \in \mathbb{K}$ and ${Z'_k}^H E_k$ has full row rank r_f. Transforming with the transpose of this sequence from the left yields

$$\begin{aligned}
\{(E_k, A_k)\}_{k\in\mathbb{K}} \;&\sim\; \left\{\left(\begin{bmatrix} {Z'_k}^H E_k \\ 0 \end{bmatrix}, \begin{bmatrix} {Z'_k}^H A_k \\ {Z_k}^H A_k \end{bmatrix}\right)\right\}_{k\in\mathbb{K}} \\
&\sim\; \left\{\left(\begin{bmatrix} E_k^{(1)} & E_k^{(2)} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A_k^{(1)} & A_k^{(2)} \\ 0 & I_{h_f} \\ 0 & 0 \end{bmatrix}\right)\right\}_{k\in\mathbb{K}} \\
&\sim\; \left\{\left(\begin{bmatrix} E_k^{(1)} & E_k^{(2)} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A_k^{(1)} & 0 \\ 0 & I_{h_f} \\ 0 & 0 \end{bmatrix}\right)\right\}_{k\in\mathbb{K}}.
\end{aligned}$$

□

Writing down the equations from (1.5) connected with the form (3.13) one obtains

$$\begin{aligned}
E_k^{(1)} x_1^{k+1} + E_k^{(2)} x_2^{k+1} &= A_k^{(1)} x_1^k + f_1^k, \\
0 &= x_2^k + f_2^k, \\
0 &= f_3^k.
\end{aligned}$$

Assuming that $\mathbb{K} = \mathbb{N}_0$, this system is equivalent to

$$\begin{aligned}
E_k^{(1)} x_1^{k+1} &= A_k^{(1)} x_1^k + \tilde{f}_1^k, \\
0 &= x_2^k + f_2^k, \\
0 &= f_3^k.
\end{aligned}$$

(where $\tilde{f}_1^k = f_1^k + E_k^{(2)} f_2^{k+1}$) which is connected with the sequence of matrix pairs

$$\left\{\left(\begin{bmatrix} E_k^{(1)} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A_k^{(1)} & 0 \\ 0 & I_{h_f} \\ 0 & 0 \end{bmatrix}\right)\right\}. \tag{3.14}$$

Since this step is reversible the set of solution sequences is not altered. One also may notice that the new right hand side $\tilde{f}^k$ can depend on the right hand side of the former next right hand side f^{k+1}. Thus, one can view this step as an index reduction.
Analogous to [8] Theorem 3.14, in the following Theorem it is shown, that the so obtained reduced sequences of matrix pairs are still globally equivalent, if the original sequences have been.

Theorem 3.11. *Assume that the sequences of matrix pairs*

$$\{(E_k, A_k)\}_{k\in\mathbb{Z}} = \left\{ \left(\begin{bmatrix} E_k^{(1)} & E_k^{(2)} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A_k^{(1)} & 0 \\ 0 & I_{h_f} \\ 0 & 0 \end{bmatrix} \right) \right\}_{k\in\mathbb{Z}}$$

and

$$\{(\tilde{E}_k, \tilde{A}_k)\}_{k\in\mathbb{Z}} = \left\{ \left(\begin{bmatrix} \tilde{E}_k^{(1)} & \tilde{E}_k^{(2)} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} \tilde{A}_k^{(1)} & 0 \\ 0 & I_{h_f} \\ 0 & 0 \end{bmatrix} \right) \right\}_{k\in\mathbb{Z}}$$

are globally equivalent on $\mathbb{Z}$ and in the form (3.13). In particular, suppose that (3.12) holds and that all $\left[E_k^{(1)}\, E_k^{(2)}\right]$ and all $\left[\tilde{E}_k^{(1)}\, \tilde{E}_k^{(2)}\right]$ have full row rank r_f. Then the sequences of matrix pairs $\{(E_k^{(1)}, A_k^{(1)})\}$ and $\{(\tilde{E}_k^{(1)}, \tilde{A}_k^{(1)})\}$ are also globally equivalent on $\mathbb{Z}$.

Proof. By assumption, there exist two pointwise nonsingular matrix sequences

$$\begin{aligned} \{P_k\}_{k\in\mathbb{Z}} &\subset \mathbb{C}^{n,n}, \\ \{Q_k\}_{k\in\mathbb{Z}} &\subset \mathbb{C}^{m,m}, \end{aligned}$$

such that

$$P_k E_k = \tilde{E}_k Q_{k+1}, \tag{3.15a}$$
$$P_k A_k = \tilde{A}_k Q_k, \tag{3.15b}$$

for all $k \in \mathbb{Z}$. By partitioning the transforming matrices appropriately we get

$$\begin{aligned}
\tilde{A}_k Q_k &= \begin{bmatrix} \tilde{A}_k^{(1)} & 0 \\ 0 & I_{h_f} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} Q_k^{(1,1)} & Q_k^{(1,2)} \\ Q_k^{(2,1)} & Q_k^{(2,2)} \end{bmatrix} \\
&= \begin{bmatrix} \tilde{A}_k^{(1)} Q_k^{(1,1)} & \tilde{A}_k^{(1)} Q_k^{(1,2)} \\ Q_k^{(2,1)} & Q_k^{(2,2)} \\ 0 & 0 \end{bmatrix} = \\
P_k A_k &= \begin{bmatrix} P_k^{(1,1)} & P_k^{(1,2)} & P_k^{(1,3)} \\ P_k^{(2,1)} & P_k^{(2,2)} & P_k^{(2,3)} \\ P_k^{(3,1)} & P_k^{(3,2)} & P_k^{(3,3)} \end{bmatrix} \begin{bmatrix} A_k^{(1)} & 0 \\ 0 & I_{h_f} \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} P_k^{(1,1)} A_k^{(1)} & P_k^{(1,2)} \\ P_k^{(2,1)} A_k^{(1)} & P_k^{(2,2)} \\ P_k^{(3,1)} A_k^{(1)} & P_k^{(3,2)} \end{bmatrix},
\end{aligned} \tag{3.16}$$

and

$$\tilde{E}_k Q_{k+1} = \begin{bmatrix} \tilde{E}_k^{(1)} & \tilde{E}_k^{(2)} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} Q_{k+1}^{(1,1)} & Q_{k+1}^{(1,2)} \\ Q_{k+1}^{(2,1)} & Q_{k+1}^{(2,2)} \end{bmatrix}$$

$$
\begin{aligned}
&= \begin{bmatrix} \tilde{E}_k^{(1)} Q_{k+1}^{(1,1)} + \tilde{E}_k^{(2)} Q_{k+1}^{(2,1)} & \tilde{E}_k^{(1)} Q_{k+1}^{(1,2)} + \tilde{E}_k^{(2)} Q_{k+1}^{(2,2)} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = \\
P_k E_k &= \begin{bmatrix} P_k^{(1,1)} & P_k^{(1,2)} & P_k^{(1,3)} \\ P_k^{(2,1)} & P_k^{(2,2)} & P_k^{(2,3)} \\ P_k^{(3,1)} & P_k^{(3,2)} & P_k^{(3,3)} \end{bmatrix} \begin{bmatrix} E_k^{(1)} & E_k^{(2)} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad (3.17) \\
&= \begin{bmatrix} P_k^{(1,1)} E_k^{(1)} & P_k^{(1,1)} E_k^{(2)} \\ P_k^{(2,1)} E_k^{(1)} & P_k^{(2,1)} E_k^{(2)} \\ P_k^{(3,1)} E_k^{(1)} & P_k^{(3,1)} E_k^{(2)} \end{bmatrix}.
\end{aligned}
$$

From (3.16) we obtain that

$$P_k^{(3,2)} = 0 \text{ for all } k \in \mathbb{Z}.$$

Let $p \in \mathbb{C}^{1,r_f}$ be any row of the matrices $P_k^{(2,1)}, P_k^{(3,1)}$. Then from (3.17) we get

$$p[E_k^{(1)} \; E_k^{(2)}] = 0.$$

But since all matrices $[E_k^{(1)} \; E_k^{(2)}]$ have full row rank as stated in Theorem 3.10, it follows that also $p = 0$. Thus, we get that

$$P_k^{(2,1)} = 0, \, P_k^{(3,1)} = 0 \text{ for all } k \in \mathbb{Z},$$

which means that the left transforming matrices take the form

$$P_k = \begin{bmatrix} P_k^{(1,1)} & P_k^{(1,2)} & P_k^{(1,3)} \\ 0 & P_k^{(2,2)} & P_k^{(2,3)} \\ 0 & 0 & P_k^{(3,3)} \end{bmatrix}.$$

Hence, the diagonal matrices $P_k^{(1,1)}$, $P_k^{(2,2)}$, $P_k^{(3,3)}$ have to be nonsingular. Since we also get from (3.16) that

$$Q_k^{(2,1)} = P_k^{(2,1)} A_k^{(1)} = 0,$$

it follows that all matrices $Q_k^{(1,1)}$, $Q_k^{(2,2)}$ are invertible. With this, from (3.17) and (3.16) we finally get that

$$
\begin{aligned}
P_k^{(1,1)} E_k^{(1)} &= \tilde{E}_k^{(1)} Q_{k+1}^{(1,1)} + \tilde{E}_k^{(2)} \underbrace{Q_{k+1}^{(2,1)}}_{=0} \\
&= \tilde{E}_k^{(1)} Q_{k+1}^{(1,1)} \\
&\text{and} \\
P_k^{(1,1)} A_k^{(1)} &= \tilde{A}_k^{(1)} Q_k^{(1,1)},
\end{aligned}
$$

which proves the claim by employing $\{P_k^{(1,1)}\}_{k\in\mathbb{Z}}$ and $\{Q_k^{(1,1)}\}_{k\in\mathbb{Z}}$ as transforming matrix sequences. □

Corollary 3.12. *Under the assumptions of Theorem 3.11 the sequences of matrix pairs*

$$\left\{\left(\begin{bmatrix} E_k^{(1)} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A_k^{(1)} & 0 \\ 0 & I_{h_f} \\ 0 & 0 \end{bmatrix}\right)\right\}$$

and

$$\left\{\left(\begin{bmatrix} \tilde{E}_k^{(1)} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} \tilde{A}_k^{(1)} & 0 \\ 0 & I_{h_f} \\ 0 & 0 \end{bmatrix}\right)\right\}$$

are globally equivalent on $\mathbb{Z}$.

Proof. Using the matrices from the proof of Theorem 3.11 one immediately sees that global equivalence is achieved by employing the sequences of transforming matrices

$$\left\{\begin{bmatrix} P_k^{(1,1)} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}\right\} \text{ and } \left\{\begin{bmatrix} Q^{(1,1)} & 0 \\ 0 & I \end{bmatrix}\right\}.$$

□

Remark 3.13. The preceding results allow for an inductive procedure closely related to the corresponding procedure for continuous-time systems [8]. For an original sequence of matrix pairs $\{(E_k, A_k)\}_{k\in\mathbb{Z}} =: \{(E_{k,0}, A_{k,0})\}_{k\in\mathbb{Z}}$ we define a sequence (of sequences of matrix pairs) $\{\{(E_{k,i}, A_{k,i})\}_{k\in\mathbb{Z}}\}_{i\in\mathbb{N}_0}$ by the following procedure. First we reduce $\{(E_{k,i}, A_{k,i})\}_{k\in\mathbb{Z}}$ by Lemma 3.10 to the from (3.13) assuming that the local invariants $r_f =: r_{f,i}$ and $h_f =: h_{f,i}$ are constant for all matrix pairs on the whole interval $\mathbb{Z}$. Then we reduce the so obtained sequence of matrix pairs to the form (3.14) which yields the next sequence of matrix pairs $\{(E_{k,i+1}, A_{k,i+1})\}_{k\in\mathbb{Z}}$.

This whole iterative process (although derived from [8]) is very similar to Luenberger's shuffle algorithm, which is described in [13] for discrete-time discrete descriptor systems with constant coefficients.
Observe that we have to have the constant rank assumptions of the form (3.12) for every step of the procedure. The so obtained sequence of local invariants (which are also global invariants) $\{(r_{f,i}, h_{f,i})\}_{i\in\mathbb{N}_0}$ is characteristic for a given equivalence class of sequences of matrix pairs, due to Corollary 3.12. Several properties of this sequence are summed up in the following Lemma.

Lemma 3.14. *Let the sequences $\{(r_{f,i}, h_{f,i})\}_{i\in\mathbb{N}_0}$ and $\{\{(E_{k,i}, A_{k,i})\}_{k\in\mathbb{Z}}\}_{i\in\mathbb{N}_0}$ be defined as in Remark 3.13. In particular, let the constant rank assumptions (3.12) hold. Defining the quantities*

$$h_{f,-1} := 0, \tag{3.18a}$$

$$a_i := h_{f,i} - h_{f,i-1} \qquad \forall i \in \mathbb{N}_0, \tag{3.18b}$$

$$\begin{aligned}
d_i &:= r_{f,i} + h_{f,i} && \forall i \in \mathbb{N}_0, && \text{(3.18c)}\\
s_i &:= r_{f,i} - r_{f,i+1} && \forall i \in \mathbb{N}_0, && \text{(3.18d)}\\
w_0 &:= m - r_{f,0} - h_{f,0}, && && \text{(3.18e)}\\
w_i &:= s_{i-1} - a_i && \forall i \in \mathbb{N}_0, && \text{(3.18f)}\\
v_i &:= m - r_{f,i} - h_{f,i} && \forall i \in \mathbb{N}_0, && \text{(3.18g)}
\end{aligned}$$

there exist $\xi, \mu \in \mathbb{N}_0$ so that:

$$\begin{aligned}
r_{f,i} &\geq r_{f,i+1} && \forall i \in \mathbb{N}_0, && \text{(3.19a)}\\
d_i &\geq d_{i+1} && \forall i \in \mathbb{N}_0, && \text{(3.19b)}\\
s_j &= 0 && \forall j \geq \mu, && \text{(3.19c)}\\
h_{f,i} &\leq h_{f,i+1} && \forall i \in \mathbb{N}_0, && \text{(3.19d)}\\
a_i &\geq a_{i+1} && \forall i \in \mathbb{N}_0, && \text{(3.19e)}\\
d_i &\geq h_{f,i} && \forall i \in \mathbb{N}_0, && \text{(3.19f)}\\
a_j &= 0 && \forall j \geq \xi, && \text{(3.19g)}\\
v_i &= v_0 + w_1 + \ldots + w_i && \forall i \in \mathbb{N}_0, && \text{(3.19h)}\\
s_i &\leq a_i && \forall i \in \mathbb{N}_0, && \text{(3.19i)}\\
a_{i+1} &\leq s_i && \forall i \in \mathbb{N}_0, && \text{(3.19j)}\\
s_i &\geq s_{i+1} && \forall i \in \mathbb{N}_0, && \text{(3.19k)}\\
a_i - w_{i+1} &\leq a_{i-1} - w_i && \forall i \in \mathbb{N}, && \text{(3.19l)}\\
a_i, d_i, s_i, w_i, v_i &\geq 0 && \forall i \in \mathbb{N}_0. && \text{(3.19m)}
\end{aligned}$$

For any $\mu \in \mathbb{N}_0$ with the property (3.19c) we also have

$$a_{i+1} = 0,\ r_{f,\mu} = r_{f,i},\ d_\mu = d_i \qquad \forall i \geq \mu. \tag{3.20}$$

Proof. Let $i \in \mathbb{N}_0$ be any non-negative integer. Then we know from Lemma 3.10 that

$$\begin{aligned}
\{(E_{k,i}, A_{k,i})\}_{k\in\mathbb{Z}} &\sim \left\{ \left(\begin{bmatrix} E_{k,i}^{(1)} & E_{k,i}^{(2)} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A_{k,i}^{(1)} & 0 \\ 0 & I_{h_{f,i}} \\ 0 & 0 \end{bmatrix} \right) \right\}_{k\in\mathbb{Z}} \\
\overset{\text{Remark 3.13}}{\Rightarrow} \{(E_{k,i+1}, A_{k,i+1})\}_{k\in\mathbb{Z}} &\sim \left\{ \left(\begin{bmatrix} E_{k,i}^{(1)} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A_{k,i}^{(1)} & 0 \\ 0 & I_{h_{f,i}} \\ 0 & 0 \end{bmatrix} \right) \right\}_{k\in\mathbb{Z}} \\
\Rightarrow r_{f,i} = \operatorname{rank}\left([E_{k,i}^{(1)}\ E_{k,i}^{(2)}]\right) &\geq \operatorname{rank}\left(\begin{bmatrix} E_{k,i}^{(1)} & 0 \end{bmatrix}\right) = r_{f,i+1} \Rightarrow \text{(3.19a)}.
\end{aligned}$$

Since $[E_{k,i}^{(1)}\ E_{k,i}^{(2)}]$ has full row rank $r_{f,i}$, we get

$$\dim\left(\operatorname{range}\left(E_{k,i}^{(1)}\right)\right) + \dim\left(\operatorname{corange}\left(E_{k,i}^{(1)}\right)\right) = r_{f,i}$$

and independent of this

$$\dim\left(\text{range}\left(E_{k,i}^{(1)}\right)\right) = r_{f,i+1}.$$

For $k \in \mathbb{Z}$ let Z_k be a basis of corange $\left(E_{k,i}^{(1)}\right)$. Then we know that

$$h_{f,i+1} = h_{f,i} + \text{rank}\left(Z_k^H A_{k,i}^{(1)}\right) \leq h_{f,i} + \dim\left(\text{corange}\left(E_{k,i}^{(1)}\right)\right). \tag{3.21}$$

Combining these equations yields

$$\begin{aligned} r_{f,i} + h_{f,i} &= \dim\left(\text{range}\left(E_{k,i}^{(1)}\right)\right) + \underbrace{\dim\left(\text{corange}\left(E_{k,i}^{(1)}\right)\right) + h_{f,i}}_{\overset{(3.21)}{\geq} h_{f,i+1}} \\ &\geq r_{f,i+1} + h_{f,i+1} \Rightarrow (3.19\text{b}). \end{aligned}$$

Since the sequence $\{r_{f,i}\}$ is non-increasing and bounded by zero, it becomes stationary at some point μ, which implies (3.19c). From (3.21) one can also see (3.19d). (3.19f) follows from (3.18c).
Now we show via induction that for all $i \in \mathbb{N}_0$ we have

$$\{(E_{k,i}, A_{k,i})\}_{k\in\mathbb{Z}} \sim \left\{\left(\begin{bmatrix} E_{k,i}^{(1)} & E_{k,i}^{(2)} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} A_{k,i}^{(1)} & 0 & 0 \\ 0 & I_{a_i} & 0 \\ 0 & 0 & I_{h_{f,i-1}} \\ 0 & 0 & 0 \end{bmatrix}\right)\right\}_{k\in\mathbb{Z}}, \tag{3.22}$$

where all $[E_{k,i}^{(1)}\ E_{k,i}^{(2)}]$ have full row rank.
Induction basis: $i = 0$
Here $h_{f,-1} = 0$ and because of Lemma 3.10 we get

$$\{(E_{k,0}, A_{k,0})\}_{k\in\mathbb{Z}} = \{(E_k, A_k)\}_{k\in\mathbb{Z}} \sim \left\{\left(\begin{bmatrix} E_{k,0}^{(1)} & E_{k,0}^{(2)} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A_{k,0}^{(1)} & 0 \\ 0 & I_{a_0} \\ 0 & 0 \end{bmatrix}\right)\right\}_{k\in\mathbb{Z}},$$

since $a_0 = h_{f,0}$.
Induction step: $i \to i+1$ with the help of (3.22).
Since (3.22) is a special form of (3.13), one can immediately perform the reduction to (3.14) by

$$\begin{aligned} \{(E_{k,i+1}, A_{k,i+1})\}_{k\in\mathbb{Z}} &\sim \left\{\left(\begin{bmatrix} E_{k,i}^{(1)} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} A_{k,i}^{(1)} & 0 & 0 \\ 0 & I_{a_i} & 0 \\ 0 & 0 & I_{h_{f,i-1}} \\ 0 & 0 & 0 \end{bmatrix}\right)\right\}_{k\in\mathbb{Z}} \\ &\sim \left\{\left(\begin{bmatrix} E_{k,i}^{(1)} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A_{k,i}^{(1)} & 0 \\ 0 & I_{a_i+h_{f,i-1}} \\ 0 & 0 \end{bmatrix}\right)\right\}_{k\in\mathbb{Z}} \end{aligned}$$

$$\sim \left\{ \left(\begin{bmatrix} E^{(1)}_{k,i+1} & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A^{(1)}_{k,i+1} & 0 \\ \hat{A}^{(2)}_{k,i+1} & 0 \\ 0 & I_{h_{f,i}} \\ 0 & 0 \end{bmatrix} \right) \right\}_{k\in\mathbb{Z}},$$

where all $E^{(1)}_{k,i+1}$ have full row rank $r_{f,i+1}$. Adapting the indexing we can proceed with

$$\begin{aligned} \{(E_{k,i+1}, A_{k,i+1})\}_{k\in\mathbb{Z}} &\sim \left\{ \left(\begin{bmatrix} E^{(1)}_{k,i+1} & E^{(2)}_{k,i+1} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} A^{(1)}_{k,i+1} & A^{(2)}_{k,i+1} & 0 \\ 0 & I_{a_{i+1}} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_{h_{f,i}} \\ 0 & 0 & 0 \end{bmatrix} \right) \right\}_{k\in\mathbb{Z}} \\ &\sim \left\{ \left(\begin{bmatrix} E^{(1)}_{k,i+1} & E^{(2)}_{k,i+1} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} A^{(1)}_{k,i+1} & 0 & 0 \\ 0 & I_{a_{i+1}} & 0 \\ 0 & 0 & I_{h_{f,i}} \\ 0 & 0 & 0 \end{bmatrix} \right) \right\}_{k\in\mathbb{Z}}, \end{aligned}$$

which completes the induction. From the form (3.22) we obtain $r_{f,i} - r_{f,i+1} \leq a_i$ for all $i \in \mathbb{N}_0$. From the induction step we can further see that $a_{i+1} = \operatorname{rank}\left(\hat{A}^{(2)}_{k,i+1}\right) \leq r_{f,i} - r_{f,i+1}$ for all $i \in \mathbb{N}_0$, since $\hat{A}^{(2)}_{k,i+1}$ only has $r_{f,i} - r_{f,i+1}$ rows. This shows (3.19e).
Let μ be as in (3.19c). Then (3.22) implies that

$$\{(E_{k,\mu}, A_{k,\mu})\}_{k\in\mathbb{Z}} \sim \left\{ \left(\begin{bmatrix} E^{(1)}_{k,\mu} & E^{(2)}_{k,\mu} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} A^{(1)}_{k,\mu} & 0 & 0 \\ 0 & I_{a_\mu} & 0 \\ 0 & 0 & I_{h_{f,\mu-1}} \\ 0 & 0 & 0 \end{bmatrix} \right) \right\}_{k\in\mathbb{Z}}.$$

Further reduction steps show that for all $j \geq 1$ we have

$$\{(E_{k,\mu+j}, A_{k,\mu+j})\}_{k\in\mathbb{Z}} \sim \left\{ \left(\begin{bmatrix} E^{(1)}_{k,\mu} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A^{(1)}_{k,\mu} & 0 \\ 0 & I_{h_{f,\mu}} \\ 0 & 0 \end{bmatrix} \right) \right\}_{k\in\mathbb{Z}}, \tag{3.23}$$

since $r_{f,\mu+j} = r_{f,\mu}$ which means that all $E^{(1)}_{k,\mu}$ have full row rank. Thus, applying a reduction step to (3.23) does not change anything. This shows that $h_{f,\mu+j} = h_{f,\mu}$ which is equivalent to $a_{\mu+j} = 0$. From this (3.20) follows.
Note that there exists a positive integer ξ for which the sequence $h_{f,i}$ gets stationary, i.e., $h_{f,i} = h_f \in \mathbb{N}_0$ for all $i \geq \xi$, which follows from the boundedness of the sequence ($h_{f,i} \leq m$) and because of (3.19d). This implies (3.19g).
(3.19h) follows since on the one hand we have

$$\begin{aligned} v_i - v_0 &= m - r_{f,i} - h_{f,i} - (m - r_{f,0} - h_{f,0}) \\ &= (r_{f,0} + h_{f,0}) - (r_{f,i} + h_{f,i}), \end{aligned}$$

and on the other hand we have

$$\begin{aligned} w_1 + \ldots + w_i &\overset{(3.18f)}{=} s_0 + \ldots + s_{i-1} - a_1 - \ldots - a_i \\ &\overset{(3.18d)}{=} r_{f,0} - r_{f,i} - (a_1 + \ldots + a_i) \\ &\overset{(3.18b)}{=} r_{f,0} - r_{f,i} - (h_{f,i} - h_{f,0}). \end{aligned}$$

(3.19i) can again be seen from the form (3.22). For this, note that we have

$$\begin{aligned} s_i &\overset{(3.18d)}{=} r_{f,i} - r_{f,i+1} \\ &= \operatorname{rank}\left(\begin{bmatrix} E_{k,i}^{(1)} & E_{k,i}^{(2)} \end{bmatrix}\right) - \operatorname{rank}\left(E_{k,i}^{(1)}\right) \\ &\leq a_i, \end{aligned}$$

where the last inequality holds, since the $E_{k,i}^{(2)}$ matrices only have a_i columns as one can see from the block structure of (3.22). (3.19l) can be obtained in the following way. Let $i \in \mathbb{N}$ be arbitrarily. Then we have that

$$\begin{aligned} & w_i + a_i \overset{(3.18f)}{=} s_{i-1} \overset{(3.19i)}{\leq} a_{i-1} \leq a_{i-1} + w_{i+1} \\ \Rightarrow & a_i - w_{i+1} \leq a_{i-1} - w_i, \end{aligned}$$

where we have used that all w_i are non-negative, which is shown below.
To prove the non-negativity of all constants defined in (3.18) without using (3.19l), first observe that $a_i \geq 0$, since the sequence $\{h_{f,i}\}_{i\geq -1}$ is non-decreasing. All d_i are non-negative, since all $r_{f,i}$ and all $h_{f,i}$ are non-negative. All s_i are non-negative because the sequence $\{r_{f,i}\}$ is non-increasing.
We then see that $v_i = m - d_i$ and thus $\{v_i\}$ is a non-decreasing sequence. Since $v_0 = m - r_{f,0} - h_{f,0} = m - \operatorname{rank}(E_k) - \operatorname{rank}\left(Z_k^H A_k\right) \geq 0$ (where Z_k is a basis of $\operatorname{corange}(E_k)$) all v_i have to be non-negative.
Also note, that for all $i \geq 1$ we have

$$\begin{aligned} v_i - v_{i-1} &= m - r_{f,i} - h_{f,i} - (m - r_{f,i-1} - h_{f,i-1}) \\ &= r_{f,i-1} - r_{f,i} - (h_{f,i} - h_{f,i-1}) \\ &= s_{i-1} - a_i = w_i, \end{aligned} \tag{3.24}$$

which shows the non-negativity of all w_i, since $\{v_i\}$ is non-decreasing.
To finally show (3.19j) and (3.19k) note that for all $i \in \mathbb{N}_0$ we have

$$s_{i+1} \overset{(3.19i)}{\leq} a_{i+1} \overset{(3.19m)}{\leq} a_{i+1} + w_{i+1} \overset{(3.18f)}{=} s_i.$$

□

The previous Lemma 3.14 leads to the following Definition.

Definition 3.15. Let $\{(E_k, A_k)\}_{k\in\mathbb{Z}}$ be a sequence of matrix pairs. Let the sequence $\{(r_{f,i}, h_{f,i})\}_{i\in\mathbb{N}_0}$ (as described in Remark 3.13) be well defined. In particular, let (3.12) hold for every entry $\{(E_{k,i}, A_{k,i})\}_{k\in\mathbb{Z}}$ of the sequence (of sequences of matrix pairs) in Remark 3.13. Then, with the definitions (3.18) we call

$$\mu = \min\{i \in \mathbb{N}_0 \mid s_i = 0\} \tag{3.25}$$

the *strangeness index* of the sequence of matrix pairs $\{(E_k, A_k)\}_{k\in\mathbb{Z}}$ and of the associated descriptor system (1.5). In the case that $\mu = 0$ we call $\{(E_k, A_k)\}_{k\in\mathbb{Z}}$ and (1.5) *strangeness-free.*

In the proof of Lemma 3.14 we were able to see that (under some constant rank assumptions) every sequence of matrix pairs $\{(E_{k,i}, A_{k,i})\}_{k\in\mathbb{Z}}$ is equivalent to a sequence of the form (3.22). With $i = \mu$ we have $s_\mu = 0$ and thus, from (3.23) we see that

$$\{(E_{k,\mu+1}, A_{k,\mu+1})\}_{k\in\mathbb{Z}} \sim \left\{ \left(\begin{bmatrix} E^{(1)}_{k,\mu} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A^{(1)}_{k,\mu} & 0 \\ 0 & I_{h_{f,\mu}} \\ 0 & 0 \end{bmatrix} \right) \right\}_{k\in\mathbb{Z}},$$

with all $E^{(1)}_{k,\mu}$ having full row rank $r_{f,\mu}$. Finally, one can further reduce all the matrices $E^{(1)}_{k,\mu}$ to echelon form (3.3) $[\, I_{r_{f,\mu}}\; 0\,]$ by global equivalence achieving (with adapted indexing)

$$\{(E_{k,\mu+1}, A_{k,\mu+1})\}_{k\in\mathbb{Z}} \sim \left\{ \left(\begin{bmatrix} I_{r_{f,\mu}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} A^{(1)}_{k,\mu} & 0 & A^{(2)}_{k,\mu} \\ 0 & I_{h_{f,\mu}} & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) \right\}_{k\in\mathbb{Z}}, \tag{3.26}$$

which can be regarded as a canonical from. One notices that in general not only μ but $\mu + 1$ reduction steps are necessary to get to the canonical form, although after μ reduction steps a strangeness-free sequence has already been reached. This situation can be avoided by introducing a further constant rank assumption in every step of the reduction process, which was described in Remark 3.13 (see [6]).
Further reductions of (3.26) are possible under additional assumptions.

Theorem 3.16. *Let $\{(E_k, A_k)\}_{k\in\mathbb{Z}} \subset \mathbb{C}^{m,n}$ be a strangeness-free sequence of matrix pairs in the form (3.26), i.e., let*

$$\{(E_k, A_k)\}_{k\in\mathbb{Z}} = \left\{ \left(\begin{bmatrix} I_{r_f} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} A^{(1)}_{k} & 0 & A^{(2)}_{k} \\ 0 & I_{h_f} & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) \right\}_{k\in\mathbb{Z}}. \tag{3.27}$$

Then the following statements hold.

1. *Let $k_0, k_f \in \mathbb{Z}$ with $k_0 \le k_f$. Set $\mathbb{K} := [k_0, k_f] \cap \mathbb{Z}$ and for $k \in \mathbb{K}$ let all $A^{(1)}_k$ be invertible. Then we have*

$$\{(E_k, A_k)\}_{k\in\mathbb{Z}} \sim \left\{ \left(\begin{bmatrix} I_{r_f} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} \tilde{A}^{(1)}_{k} & 0 & \tilde{A}^{(2)}_{k} \\ 0 & I_{h_f} & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) \right\}_{k\in\mathbb{Z}},$$

where $\tilde{A}^{(1)}_k = I_{r_f}$ for $k \in \mathbb{K}$.

2. Let all $A_k^{(1)}$ *have the same constant rank, i.e., let* $\operatorname{rank}\left(A_k^{(1)}\right) = p$ *for all* $k \in \mathbb{Z}$*. Then we have*

$$\{(E_k, A_k)\}_{k\in\mathbb{Z}} \sim \left\{\left(\begin{bmatrix} I_p & 0 & 0 & 0 \\ 0 & I_{r_f-p} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} \tilde{A}_k^{(1)} & \tilde{A}_k^{(2)} & 0 & \tilde{A}_k^{(3)} \\ 0 & 0 & 0 & \tilde{A}_k^{(4)} \\ 0 & 0 & I_{h_f} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\right)\right\}_{k\in\mathbb{Z}},$$

where all $[\,\tilde{A}_k^{(1)}\,,\,\tilde{A}_k^{(2)}\,]$ *have full row rank* p *for all* $k \in \mathbb{Z}$*.*

3. For all $k \in \mathbb{Z}$ *let* $\operatorname{rank}\left(A_k^{(2)}\right) = r_f$*. Then we have*

$$\{(E_k, A_k)\}_{k\in\mathbb{Z}} \sim \left\{\left(\begin{bmatrix} I_{r_f} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & A_k^{(2)} \\ 0 & I_{h_f} & 0 \\ 0 & 0 & 0 \end{bmatrix}\right)\right\}_{k\in\mathbb{Z}}.$$

Proof. In all cases the global equivalence transformations are only applied to the first block row and the first and third block column. Thus, it is sufficient to look at the sequence of matrix pairs

$$\left\{\left(\begin{bmatrix} I_{r_f} & 0 \end{bmatrix}, \begin{bmatrix} A_k^{(1)} & A_k^{(2)} \end{bmatrix}\right)\right\}_{k\in\mathbb{Z}}. \tag{3.28}$$

1. Since the identity matrix I_{r_f} in (3.28) has to be kept, the allowed global equivalence transformations are limited. Effectively, the transformations on $A_k^{(1)}$ that are allowed are given by

   $$A_k^{(1)} \sim P_k A_k^{(1)} P_{k-1}^{-1}. \tag{3.29}$$

   In the following we show by induction that a finite sequence of invertible matrices can be transformed to identity matrices by the equivalence relation (3.29), i.e., we show that for $k_i \leq k_f$ there exist invertible matrices $P_k \in \mathbb{C}^{n,n}$, for $k_0 - 1 \leq k \leq k_i$ with

   $$P_k A_k^{(1)} P_{k-1}^{-1} = I_{r_f} \text{ for all } k_0 \leq k \leq k_i. \tag{3.30}$$

   Induction basis: $k_i = k_0$
   Choose P_{k_0} the inverse of $A_{k_0}^{(1)}$ and $P_{k_0-1} = I_{r_f}$.
   Induction step: $k_i \to k_i + 1 \leq k_f$
   Because of (3.30) there exist invertible matrices $P_{k_0-1}, \ldots, P_{k_i}$ with

   $$\begin{aligned} P_{k_0} A_{k_0}^{(1)} P_{k_0-1}^{-1} &= I_{r_f}, \\ &\vdots \\ P_{k_i} A_{k_i}^{(1)} P_{k_i-1}^{-1} &= I_{r_f}. \end{aligned}$$

By setting $P_{k_i+1} = P_{k_i}(A^{(1)}_{k_i+1})^{-1}$ we obtain the equation

$$P_{k_i+1}A^{(1)}_{k_i+1}P^{-1}_{k_i} = I_{r_f}.$$

Thus, by induction the assertion follows.

2. First reduce all $A^{(1)}_k$ to echelon form (3.3) $P_kA^{(1)}_kQ_k$. Then we have

$$P_kA^{(1)}_k = \begin{bmatrix} \hat{A}^{(1)}_k \\ 0 \end{bmatrix}$$

with $\hat{A}^{(1)}_k$ having full row rank p, since all Q_k were invertible. From this we get with global equivalence

$$\begin{aligned}
\left(\begin{bmatrix} I_{r_f} & 0 \end{bmatrix}, \begin{bmatrix} A^{(1)}_k & A^{(2)}_k \end{bmatrix}\right) &\sim \left(\begin{bmatrix} P_k & 0 \end{bmatrix}, \begin{bmatrix} P_kA^{(1)}_k & P_kA^{(2)}_k \end{bmatrix}\right) \\
&\sim \left(\begin{bmatrix} P_kP_k{}^{-1} & 0 \end{bmatrix}, \begin{bmatrix} P_kA^{(1)}_kP_{k-1}{}^{-1} & P_kA^{(2)}_k \end{bmatrix}\right) \\
&= \left(\begin{bmatrix} I_p & 0 & 0 \\ 0 & I_{r_f-p} & 0 \end{bmatrix}, \begin{bmatrix} \tilde{A}^{(1)}_k & \tilde{A}^{(2)}_k & \tilde{A}^{(3)}_k \\ 0 & 0 & \tilde{A}^{(4)}_k \end{bmatrix}\right),
\end{aligned}$$

with $\hat{A}^{(1)}_kP^{-1}_{k-1} = [\,\tilde{A}^{(1)}_k\,,\ \tilde{A}^{(2)}_k\,]$ having full row rank p for all $k \in \mathbb{Z}$.

3. Without loss of generality we may assume that for all $k \in \mathbb{Z}$ we have $A^{(2)}_k = [\,\tilde{A}^{(2)}_k\,,\ \tilde{A}^{(3)}_k\,]$ with $\tilde{A}^{(2)}_k$ being invertible (otherwise permute columns, which is a global equivalence transformation). Hence it follows that

$$\begin{aligned}
(E_k, A_k) &= \left(\begin{bmatrix} I_{r_f} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} A^{(1)}_k & 0 & \tilde{A}^{(2)}_k & \tilde{A}^{(3)}_k \\ 0 & I_{h_f} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\right) \\
&\sim \left(\begin{bmatrix} I_{r_f} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} A^{(1)}_k & 0 & \tilde{A}^{(2)}_k & \tilde{A}^{(3)}_k \\ 0 & I_{h_f} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} I_{r_f} & 0 & 0 & 0 \\ 0 & I_{h_f} & 0 & 0 \\ -(\tilde{A}^{(2)}_k)^{-1}\tilde{A}^{(1)}_k & 0 & I_{r_f} & 0 \\ 0 & 0 & 0 & I_{n-r_f-a-r_f} \end{bmatrix}\right) \\
&= \left(\begin{bmatrix} I_{r_f} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & A^{(2)}_k \\ 0 & I_{h_f} & 0 \\ 0 & 0 & 0 \end{bmatrix}\right)
\end{aligned}$$

and thus the assertion is proved.

□

Note that in the proof of part 1 of Theorem 3.16, P_{k_i+1} may be changed, while all other elements of the matrix sequence $\{P_k\}$ are not altered. Thus, part 1 of Theorem 3.16 can be extended to the case where either $k_0 = -\infty$ or $k_f = \infty$.

Example 3.17. Consider the constant sequence of matrix pairs consisting of a regular matrix pencil in Kronecker canonical form

$$\{(E_k, A_k)\}_{k\in\mathbb{Z}} := \left\{ \left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \right\}_{k\in\mathbb{Z}}. \tag{3.31}$$

This pencil only has one Kronecker block of size 3 corresponding to the infinite eigenvalue with nilpotency index 3. Using Definition 2.4 pencil (3.31) has Kronecker index 3. To determine the strangeness index we follow the procedure leading to Definition 3.15. Using the sign $\overset{\text{red}}{\sim}$ to denote reducing a sequence of matrix pairs from (3.13) to (3.14) we have that

$$\begin{aligned} &\left\{ \left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \right\}_{k\in\mathbb{Z}} \overset{\text{red}}{\sim} \left\{ \left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \right\}_{k\in\mathbb{Z}} \\ \overset{\text{red}}{\sim} &\left\{ \left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \right\}_{k\in\mathbb{Z}}. \end{aligned}$$

The sequence of characteristic values (along with some of those defined in (3.18)) is thereby given as

$$\begin{aligned} (r_{f,0}, h_{f,0}, a_0, s_0) &= (2,1,1,1), \\ (r_{f,1}, h_{f,1}, a_1, s_1) &= (1,2,1,1), \\ (r_{f,2}, h_{f,2}, a_2, s_2) &= (0,3,1,0), \\ (r_{f,3}, h_{f,3}, a_3, s_3) &= (0,3,0,0), \\ (r_{f,4}, h_{f,4}, a_4, s_4) &= (0,3,0,0), \\ &\vdots \end{aligned}$$

which shows that the strangeness index of this sequence of matrix pairs is 2. Thus, this strangeness index shows the same behavior as the one defined in [8] for constant coefficient matrix pairs.
One also can guess that increasing the order of the nilpotent block will increase the strangeness index accordingly.

Example 3.18. To determine the strangeness index of Example 3.2 we observe that

$$\begin{aligned} &\left\{ \left(\begin{bmatrix} 0 & 0 \\ -1 & k \end{bmatrix}, \begin{bmatrix} -1 & k-1 \\ 0 & 0 \end{bmatrix} \right) \right\}_{k\in\mathbb{Z}} \sim \left\{ \left(\begin{bmatrix} -1 & k \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ -1 & k-1 \end{bmatrix} \right) \right\}_{k\in\mathbb{Z}} \\ \sim &\left\{ \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right) \right\}_{k\in\mathbb{Z}} \sim \left\{ \left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right) \right\}_{k\in\mathbb{Z}} \\ \overset{red}{\sim} &\left\{ \left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right) \right\}_{k\in\mathbb{Z}}. \end{aligned}$$

The sequence of characteristic values is thereby given as

$$
\begin{aligned}
(r_{f,0}, h_{f,0}, a_0, s_0) &= (1, 1, 1, 1), \\
(r_{f,1}, h_{f,1}, a_1, s_1) &= (0, 1, 0, 0), \\
(r_{f,2}, h_{f,2}, a_2, s_2) &= (0, 1, 0, 0), \\
&\vdots
\end{aligned}
$$

showing that the strangeness index of this sequence of matrix pairs is 1.

Example 3.19. To determine the strangeness index of Example 3.3 we observe that

$$
\begin{aligned}
\left\{\left(\begin{bmatrix} 0 & 0 \\ 1 & -k \end{bmatrix}, \begin{bmatrix} -1 & k \\ 0 & 0 \end{bmatrix}\right)\right\}_{k\in\mathbb{Z}} &\sim \left\{\left(\begin{bmatrix} 1 & -k \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ -1 & k \end{bmatrix}\right)\right\}_{k\in\mathbb{Z}} \\
&\sim \left\{\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix}\right)\right\}_{k\in\mathbb{Z}} \\
&\sim \left\{\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right)\right\}_{k\in\mathbb{Z}} \\
&\overset{red}{\sim} \left\{\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right)\right\}_{k\in\mathbb{Z}}.
\end{aligned}
$$

The sequence of characteristic values is thereby given as

$$
\begin{aligned}
(r_{f,0}, h_{f,0}, a_0, s_0) &= (1, 1, 1, 0), \\
(r_{f,1}, h_{f,1}, a_1, s_1) &= (1, 1, 0, 0), \\
(r_{f,2}, h_{f,2}, a_2, s_2) &= (1, 1, 0, 0), \\
&\vdots
\end{aligned}
$$

showing that the strangeness index of this sequence of matrix pairs is 0 although the matrix pairs in this example are all singular while the solution is uniquely determined by the right hand side.

Analogously to [8] one can derive a canonical form for sequences of matrix pairs with well defined strangeness index without the use of shifts. For convenience, we denote unspecified blocks in a matrix by $*$.

Theorem 3.20. *Let the strangeness index μ of the sequence of matrix pairs $\{(E_k, A_k)\}_{k\in\mathbb{Z}}$ as in (3.25) be well defined. Then, with the definitions from (3.18), $\{(E_k, A_k)\}_{k\in\mathbb{Z}}$ is globally equivalent to a sequence of the form*

$$
\left\{\left(\begin{bmatrix} I_{r_{f,\mu}} & 0 & W_k \\ 0 & 0 & F_k \\ 0 & 0 & G_k \end{bmatrix}, \begin{bmatrix} * & * & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_{h_{f,\mu}} \end{bmatrix}\right)\right\}_{k\in\mathbb{Z}}, \tag{3.32}
$$

with

$$F_k = \begin{bmatrix} 0 & F_k^{(\mu)} & & * \\ & \ddots & \ddots & \\ & & \ddots & F_k^{(1)} \\ & & & 0 \end{bmatrix}, \; G_k = \begin{bmatrix} 0 & G_k^{(\mu)} & & * \\ & \ddots & \ddots & \\ & & \ddots & G_k^{(1)} \\ & & & 0 \end{bmatrix}, \tag{3.33}$$

where all $F_k^{(i)}$ and $G_k^{(i)}$ have sizes $w_i \times a_{i-1}$ and $a_i \times a_{i-1}$, respectively, and all $W_k = [\, * \cdots * \,]$ are partitioned accordingly. In particular, all $F_k^{(i)}$ and $G_k^{(i)}$ together have full row rank, i.e.,

$$\operatorname{rank}\left(\begin{bmatrix} F_k^{(i)} \\ G_k^{(i)} \end{bmatrix}\right) = a_i + w_i = s_{i-1} \overset{\text{(3.19i)}}{\leq} a_{i-1} \qquad \forall k \in \mathbb{Z}. \tag{3.34}$$

Proof. With an inductive argument we show that

$$\{(E_k, A_k)\}_{k\in\mathbb{Z}} \sim$$

$$\{(\tilde{E}_{k,i}, \tilde{A}_{k,i})\}_{k\in\mathbb{Z}} := \left\{\left(\begin{bmatrix} E_{k,i}^{(1,1)} & E_{k,i}^{(1,2)} & * & \cdots & * \\ 0 & 0 & F_{k,i} & & * \\ \vdots & \vdots & \ddots & \ddots & \\ \vdots & \vdots & & \ddots & F_{k,1} \\ 0 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & G_{k,i} & & * \\ \vdots & \vdots & \ddots & \ddots & \\ \vdots & \vdots & & \ddots & G_{k,1} \\ 0 & 0 & \cdots & \cdots & 0 \end{bmatrix}, \begin{bmatrix} A_{k,i}^{(1)} & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & & & \vdots \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & \cdots & \cdots & 0 \\ 0 & I_{a_i} & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & I_{a_0} \end{bmatrix}\right)\right\}, \tag{3.35}$$

where:

1. All $[\, E_{k,i}^{(1,1)} \,,\, E_{k,i}^{(1,2)} \,]$ are of full row rank $r_{f,i}$ and $\operatorname{rank}\left(E_{k,i}^{(1,1)}\right) = r_{f,i+1}$.
2. With $Z_{k,i}$ being bases of $\operatorname{corange}\left(E_{k,i}^{(1,1)}\right)$, we have that $\operatorname{rank}\left(Z_{k,i}^H A_{k,i}^{(1)}\right) = a_{i+1}$ for all $k \in \mathbb{Z}$.
3. $\operatorname{rank}\left(\begin{bmatrix} F_{k,j} \\ G_{k,j} \end{bmatrix}\right)$ is full for all $j \in \{1, \ldots, i\}$ and for all $k \in \mathbb{Z}$.

Induction basis: $i = 0$

From Lemma 3.10 we know that

$$\begin{aligned} \{(E_k, A_k)\} &\sim \left\{\left(\begin{bmatrix} E_{k,0}^{(1,1)} & E_{k,0}^{(1,2)} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A_{k,0}^{(1)} & 0 \\ 0 & I_{h_{f,0}} \\ 0 & 0 \end{bmatrix}\right)\right\} \\ &\sim \left\{\left(\begin{bmatrix} E_{k,0}^{(1,1)} & E_{k,0}^{(1,2)} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A_{k,0}^{(1)} & 0 \\ 0 & 0 \\ 0 & I_{a_0} \end{bmatrix}\right)\right\}, \end{aligned}$$

with $[\, E_{k,0}^{(1,1}\, ,\, E_{k,0}^{(1,2)}\,] = r_{f,0}$ from which the first part of 1. immediately follows. If we perform the step from (3.13) to (3.14), then we find that

$$r_{f,1} = \operatorname{rank}\left(E_{k,0}^{(1,1)}\right)$$

holds and thus 1. is shown. To see 2. let $Z_{k,0}$ be bases of corange $\left(E_{k,0}^{(1,1)}\right)$ for all $k \in \mathbb{Z}$. Then

$$\begin{bmatrix} Z_{k,0} & 0 & 0 \\ 0 & I_{w_0} & 0 \\ 0 & 0 & I_{a_0} \end{bmatrix} \text{ is a basis of corange } \left(\begin{bmatrix} E_{k,0}^{(1,1)} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}\right),$$

whereas the latter matrix is the one in (3.14). Thus,

$$h_{f,1} = \operatorname{rank}\left(\begin{bmatrix} Z_{k,0}^H & 0 & 0 \\ 0 & I_{w_0} & 0 \\ 0 & 0 & I_{a_0} \end{bmatrix} \begin{bmatrix} A_{k,0}^{(1)} & 0 \\ 0 & 0 \\ 0 & I_{a_0} \end{bmatrix}\right) = \operatorname{rank}\left(Z_{k,0}^H A_{k,0}^{(1)}\right) + a_0.$$

This gives

$$\operatorname{rank}\left(Z_{k,0}^H A_{k,0}^{(1)}\right) = h_{f,1} - a_0 = h_{f,1} - h_{f,0} = a_1.$$

Induction step: $i \to i+1$ with the help of (3.35)

Applying Lemma 3.10 to the sequence of matrix pairs $\{(E_{k,i}^{(1,1)}, A_{k,i}^{(1)})\}_{k\in\mathbb{Z}}$ (in (3.35)) one obtains that

$$\{(E_{k,i}^{(1,1)}, A_{k,i}^{(1)})\}_{k\in\mathbb{Z}} \sim \left\{ \begin{bmatrix} E_{k,i+1}^{(1,1)} & E_{k,i+1}^{(1,2)} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A_{k,i+1}^{(1)} & 0 \\ 0 & 0 \\ 0 & I_{a_{i+1}} \end{bmatrix} \right\}, \tag{3.36}$$

where $[\, E_{k,i+1}^{(1,1)}\, ,\, E_{k,i+1}^{(1,2)}\,]$ is of full row rank $r_{f,i+1}$ due to part 1. and 2. of the inductive assumption. Applying the transformation corresponding to (3.36) to the original sequence of matrix pairs (3.35) yields that (E_k, A_k) is globally equivalent to

$$\left\{\left(\begin{bmatrix} E_{k,i+1}^{(1,1)} & E_{k,i+1}^{(1,2)} & E_{k,i+1}^{(1,3)} & * & \cdots & * \\ 0 & 0 & E_{k,i+1}^{(2,3)} & * & & * \\ 0 & 0 & E_{k,i+1}^{(3,3)} & * & & * \\ 0 & 0 & 0 & F_{k,i} & & * \\ \vdots & \vdots & \vdots & \ddots & \ddots & \\ \vdots & \vdots & \vdots & & \ddots & F_{k,1} \\ 0 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 0 & G_{k,i} & & * \\ \vdots & \vdots & \vdots & \ddots & \ddots & \\ \vdots & \vdots & \vdots & & \ddots & G_{k,1} \\ 0 & 0 & 0 & \cdots & \cdots & 0 \end{bmatrix}, \begin{bmatrix} A_{k,i+1}^{(1)} & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & I_{a_{i+1}} & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & & & \vdots \\ \vdots & \vdots & \vdots & & & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & I_{a_i} & 0 & \cdots & 0 \\ \vdots & 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 0 & I_{a_0} \end{bmatrix}\right)\right\}$$

$$
\sim \left\{ \left(\begin{bmatrix} E^{(1,1)}_{k,i+1} & E^{(1,2)}_{k,i+1} & E^{(1,3)}_{k,i+1} & * & \cdots & * \\ 0 & 0 & F_{k,i+1} & * & & * \\ 0 & 0 & 0 & F_{k,i} & & * \\ \vdots & \vdots & \vdots & \ddots & \ddots & \\ \vdots & \vdots & \vdots & & \ddots & F_{k,1} \\ 0 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & G_{k,i+1} & * & & * \\ 0 & 0 & 0 & G_{k,i} & & * \\ \vdots & \vdots & \vdots & \ddots & \ddots & \\ \vdots & \vdots & \vdots & & \ddots & G_{k,1} \\ 0 & 0 & 0 & \cdots & \cdots & 0 \end{bmatrix}, \begin{bmatrix} A^{(1)}_{k,i+1} & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & & & \vdots \\ \vdots & \vdots & \vdots & & & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & I_{a_{i+1}} & 0 & 0 & \cdots & 0 \\ 0 & 0 & I_{a_i} & 0 & \cdots & 0 \\ \vdots & 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 0 & I_{a_0} \end{bmatrix} \right) \right\}, \tag{3.37}
$$

by defining $F_{k,i+1} := E^{(2,3)}_{k,i+1}$ and $G_{k,i+1} := E^{(3,3)}_{k,i+1}$. Due to the nature of global equivalence it follows that

$$
\begin{aligned}
r_{f,i} &= \operatorname{rank}\left(\begin{bmatrix} E^{(1,1)}_{k,i} & E^{(1,2)}_{k,i} \end{bmatrix} \right) \\
&= \operatorname{rank}\left(\begin{bmatrix} E^{(1,1)}_{k,i+1} & E^{(1,2)}_{k,i+1} & E^{(1,3)}_{k,i+1} \\ 0 & 0 & E^{(2,3)}_{k,i+1} \\ 0 & 0 & E^{(3,3)}_{k,i+1} \end{bmatrix} \right).
\end{aligned}
$$

With regard to the fact that all $[\, E^{(1,1)}_{k,i+1} \,,\, E^{(1,2)}_{k,i+1} \,]$ also have full rank $r_{f,i+1}$ one observes that

$$
\operatorname{rank}\left(\begin{bmatrix} E^{(2,3)}_{k,i+1} \\ E^{(3,3)}_{k,i+1} \end{bmatrix} \right) = \operatorname{rank}\left(\begin{bmatrix} F_{k,i+1} \\ G_{k,i+1} \end{bmatrix} \right) = r_{f,i} - r_{f,i+1} \overset{(3.18\text{d})}{=} s_i,
$$

which means that part 3 of the inductive assumption is shown. Further, we notice that $G_{k,i+1}$ is in the same block row as $I_{a_{i+1}}$ and in the same block column as I_{a_i} which means that all $G_{k,i+1}$ are of size $a_{i+1} \times a_i$. Since $\begin{bmatrix} E^{(2,3)}_{k,i+1} \\ E^{(3,3)}_{k,i+1} \end{bmatrix}$ has $(r_{f,i} - r_{f,i+1})$ rows, it follows that all $F_{k,i+1}$ are of size $(r_{f,i} - r_{f,i+1} - a_{i+1}) \times a_i \overset{(3.18\text{f})}{=} w_{i+1} \times h_{f,i}$.

Performing $i+1$ reductions from (3.13) to (3.14) for the sequence (3.37) gives

$$
r_{f,i+1} = \operatorname{rank}\left(\begin{bmatrix} E^{(1,1)}_{k,i+1} & E^{(1,2)}_{k,i+1} \end{bmatrix} \right).
$$

Analogously performing $i+2$ reductions from (3.13) to (3.14) on the sequence of matrix pairs (3.37) one finds that

$$
r_{f,i+2} = \operatorname{rank}\left(\begin{bmatrix} E^{(1,1)}_{k,i+1} \end{bmatrix} \right).
$$

This shows part 1. To prove part 2. let $Z_{k,i+1}$ be bases of corange $\left(E_{k,i+1}^{(1,1)}\right)$. Performing again $i+2$ reductions from (3.13) to (3.14) on the sequence (3.37) and denoting the so obtained sequence by $\{(\hat{E}_k, \hat{A}_k)\}_{k\in\mathbb{Z}}$, we have that

$$\hat{Z}_k := \begin{bmatrix} Z_{k,i+1} & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & I_{w_{i+1}} & \ddots & & & & \vdots \\ \vdots & \ddots & \ddots & \ddots & & & \vdots \\ \vdots & & \ddots & I_{w_0} & \ddots & & \vdots \\ \vdots & & & \ddots & I_{a_{i+1}} & \ddots & \vdots \\ \vdots & & & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & I_{a_0} \end{bmatrix}$$

are bases of corange $\left(\hat{E}_k\right)$. Since all $\hat{A}_k$ then only contain the $A_{k,i+1}^{(1)}$ and I_{a_j} block entries, it is clear that

$$h_{f,i+2} = \operatorname{rank}\left(\hat{Z}_k^H \hat{A}_k\right) = \operatorname{rank}\left(Z_{k,i+1} A_{k,i+1}^{(1)}\right) + a_0 + \ldots + a_{i+1} \tag{3.38}$$

$$\Rightarrow \operatorname{rank}\left(Z_{k,i+1} A_{k,i+1}^{(1)}\right) = h_{f,i+2} - a_{i+1} - \ldots - a_0 \tag{3.39}$$

$$\overset{(3.18b)}{\Rightarrow} \operatorname{rank}\left(Z_{k,i+1} A_{k,i+1}^{(1)}\right) = a_{i+2}. \tag{3.40}$$

This proves (3.35). Selecting $i = \mu$ in (3.35) yields

$$\{(E_k, A_k)\} \sim \left\{\left(\begin{bmatrix} E_{k,\mu}^{(1,1)} & E_{k,\mu}^{(1,2)} & * & \cdots & * \\ 0 & 0 & F_{k,\mu} & & * \\ \vdots & \vdots & \ddots & \ddots & \\ \vdots & \vdots & & \ddots & F_{k,1} \\ 0 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & G_{k,\mu} & & * \\ \vdots & \vdots & \ddots & \ddots & \\ \vdots & \vdots & & \ddots & G_{k,1} \\ 0 & 0 & \cdots & \cdots & 0 \end{bmatrix}, \begin{bmatrix} A_{k,\mu}^{(1)} & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & & & \vdots \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & \cdots & \cdots & 0 \\ 0 & I_{a_\mu} & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & I_{a_0} \end{bmatrix}\right)\right\}$$

$$\overset{(3.18b)}{=} \left\{\left(\begin{bmatrix} E_{k,\mu}^{(1,1)} & \tilde{W}_k \\ 0 & F_k \\ 0 & G_k \end{bmatrix}, \begin{bmatrix} * & 0 \\ 0 & 0 \\ 0 & I_{h_{f,\mu}} \end{bmatrix}\right)\right\}.$$

By induction we obtain that all $\begin{bmatrix} E_{k,\mu}^{(1,1)} & E_{k,\mu}^{(1,2)} \end{bmatrix}$ have full row rank $r_{f,\mu}$ and that all $E_{k,\mu}^{(1,1)}$ have rank $r_{f,\mu+1}$. Further, from (3.20) we know that $r_{f,\mu} = r_{f,\mu+1}$. Thus, all $E_{k,\mu}^{(1,1)}$ have to have full row rank $r_{f,\mu}$ and one may reduce each of these matrices to echelon form (3.3) to obtain the form (3.32). □

Further reduction of the F_k and G_k blocks in (3.32) are possible.

Corollary 3.21. *Let the strangeness index μ of the sequence of matrix pairs $\{(E_k, A_k)\}_{k\in\mathbb{Z}}$ as in (3.25) be well defined. Then, with the definitions from (3.18), $\{(E_k, A_k)\}_{k\in\mathbb{Z}}$ is globally equivalent to a sequence of the form*

$$\left\{\left(\begin{bmatrix} I_{r_{f,\mu}} & 0 & W_k \\ 0 & 0 & C_k \\ 0 & 0 & D_k \end{bmatrix}, \begin{bmatrix} * & * & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_{h_{f,\mu}} \end{bmatrix}\right)\right\}_{k\in\mathbb{Z}}, \tag{3.41}$$

with

$$C_k = \begin{bmatrix} 0 & 0 & I_{w_\mu} & 0 & & & & & \\ & & & & I_{w_{\mu-1}} & 0 & & & \\ & & & & & & \ddots & & \\ & & & & & & & I_{w_1} & 0 \\ & & & & & & & 0 & 0 \end{bmatrix}, \tag{3.42}$$

$$D_k = \begin{bmatrix} 0 & 0 & 0 & D_k^{(\mu)} & 0 & * & \cdots & 0 & * \\ 0 & 0 & 0 & E_k^{(\mu)} & 0 & * & \cdots & 0 & * \\ & & 0 & 0 & 0 & D_k^{(\mu-1)} & \cdots & 0 & * \\ & & 0 & 0 & 0 & E_k^{(\mu-1)} & \cdots & 0 & * \\ & & & & & & \ddots & \vdots & \vdots \\ & & & & & & & 0 & D_k^{(1)} \\ & & & & & & & 0 & E_k^{(1)} \\ & & & & & & & 0 & 0 \\ & & & & & & & 0 & 0 \end{bmatrix}, \tag{3.43}$$

where all $D_k^{(i)}$ and $E_k^{(i)}$ have sizes $w_{i+1} \times (a_{i-1} - w_i)$ and $(a_i - w_{i+1}) \times (a_{i-1} - w_i)$, respectively, and all $W_k = [* \cdots *]$ are partitioned accordingly. In particular, all $D_k^{(i)}$ and $E_k^{(i)}$ together have full row rank, i.e.,

$$\operatorname{rank}\left(\begin{bmatrix} D_k^{(i)} \\ E_k^{(i)} \end{bmatrix}\right) = w_{i+1} + (a_i - w_{i+1}) = a_i \qquad \forall k \in \mathbb{Z}. \tag{3.44}$$

The diagonal blocks in D_k all are square matrices.

Proof. The only differences between the forms (3.32) and (3.41) are in the $(2,3)$ and $(3,3)$ blocks. We can transform these blocks without affecting the other structure (the W_k block is affected, but this does not matter, since its structure is not used). Thus, it is sufficient to only consider the sequence of matrix pairs which is built of the blocks mentioned before.

From Theorem 3.20 we know that

$$\left\{\left(\begin{bmatrix}F_k\\G_k\end{bmatrix},\begin{bmatrix}0\\I_{h_{f,\mu}}\end{bmatrix}\right)\right\} = \left\{\left(\begin{bmatrix} 0 & F_k^{(\mu)} & & * \\ & \ddots & \ddots & \\ & & \ddots & F_k^{(1)} \\ & & & 0 \\ 0 & G_k^{(\mu)} & & * \\ & \ddots & \ddots & \\ & & \ddots & G_k^{(1)} \\ & & & 0\end{bmatrix}, \begin{bmatrix} 0 & \cdots & \cdots & 0 \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ 0 & \cdots & \cdots & 0 \\ I_{a_\mu} & & & \\ & \ddots & & \\ & & \ddots & \\ & & & I_{a_0}\end{bmatrix}\right)\right\},$$

where all $F_k^{(i)}$ have full row rank w_i, due to (3.34). Fixing any $i \in \{1,\dots,\mu\}$ we see that all $F_k^{(i)}$ can be reduced to echelon form by transforming with invertible matrices in block row $\mu+1-i$ and in block column $\mu+2-i$. This indeed affects the matrices $G_k^{(i)}$ and the identity matrices $I_{a_{i-1}}$ (by a multiplication with an invertible matrix from the right) but the $G_k^{(i)}$ matrices still have full row rank, together with the new $F_k^{(i)}$ matrices (which now are in echelon form), i.e., (3.34) still holds. Also the identity matrices $I_{a_{i-1}}$ can be restored by transforming with invertible matrices from the left. This alters the $G_k^{(i)}$ once more but, again, condition (3.34) is not affected. From this we see that

$$\begin{aligned}
&\left\{\left(\begin{bmatrix}F_k\\G_k\end{bmatrix},\begin{bmatrix}0\\I_{h_{f,\mu}}\end{bmatrix}\right)\right\} \\
\sim &\left\{\left(\begin{bmatrix}
0 & I_{w_\mu} & 0 & * & * & \cdots & \cdots & * \\
 & & & I_{w_{\mu-1}} & 0 & & & \vdots \\
 & & & & & \ddots & * & * \\
 & & & & & & I_{w_1} & 0 \\
 & & & & & & & 0 \\
0 & G_k^{(\mu,1)} & G_k^{(\mu,2)} & * & * & \cdots & \cdots & * \\
 & & & G_k^{(\mu-1,1)} & G_k^{(\mu-1,2)} & & & \vdots \\
 & & & & & \ddots & * & * \\
 & & & & & & G_k^{(1,1)} & G_k^{(1,2)} \\
 & & & & & & & 0
\end{bmatrix}, \begin{bmatrix}0\\I_{h_{f,\mu}}\end{bmatrix}\right)\right\} \\
\sim &\left\{\left(\begin{bmatrix}
0 & I_{w_\mu} & 0 & 0 & * & & 0 & * \\
 & & & I_{w_{\mu-1}} & 0 & & \vdots & \vdots \\
 & & & & & \ddots & 0 & * \\
 & & & & & & I_{w_1} & 0 \\
 & & & & & & & 0 \\
0 & 0 & G_k^{(\mu,2)} & 0 & * & & 0 & * \\
 & & & 0 & G_k^{(\mu-1,2)} & & \vdots & \vdots \\
 & & & & & \ddots & 0 & * \\
 & & & & & & 0 & G_k^{(1,2)} \\
 & & & & & & & 0
\end{bmatrix}, \begin{bmatrix}0\\I_{h_{f,\mu}}\end{bmatrix}\right)\right\}
\end{aligned}$$

$$
\sim \left\{ \left(\begin{bmatrix}
0 & I_{w_\mu} & 0 & 0 & 0 & \cdots & 0 & 0 \\
 & & & I_{w_{\mu-1}} & 0 & & \vdots & \vdots \\
 & & & & & \ddots & 0 & 0 \\
 & & & & & & I_{w_1} & 0 \\
 & & & & & & & 0 \\
0 & 0 & G_k^{(\mu,2)} & 0 & * & & 0 & * \\
 & & & 0 & G_k^{(\mu-1,2)} & & \vdots & \vdots \\
 & & & & & \ddots & 0 & * \\
 & & & & & & 0 & G_k^{(1,2)} \\
 & & & & & & & 0
\end{bmatrix}, \begin{bmatrix} 0 \\ I_{h_{f,\mu}} + B_k \end{bmatrix} \right) \right\},
$$

where all the matrices B_k are upper triangular and nilpotent. Thus, these B_k matrices can again be eliminated by adding multiples of a row k to a row l, where always $k > l$ (i.e., by transforming from the left). By splitting the block rows in the lower part (i.e., the part corresponding to the G_k block) and using

$$
G^{(i,2)} = \begin{bmatrix} D_k^{(i)} \\ E_k^{(i)} \end{bmatrix},
$$

we finally obtain the assertion, where (3.44) follows from (3.34), since (3.34) now reads

$$
\operatorname{rank}\left(\begin{bmatrix} I_{w_i} & 0 \\ 0 & D_k^{(i)} \\ 0 & E_k^{(i)} \end{bmatrix} \right) = s_{i-1}.
$$

□

The converse of Theorem 3.20 clearly is easier to understand than Theorem 3.20 itself.

Lemma 3.22. *Let $\mu \in \mathbb{N}_0$ and let $\{\hat{a}_i\}_{i\in\mathbb{N}_0}$ be a non-increasing sequence with $\hat{a}_{\mu+1} = 0$ and $\hat{a}_\mu \neq 0$. Further let $\{\hat{w}_i\}_{i\in\mathbb{N}_0}$ be a non-negative sequence with $\hat{w}_i + \hat{a}_i \leq \hat{a}_{i-1}$. Set $\hat{h}_f = \hat{a}_0 + \ldots + \hat{a}_\mu$ and let $\hat{r}_f \in \mathbb{N}_0$ be an integer. Assume that $E_k, A_k \in \mathbb{C}^{m,n}$ are matrices such that the sequence $\{(E_k, A_k)\}_{k\in\mathbb{Z}}$ is globally equivalent to*

$$
\left\{ \left(\begin{bmatrix} I_{\hat{r}_f} & 0 & W_k \\ 0 & 0 & F_k \\ 0 & 0 & G_k \end{bmatrix}, \begin{bmatrix} * & * & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_{\hat{h}_f} \end{bmatrix} \right) \right\}_{k\in\mathbb{Z}}, \tag{3.45}
$$

with

$$
F_k = \begin{bmatrix} 0 & F_k^{(\mu)} & & * \\ & \ddots & \ddots & \\ & & \ddots & F_k^{(1)} \\ & & & 0 \end{bmatrix}, \quad G_k = \begin{bmatrix} 0 & G_k^{(\mu)} & & * \\ & \ddots & \ddots & \\ & & \ddots & G_k^{(1)} \\ & & & 0 \end{bmatrix}, \tag{3.46}
$$

where all $F_k^{(i)}$ and $G_k^{(i)}$ have sizes $\hat{w}_i \times \hat{a}_{i-1}$ and $\hat{a}_i \times \hat{a}_{i-1}$, respectively, and all $W_k = [\, * \cdots * \,]$ are partitioned accordingly. Also assume that all $F_k^{(i)}$ and $G_k^{(i)}$ together have full row rank, i.e.,

$$\operatorname{rank}\left(\begin{bmatrix} F_k^{(i)} \\ G_k^{(i)} \end{bmatrix}\right) = \hat{a}_i + \hat{w}_i \qquad \forall k \in \mathbb{Z}. \tag{3.47}$$

Then $\{(E_k, A_k)\}_{k\in\mathbb{Z}}$ has a well defined strangeness index μ as defined in (3.25). The sequence of characteristic values is thereby given as

$$r_{f,i} = \hat{r}_f + \sum_{j=i+1}^{\mu} \hat{w}_j + \sum_{j=i+1}^{\mu} \hat{a}_j, \tag{3.48}$$

$$h_{f,i} = \sum_{j=0}^{i} \hat{a}_j. \tag{3.49}$$

Proof. Performing i reductions from (3.13) to (3.14) on the sequence of pencils (3.46) shows that for the resulting sequence we have

$$\{(E_{k,i}, A_{k,i})\}_{k\in\mathbb{Z}} \sim$$
$$\left\{\left(\begin{bmatrix} I_{\hat{r}_f} & 0 & * & * & \cdots & * & 0 \\ 0 & 0 & 0 & F_k^{(\mu)} & & * & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & & 0 \\ \vdots & \vdots & \vdots & & \ddots & F_k^{(i+1)} & 0 \\ 0 & 0 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & 0 & 0 & G_k^{(\mu)} & & * & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & & 0 \\ \vdots & \vdots & \vdots & & \ddots & G_k^{(i+1)} & 0 \\ 0 & 0 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & \cdots & 0 & 0 \end{bmatrix}, \begin{bmatrix} * & * & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & \vdots & & & & \vdots \\ \vdots & \vdots & \vdots & & & & \vdots \\ \vdots & \vdots & \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & I_{\hat{a}_\mu} & & & & \\ \vdots & \vdots & & \ddots & & & \\ \vdots & \vdots & & & \ddots & & \\ 0 & 0 & & & & I_{\hat{a}_i} & \\ 0 & 0 & & & & & I_{\hat{h}_{f,i-1}} \end{bmatrix}\right)\right\},$$

where $\hat{h}_{f,i-1} := \hat{a}_0 + \ldots + \hat{a}_{i-1}$. Thus, one can see, that the rank of all $E_{k,i}$ is constant in k and given by (3.48). Once Z_k are bases of corange $(E_{k,i})$ one also notices, that the rank of $Z_k^H A_{k,i}$ is also constantly equal to $\hat{a}_i + \hat{h}_{f,i-1}$, which shows (3.49). $\square$

We derive another Lemma from Theorem 3.20 which will be needed in the next section.

Lemma 3.23. *With the assumptions and notation of Theorem 3.20 we have*

$$\operatorname{rank}(F_i G_{i+1} \ldots G_l) = \sum_{j=l-i+1}^{\mu} w_j \text{ and} \tag{3.50}$$

$$\operatorname{rank}(G_i G_{i+1} \ldots G_l) = \sum_{j=l-i+1}^{\mu} a_j \tag{3.51}$$

for all $l \geq i$. Also, we know that all non-zero rows of $F_i G_{i+1} \dots G_l$ and all non-zeros rows of $G_i G_{i+1} \dots G_l$ are linear independent.

Proof. Since all matrices F_k and G_k are strictly block-upper-triangular with $\mu+1$ block-rows and block-columns it is clear that all matrices F_k and G_k are nilpotent with nilpotency index $\mu+1$. Also we have that

$$\begin{aligned} F_{k_1} G_{k_2} \dots G_{k_{\mu+1}} &= 0 \text{ and} \\ G_{k_1} G_{k_2} \dots G_{k_{\mu+1}} &= 0 \end{aligned}$$

for any integer sequences $k_1, \dots, k_{\mu+1}$. For $l = i$ the claim is that all F_i have rank $\sum_{j=1}^{\mu} w_j$ and that all G_i have rank $\sum_{j=1}^{\mu} a_j$, which follows directly from condition (3.34). Thus, it is sufficient to show the statement of the Lemma for $i < l < \mu + i$.
For $i < l < \mu + i$ we have

$$G_{i+1} \dots G_l = \begin{bmatrix} 0 & \cdots & 0 & H_{i+1,l}^{(\mu)} & & * \\ \vdots & & & \ddots & \ddots & \\ \vdots & & & & \ddots & H_{i+1,l}^{(l-i)} \\ \vdots & & & & & 0 \\ \vdots & & & & & \vdots \\ 0 & & \cdots & \cdots & & 0 \end{bmatrix},$$

where we have used the definition $H_{p,q}^{(j)} := G_p^{(j)} \dots G_q^{(j-q+p)}$. Since $l - i < \mu$, the matrix $G_{i+1} \dots G_l$ may have two block-rows and two block-columns that are non zero. Premultiplying with F_i or G_i yields

$$F_i G_{i+1} \dots G_l = \begin{bmatrix} 0 & \cdots & 0 & F_i^{(\mu)} H_{i+1,l}^{(\mu-1)} & & * \\ \vdots & & & \ddots & \ddots & \\ \vdots & & & & \ddots & F_i^{(l-i+1)} H_{i+1,l}^{(l-i)} \\ \vdots & & & & & 0 \\ \vdots & & & & & \vdots \\ 0 & & \cdots & \cdots & & 0 \end{bmatrix} \text{ and}$$

$$G_i G_{i+1} \dots G_l = \begin{bmatrix} 0 & \cdots & 0 & H_{i,l}^{(\mu)} & & * \\ \vdots & & & \ddots & \ddots & \\ \vdots & & & & \ddots & H_{i,l}^{(l-i+1)} \\ \vdots & & & & & 0 \\ \vdots & & & & & \vdots \\ 0 & & \cdots & \cdots & & 0 \end{bmatrix}.$$

In the following we show that $\operatorname{rank}\left(H_{i,l}^{(j)}\right) = a_j$ and $\operatorname{rank}\left(F_i^{(j)} H_{i+1,l}^{(j-1)}\right) = w_j$ and also that $H_{i,l}^{(j)}$ and $F_i^{(j)} H_{i+1,l}^{(j-1)}$ are both of full row rank. This then immediately proves the claim.
We first observe, that $H_{i,l}^{(j)}$ is a product of a $a_j \times a_{j-1}$ matrix and a $a_{j-1} \times a_{j-2}$ matrix and so on. Since $\{a_i\}_{i \in \mathbb{N}_0}$ is a non-increasing sequence, due to (3.19e), this means that $H_{i,l}^{(j)}$ is a product of full row rank matrices ($G_s^{(t)}$), where the column dimension increases with every further post-multiplication. Thus, $H_{i,l}^{(j)}$ has to have full row rank a_j. With the very same argument we see that $F_i^{(j)} H_{i+1,l}^{(j-1)}$ is of full row rank w_j, since from (3.34) we know that $w_j \leq a_{j-1}$. □

In [6] index concepts for discrete-time descriptor systems are examined. There an approach similar to the following is taken.

Theorem 3.24. *Let $\{(E_k, A_k)\}_{k \in \mathbb{Z}}$ be a sequence of matrix pairs and introduce the following matrices:*

$$\begin{aligned}
&Z_k \textit{ basis of } \operatorname{corange}(E_k) = \operatorname{kernel}\left({E_k}^H\right), \\
&Z_k' \textit{ basis of } \operatorname{range}(E_k), \\
&T_k \textit{ basis of } \operatorname{kernel}(E_k), \\
&T_k' \textit{ basis of } \operatorname{cokernel}(E_k) = \operatorname{range}\left({E_k}^H\right), \\
&V_k \textit{ basis of } \operatorname{corange}\left(Z_k^H A_k T_{k-1}\right), \\
&V_k' \textit{ basis of } \operatorname{range}\left(Z_k^H A_k T_{k-1}\right),
\end{aligned}$$

for all $k \in \mathbb{Z}$. With this, define the following quantities:

$$\begin{aligned}
r_{f,k} &:= \operatorname{rank}(E_k), \\
\tilde{a}_k &:= \operatorname{rank}\left(Z_k^H A_k T_{k-1}\right), \\
\tilde{s}_k &:= \operatorname{rank}\left(V_k^H Z_k^H A_k T_{k-1}'\right),
\end{aligned}$$

for all $k \in \mathbb{Z}$. Under the assumption that

$$r_f \equiv r_{f,k}, \quad \tilde{a} \equiv \tilde{a}_k, \quad \tilde{s} \equiv \tilde{s}_k, \quad k \in \mathbb{Z}, \tag{3.52}$$

the sequence $\{(E_k, A_k)\}_{k \in \mathbb{Z}}$ is globally equivalent to

$$\left\{ \left(\begin{bmatrix} I_{\tilde{s}} & 0 & 0 & 0 \\ 0 & I_{\tilde{d}} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & A_k^{(1,2)} & 0 & A_k^{(1,4)} \\ 0 & A_k^{(2,2)} & 0 & A_k^{(2,4)} \\ 0 & 0 & I_{\tilde{a}} & 0 \\ I_{\tilde{s}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right) \right\}, \tag{3.53}$$

where the abbreviation $\tilde{d} := r_f - \tilde{s}$ has been used.

Proof. We have

$$
\begin{aligned}
\{(E_k, A_k)\}_{k\in\mathbb{Z}} &\sim \left\{ \left(\begin{bmatrix} {Z'_k}^H E_k T'_k & {Z'_k}^H E_k T_k \\ {Z_k}^H E_k T'_k & {Z_k}^H E_k T_k \end{bmatrix}, \begin{bmatrix} {Z'_k}^H A_k T'_{k-1} & {Z'_k}^H A_k T_{k-1} \\ {Z_k}^H A_k T'_{k-1} & {Z_k}^H A_k T_{k-1} \end{bmatrix} \right) \right\}_{k\in\mathbb{Z}} \\
&= \left\{ \left(\begin{bmatrix} {Z'_k}^H E_k T'_k & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} {Z'_k}^H A_k T'_{k-1} & {Z'_k}^H A_k T_{k-1} \\ {Z_k}^H A_k T'_{k-1} & {Z_k}^H A_k T_{k-1} \end{bmatrix} \right) \right\}_{k\in\mathbb{Z}} \\
&\sim \left\{ \left(\begin{bmatrix} I_{r_f} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} \tilde{A}_k^{(1,1)} & \tilde{A}_k^{(1,2)} \\ {V'_k}^H {Z_k}^H A_k T'_{k-1} & {V'_k}^H {Z_k}^H A_k T_{k-1} \\ {V_k}^H {Z_k}^H A_k T'_{k-1} & {V_k}^H {Z_k}^H A_k T_{k-1} \end{bmatrix} \right) \right\}_{k\in\mathbb{Z}} \\
&= \left\{ \left(\begin{bmatrix} I_{r_f} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} \tilde{A}_k^{(1,1)} & \tilde{A}_k^{(1,2)} \\ {V'_k}^H {Z_k}^H A_k T'_{k-1} & {V'_k}^H {Z_k}^H A_k T_{k-1} \\ {V_k}^H {Z_k}^H A_k T'_{k-1} & 0 \end{bmatrix} \right) \right\}_{k\in\mathbb{Z}} \\
&\sim \left\{ \left(\begin{bmatrix} I_{r_f} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} \tilde{A}_k^{(1,1)} & \tilde{A}_k^{(1,2)} & \tilde{A}_k^{(1,3)} \\ {V'_k}^H {Z_k}^H A_k T'_{k-1} & I_{\tilde{a}} & 0 \\ {V_k}^H {Z_k}^H A_k T'_{k-1} & 0 & 0 \end{bmatrix} \right) \right\}_{k\in\mathbb{Z}} \\
&\sim \left\{ \left(\begin{bmatrix} I_{r_f} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} \tilde{A}_k^{(1,1)} & 0 & \tilde{A}_k^{(1,3)} \\ 0 & I_{\tilde{a}} & 0 \\ {V_k}^H {Z_k}^H A_k T'_{k-1} & 0 & 0 \end{bmatrix} \right) \right\}_{k\in\mathbb{Z}} \\
&\sim \left\{ \left(\begin{bmatrix} I_{\tilde{s}} & 0 & 0 & 0 \\ 0 & I_{\tilde{d}} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} \tilde{A}_k^{(1,1)} & \tilde{A}_k^{(1,2)} & 0 & \tilde{A}_k^{(1,4)} \\ \tilde{A}_k^{(2,1)} & \tilde{A}_k^{(2,2)} & 0 & \tilde{A}_k^{(2,4)} \\ 0 & 0 & I_{\tilde{a}} & 0 \\ I_{\tilde{s}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right) \right\}_{k\in\mathbb{Z}} \\
&\sim \left\{ \left(\begin{bmatrix} I_{\tilde{s}} & 0 & 0 & 0 \\ 0 & I_{\tilde{d}} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \tilde{A}_k^{(1,2)} & 0 & \tilde{A}_k^{(1,4)} \\ 0 & \tilde{A}_k^{(2,2)} & 0 & \tilde{A}_k^{(2,4)} \\ 0 & 0 & I_{\tilde{a}} & 0 \\ I_{\tilde{s}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right) \right\}_{k\in\mathbb{Z}}.
\end{aligned}
$$

□

By further reducing (3.53) to the form

$$
\left\{ \left(\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & I_{\tilde{d}} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & A_k^{(1,2)} & 0 & A_k^{(1,4)} \\ 0 & A_k^{(2,2)} & 0 & A_k^{(2,4)} \\ 0 & 0 & I_{\tilde{a}} & 0 \\ I_{\tilde{s}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right) \right\}, \tag{3.54}
$$

[6] then defines a sequence of characteristic values which we denote by

$$
\{(r_f^{(i)}, \tilde{a}^{(i)}, \tilde{s}^{(i)})\}_{i\in\mathbb{N}_0} \tag{3.55}
$$

and a sequence of sequences of matrix pairs similar to Remark 3.13. In the following we compare the approach from [6] (as in Theorem 3.24) to our approach.

Lemma 3.25. *With the assumptions and symbols from the preceding Theorem 3.24 and $h_{f,k}$ defined as in (3.11c) we have that*

$$h_{f,k} = \tilde{a}_k + \tilde{s}_k = \tilde{a} + \tilde{s} =: h_f.$$

In particular, under the assumptions of Theorem 3.24, assumption (3.12) is satisfied (and thus all assumptions of Lemma 3.10).

Proof. Using the notation of Theorem 3.24, we see that V_k is a basis of kernel $\left(T_{k-1}^H A_k^H Z_k\right)$, from which it is clear that $T_{k-1}^H A_k^H Z_k V_k = 0$. Transposing the very last identity yields

$$V_k^H Z_k^H A_k T_{k-1} = 0. \tag{3.56}$$

Also we know that

$$V_k'^H Z_k^H A_k T_{k-1} \text{ has full row rank } \tilde{a}_k, \tag{3.57}$$

since V_k' is a basis of range $\left(Z_k^H A_k T_{k-1}\right)$. Hence, we finally obtain

$$\begin{aligned}
h_{f,k} &= \operatorname{rank}\left(Z_k^H A_k\right) \\
&= \operatorname{rank}\left(\begin{bmatrix} V_k'^H Z_k^H A_k T_{k-1} & V_k'^H Z_k^H A_k T_{k-1}' \\ V_k^H Z_k^H A_k T_{k-1} & V_k^H Z_k^H A_k T_{k-1}' \end{bmatrix}\right) \\
&\overset{(3.56)}{=} \operatorname{rank}\left(\begin{bmatrix} V_k'^H Z_k^H A_k T_{k-1} & V_k'^H Z_k^H A_k T_{k-1}' \\ 0 & V_k^H Z_k^H A_k T_{k-1}' \end{bmatrix}\right) \\
&\overset{(3.57)}{=} \operatorname{rank}\left(\begin{bmatrix} V_k'^H Z_k^H A_k T_{k-1} & 0 \\ 0 & V_k^H Z_k^H A_k T_{k-1}' \end{bmatrix}\right) \\
&= \operatorname{rank}\left(V_k'^H Z_k^H A_k T_{k-1}\right) + \operatorname{rank}\left(V_k^H Z_k^H A_k T_{k-1}'\right) \\
&= \tilde{a}_k + \tilde{s}_k.
\end{aligned}$$

The result then follows, since $r_{f,k}$, $\tilde{a}_k$, and $\tilde{s}_k$ are constant for all $k \in \mathbb{Z}$. ☐

Remark 3.26. Note that (3.53) can be transformed to a sequence of matrix pairs of the form (3.13) by simple block column permutations. From this we see that reducing (3.13) to (3.14) is quite the same as reducing (3.53) to (3.54). Hence, Lemma 3.25 suggests that our approach is more general than the one in [6], since it only requires $h_{f,k} = \tilde{a}_k + \tilde{s}_k$ to be constant in every step of the reduction procedure.

Remark 3.27. Suppose that a sequence of matrix pairs $\{(E_k, A_k)\}_{k\in\mathbb{Z}}$ satisfies the assumptions of Theorem 3.24. Further suppose, that the associated reduced sequence (3.54) also satisfies the constant rank assumptions of Theorem (3.24). Then we know that the first two elements of the sequence $\{(r_{f,i})\}_{i\in\mathbb{N}_0}$ (as defined in Remark 3.13) are given by $r_{f,0} = \tilde{d} + \tilde{s}$ and $r_{f,1} = \tilde{d}$. Thus, from (3.18d) it follows that $s_0 = r_{f,0} - r_{f,1} = \tilde{s} = \tilde{s}^{(0)}$, where $\tilde{s}^{(0)}$ is as in (3.55). We conjecture that it is possible to show that for the sequence of characteristic values (3.55) we have $\tilde{s}^{(i)} = s_i$ for all $i \in \mathbb{N}_0$, where s_i is as defined in (3.18d).

3.2.1 Local and global invariants

Analogously to [8] in this subsection the connections between the global invariants of a given sequence of matrix pairs and the local invariants of corresponding inflated descriptor systems are investigated. The inflated descriptor system of a sequence of matrix pairs $\{(E_k, A_k)\}_{k\in\mathbb{Z}}$ of stage $l \in \mathbb{N}$ is given by

$$M_k^{(l)}(z^{(l)})^k = N_k^{(l)}(z^{(l)})^k + (g^{(l)})^k, \tag{3.58}$$

where

$$M_k^{(l)} = \begin{bmatrix} E_k & & & \\ -A_{k+1} & E_{k+1} & & \\ & \ddots & \ddots & \\ & & -A_{k+l} & E_{k+l} \end{bmatrix},$$

$$N_k^{(l)} = \begin{bmatrix} A_k & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix},$$

$$(z^{(l)})^k = \begin{bmatrix} x^k \\ \vdots \\ x^{k+l} \end{bmatrix},$$

$$(g^{(l)})^k = \begin{bmatrix} f^k \\ \vdots \\ f^{k+l} \end{bmatrix}.$$

This corresponds to combining l steps of the original descriptor system in one step. For every $l \in \mathbb{N}$ and every $k \in \mathbb{Z}$, we can determine the local characteristic values (3.6) of the matrix pair $(M_k^{(l)}, N_k^{(l)})$. The next Theorem shows that those local characteristic values are invariant under global equivalence transformations of the original sequence of matrix pairs.

Theorem 3.28. *Let $\{(E_k, A_k)\}_{k\in\mathbb{Z}}$ and $\{(\tilde{E}_k, \tilde{A}_k)\}_{k\in\mathbb{Z}}$ be two globally equivalent sequences of matrix pairs, i.e., suppose there exist pointwise invertible matrix sequences $\{P_k\}_{k\in\mathbb{Z}}$ and $\{Q_k\}_{k\in\mathbb{Z}}$ with*

$$\begin{aligned} E_k &= P_k \tilde{E}_k Q_{k+1}, \\ A_k &= P_k \tilde{A}_k Q_k. \end{aligned}$$

Then for any $k \in \mathbb{Z}$ and for any $l \in \mathbb{N}$ the matrix pairs $(M_k^{(l)}, N_k^{(l)})$ and $(\tilde{M}_k^{(l)}, \tilde{N}_k^{(l)})$ (corresponding to (3.58)) are locally equivalent as in Definition (3.6).

Proof. Let $k \in \mathbb{Z}$ and $l \in \mathbb{N}$ be fixed. Define the matrices

$$\Pi := \begin{bmatrix} P_k & & \\ & \ddots & \\ & & P_{k+l} \end{bmatrix},$$

$$\Theta := \begin{bmatrix} Q_{k+1} & & \\ & \ddots & \\ & & Q_{k+l+1} \end{bmatrix},$$

$$\Psi := \begin{bmatrix} Q_k & & \\ & \ddots & \\ & & Q_{k+l} \end{bmatrix}.$$

Then we have

$$\begin{aligned}
\Pi \tilde{M}_k^{(l)} \Theta &= \begin{bmatrix} P_k \tilde{E}_k & & & \\ -P_{k+1}\tilde{A}_{k+1} & P_{k+1}\tilde{E}_{k+1} & & \\ & \ddots & \ddots & \\ & & -P_{k+l}\tilde{A}_{k+l} & P_{k+l}\tilde{E}_{k+l} \end{bmatrix} \Theta \\
&= \begin{bmatrix} P_k \tilde{E}_k Q_{k+1} & & & \\ -P_{k+1}\tilde{A}_{k+1}Q_{k+1} & P_{k+1}\tilde{E}_{k+1}Q_{k+2} & & \\ & \ddots & \ddots & \\ & & -P_{k+l}\tilde{A}_{k+l}Q_{k+l} & P_{k+l}\tilde{E}_{k+l}Q_{k+l+1} \end{bmatrix} \\
&= \begin{bmatrix} E_k & & & \\ -A_{k+1} & E_{k+1} & & \\ & \ddots & \ddots & \\ & & -A_{k+l} & E_{k+l} \end{bmatrix} = M_k^{(l)}.
\end{aligned}$$

For the other matrix we also get

$$\Pi \tilde{N}_k^{(l)} \Psi = \begin{bmatrix} P_k \tilde{A}_k Q_k & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} = N_k^{(l)}$$

from which the local equivalence follows. □

To make a statement about the relation between the global characteristics of the original sequence of matrix pairs and the local characteristics of its inflated descriptor systems as in [8] we need the following Lemma.

Lemma 3.29. *With the assumptions and notation of Theorem 3.20 we have*

$$\operatorname{rank}\left(\begin{bmatrix} G_k \dots G_{k+l} \\ F_k G_{k+1} \dots G_{k+l} \\ \vdots \\ F_{k+l-1} G_{k+l} \\ F_{k+l} \end{bmatrix}\right) = \sum_{j=l+1}^{\mu} a_j + \sum_{i=1}^{l+1} \sum_{j=i}^{\mu} w_j, \tag{3.59}$$

for all $k \in \mathbb{Z}$ and all $l \in \mathbb{N}_0$.

Proof. We use induction to prove the claim.
Induction basis: $l = 0$
Here (3.59) reads

$$\operatorname{rank}\left(\begin{bmatrix} G_k \\ F_k \end{bmatrix}\right) = \sum_{j=1}^{\mu} a_j + \sum_{j=1}^{\mu} w_j,$$

which follows directly from Theorem 3.20.
Induction step: $l \to l+1$ with the help of (3.59).
Since all non-zero rows of $\begin{bmatrix} G_{k+l+1} \\ F_{k+l+1} \end{bmatrix}$ are linear independent we know, that

$$\begin{aligned}
\operatorname{rank}\left(\begin{bmatrix} G_k \dots G_{k+l+1} \\ F_k G_{k+1} \dots G_{k+l+1} \\ \vdots \\ F_{k+l} G_{k+l+1} \\ F_{k+l+1} \end{bmatrix}\right) &= \operatorname{rank}\left(\begin{bmatrix} \begin{bmatrix} G_k \dots G_{k+l} \\ F_k G_{k+1} \dots G_{k+l} \\ \vdots \\ F_{k+l-1} G_{k+l} \\ F_{k+l} \end{bmatrix} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} G_{k+l+1} \\ F_{k+l+1} \end{bmatrix}\right) \\
&= \operatorname{rank}\left(\begin{bmatrix} G_k \dots G_{k+l} \\ F_k G_{k+1} \dots G_{k+l} \\ \vdots \\ F_{k+l-1} G_{k+l} \\ F_{k+l} \end{bmatrix} G_{k+l+1}\right) + \operatorname{rank}\left(F_{k+l+1}\right).
\end{aligned} \tag{3.60}$$

To determine the rank of

$$\underbrace{\begin{bmatrix} G_k \dots G_{k+l} \\ F_k G_{k+1} \dots G_{k+l} \\ \vdots \\ F_{k+l-1} G_{k+l} \\ F_{k+l} \end{bmatrix}}_{=:H_1} \underbrace{G_{k+l+1}}_{=:H_2}$$

we first notice the special structure of the block entries of the matrices H_1 and H_2, which was presented in the proof of Lemma 3.23. Lemma 3.23 also shows, that all non-zero rows of H_1 are linear independent, since due to the inductive assumption we have

$$\begin{aligned}\operatorname{rank}(H_1) =&\operatorname{rank}(G_k \dots G_{k+l}) + \\ &\operatorname{rank}(F_k G_{k+1} \dots G_{k+l}) + \dots + \\ &\operatorname{rank}(F_{k+l-1} G_{k+l}) + \\ &\operatorname{rank}(F_{k+l}) .\end{aligned}$$

By multiplying H_1 with H_2 from the right more rows that only contain zeros are generated. However, the non-zero rows of $H_1 H_2$ are still linear independent. To see this, note that

$$\begin{aligned}\operatorname{range}(G_k G_{k+1} \dots G_{k+l} G_{k+l+1}) &\subset \operatorname{range}(G_k G_{k+1} \dots G_{k+l}) , \\ \operatorname{range}(F_i G_{i+1} \dots G_{k+l} G_{k+l+1}) &\subset \operatorname{range}(F_i G_{i+1} \dots G_{k+l}) , \text{ for all } i = k, \dots, l,\end{aligned}$$

and remember the special structure presented in the proof of Lemma 3.23. This proves that

$$\begin{aligned}\operatorname{rank}(H_1 H_2) =&\operatorname{rank}(G_k \dots G_{k+l} G_{k+l+1}) + \\ &\operatorname{rank}(F_k G_{k+1} \dots G_{k+l} G_{k+l+1}) + \dots + \\ &\operatorname{rank}(F_{k+l-1} G_{k+l} G_{k+l+1}) + \\ &\operatorname{rank}(F_{k+l} G_{k+l+1}) \\ =& \sum_{j=l+2}^{\mu} a_j + \sum_{j=l+2}^{\mu} w_j + \dots + \sum_{j=3}^{\mu} w_j + \sum_{j=2}^{\mu} w_j . \end{aligned} \tag{3.61}$$

Thus, we have

$$\begin{aligned}\operatorname{rank}\left(\begin{bmatrix} G_k \dots G_{k+l+1} \\ F_k G_{k+1} \dots G_{k+l+1} \\ \vdots \\ F_{k+l} G_{k+l+1} \\ F_{k+l+1} \end{bmatrix}\right) &\overset{(3.60)}{=} \operatorname{rank}(H_1 H_2) + \operatorname{rank}(F_{k+l+1}) \\ &\overset{(3.61)}{=} \sum_{j=l+2}^{\mu} a_j + \sum_{j=l+2}^{\mu} w_j + \dots + \sum_{j=3}^{\mu} w_j + \sum_{j=2}^{\mu} w_j + \sum_{j=1}^{\mu} w_j ,\end{aligned}$$

which proves the claim. □

Lemma 3.29 now will be used to show the following Theorem.

Theorem 3.30. *With $E_k, A_k \in \mathbb{C}^{m,n}$ let the strangeness index μ of $\{(E_k, A_k)\}_{k\in\mathbb{Z}}$ as in (3.25) be well defined with global characteristic values $(r_{f,i}, h_{f,i}), i \in \mathbb{N}_0$. Fix any $\hat{k} \in \mathbb{Z}$ and for any $l \in \mathbb{N}_0$, let $(M_{\hat{k}}^{(l)}, N_{\hat{k}}^{(l)})$ be the inflated descriptor system corresponding to (3.58) with local characteristic values $(\tilde{r}_{f,l}, \tilde{h}_{f,l})$ as in (3.6). Then,*

$$\tilde{r}_{f,l} = (l+1)m - \sum_{i=0}^{l} a_i - \sum_{i=0}^{l} v_i \overset{(3.18)}{=} \sum_{i=0}^{l} r_{f,i} + \sum_{i=0}^{l-1} h_{f,i}, \tag{3.62}$$

$$\tilde{h}_{f,l} = \sum_{i=0}^{l} a_i \overset{(3.18b)}{=} h_{f,l}, \tag{3.63}$$

with v_i defined in (3.18g).

Proof. By Theorem 3.28, we may assume that the sequence $\{(E_k, A_k)\}_{k\in\mathbb{Z}}$ is already in canonical form (3.32). With this and by using $k = \hat{k}$, from (3.58) we get that

$$
\begin{aligned}
&(M^{(l)}, N^{(l)}) := (M^{(l)}_{\hat{k}}, N^{(l)}_{\hat{k}}) \\
&= \left(\begin{bmatrix} E_k & & & \\ -A_{k+1} & E_{k+1} & & \\ & \ddots & \ddots & \\ & & -A_{k+l} & E_{k+l} \end{bmatrix}, \begin{bmatrix} A_k & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \right) \\
&= \left(\left[\begin{array}{ccc|ccc|ccc|ccc}
I_{r_{f,\mu}} & 0 & W_k & & & & & & & & & \\
0 & 0 & F_k & & & & & & & & & \\
0 & 0 & G_k & & & & & & & & & \\ \hline
* & * & 0 & I_{r_{f,\mu}} & 0 & W_{k+1} & & & & & & \\
0 & 0 & 0 & 0 & 0 & F_{k+1} & & & & & & \\
0 & 0 & -I_{h_{f,\mu}} & 0 & 0 & G_{k+1} & & & & & & \\ \hline
& & & \ddots & & & \ddots & & & & & \\
& & & & \ddots & & & \ddots & & & & \\
& & & & & \ddots & & & \ddots & & & \\ \hline
& & & & & & * & * & 0 & I_{r_{f,\mu}} & 0 & W_{k+l} \\
& & & & & & 0 & 0 & 0 & 0 & 0 & F_{k+l} \\
& & & & & & 0 & 0 & -I_{h_{f,\mu}} & 0 & 0 & G_{k+l}
\end{array} \right], N^{(l)} \right) \\
&\sim \left(\left[\begin{array}{ccc|ccc|ccc|ccc}
I_{r_{f,\mu}} & 0 & 0 & & & & & & & & & \\
0 & 0 & F_k & & & & & & & & & \\
0 & 0 & G_k & & & & & & & & & \\ \hline
0 & 0 & 0 & I_{r_{f,\mu}} & 0 & 0 & & & & & & \\
0 & 0 & 0 & 0 & 0 & F_{k+1} & & & & & & \\
0 & 0 & -I_{h_{f,\mu}} & 0 & 0 & G_{k+1} & & & & & & \\ \hline
& & & \ddots & & & \ddots & & & & & \\
& & & & \ddots & & & \ddots & & & & \\
& & & & & \ddots & & & \ddots & & & \\ \hline
& & & & & & 0 & 0 & 0 & I_{r_{f,\mu}} & 0 & 0 \\
& & & & & & 0 & 0 & 0 & 0 & 0 & F_{k+l} \\
& & & & & & 0 & 0 & -I_{h_{f,\mu}} & 0 & 0 & G_{k+l}
\end{array} \right], N^{(l)} \right),
\end{aligned}
$$

where we have to start eliminating the $*$ and W_k blocks from the bottom of the matrix. Again starting from the bottom, we eliminate the $F_{k+l-1}, \ldots, F_k$ and $G_{k+l-1}, \ldots, G_k$ blocks with help of the $-I_{h_{f,\mu}}$ blocks, which are below these blocks. By doing so we obtain that $(M^{(l)}, N^{(l)})$ is locally equivalent to

$$
\left(\left[\begin{array}{ccc|ccc|c|ccc|cc|c}
I_{r_{f,\mu}} & 0 & 0 & & & & & & & & & & \\
0 & 0 & F_k & & & & & & & & & & \\
0 & 0 & G_k & & & & & & & & & & \\
\hline
0 & 0 & 0 & I_{r_{f,\mu}} & 0 & 0 & & & & & & & \\
0 & 0 & 0 & 0 & 0 & F_{k+1} & & & & & & & \\
0 & 0 & -I_{h_{f,\mu}} & 0 & 0 & G_{k+1} & & & & & & & \\
\hline
 & & & 0 & 0 & 0 & \ddots & & & & & & \\
 & & & 0 & 0 & 0 & \ddots & & & & & & \\
 & & & 0 & 0 & -I_{h_{f,\mu}} & \ddots & & & & & & \\
\hline
 & & & & & & \ddots & I_{r_{f,\mu}} & 0 & 0 & & & \\
 & & & & & & \ddots & 0 & 0 & F_{k+l-1} & & & \\
 & & & & & & \ddots & 0 & 0 & G_{k+l-1} & & & \\
\hline
 & & & & & & & 0 & 0 & 0 & I_{r_{f,\mu}} & 0 & 0 \\
 & & & & & & & 0 & 0 & 0 & 0 & 0 & F_{k+l} \\
 & & & & & & & 0 & 0 & -I_{h_{f,\mu}} & 0 & 0 & G_{k+l}
\end{array}\right], N^{(l)}\right)
$$

$$
\sim \left(\left[\begin{array}{ccc|ccc|c|ccc|cc|c}
I_{r_{f,\mu}} & 0 & 0 & & & & & & & & & & \\
0 & 0 & F_k & & & & & & & & & & \\
0 & 0 & G_k & & & & & & & & & & \\
\hline
0 & 0 & 0 & I_{r_{f,\mu}} & 0 & 0 & & & & & & & \\
0 & 0 & 0 & 0 & 0 & F_{k+1} & & & & & & & \\
0 & 0 & -I_{h_{f,\mu}} & 0 & 0 & G_{k+1} & & & & & & & \\
\hline
 & & & 0 & 0 & 0 & \ddots & & & & & & \\
 & & & 0 & 0 & 0 & \ddots & & & & & & \\
 & & & 0 & 0 & -I_{h_{f,\mu}} & \ddots & & & & & & \\
\hline
 & & & & & & \ddots & I_{r_{f,\mu}} & 0 & 0 & & & 0 \\
 & & & & & & \ddots & 0 & 0 & 0 & & & F_{k+l-1}G_{k+l} \\
 & & & & & & \ddots & 0 & 0 & 0 & & & G_{k+l-1}G_{k+l} \\
\hline
 & & & & & & & 0 & 0 & 0 & I_{r_{f,\mu}} & 0 & 0 \\
 & & & & & & & 0 & 0 & 0 & 0 & 0 & F_{k+l} \\
 & & & & & & & 0 & 0 & -I_{h_{f,\mu}} & 0 & 0 & 0
\end{array}\right], N^{(l)}\right)
$$

$$
\sim \ldots \sim \left(\left[\begin{array}{ccc|ccc|c|ccc|cc|c}
I_{r_{f,\mu}} & 0 & 0 & & & & & & & & & & 0 \\
0 & 0 & 0 & & & & & & & & & & F_k R_k^{(1,l)} \\
0 & 0 & 0 & & & & & & & & & & G_k R_k^{(1,l)} \\
\hline
0 & 0 & 0 & I_{r_{f,\mu}} & 0 & 0 & & & & & & & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & & & & & & & F_{k+1} R_k^{(2,l)} \\
0 & 0 & -I_{h_{f,\mu}} & 0 & 0 & 0 & & & & & & & 0 \\
\hline
 & & & 0 & 0 & 0 & \ddots & & & & & & \\
 & & & 0 & 0 & 0 & \ddots & & & & & & \vdots \\
 & & & 0 & 0 & -I_{h_{f,\mu}} & \ddots & & & & & & \\
\hline
 & & & & & & \ddots & I_{r_{f,\mu}} & 0 & 0 & & & 0 \\
 & & & & & & \ddots & 0 & 0 & 0 & & & F_{k+l-1} R_k^{(l,l)} \\
 & & & & & & \ddots & 0 & 0 & 0 & & & 0 \\
\hline
 & & & & & & & 0 & 0 & 0 & I_{r_{f,\mu}} & 0 & 0 \\
 & & & & & & & 0 & 0 & 0 & 0 & 0 & F_{k+l} \\
 & & & & & & & 0 & 0 & -I_{h_{f,\mu}} & 0 & 0 & 0
\end{array} \right], N^{(l)} \right), \tag{3.64}
$$

by using the definition $R_k^{(i,j)} := \prod_{l=i}^{j} G_{k+l}$. By Lemma 3.29 we can determine the rank of $M^{(l)}$ as the sum of the ranks of the block columns. Thus, we have

$$
\begin{aligned}
\tilde{r}_{f,l} =& (l+1) r_{f,\mu} + l h_{f,\mu} + \operatorname{rank} \left(\begin{bmatrix} R_k^{(0,l)} \\ F_k R_k^{(1,l)} \\ \vdots \\ F_{k+l} R_k^{(l+1,l)} \end{bmatrix} \right) \\
\overset{\text{Lemma 3.29}}{=}& (l+1) r_{f,\mu} + l h_{f,\mu} + \sum_{j=l+1}^{\mu} a_j + \sum_{j=l+1}^{\mu} w_j + \ldots + \sum_{j=1}^{\mu} w_j .
\end{aligned} \tag{3.65}
$$

With (3.65) we show (3.62) by induction. For $l = 0$ the inductive argument follows directly from (3.18g), since $(E_{\hat{k}}, A_{\hat{k}}) = (M_{\hat{k}}^{(0)}, N_{\hat{k}}^{(0)})$ and $a_0 = h_{f,0}$ because of (3.18b). Assume that (3.62) holds for any $l < \mu$. Then we obtain

$$
\begin{aligned}
\tilde{r}_{f,l+1} \overset{(3.65)}{=}& (l+2) r_{f,\mu} + (l+1) h_{f,\mu} + \sum_{j=l+2}^{\mu} a_j + \sum_{j=l+2}^{\mu} w_j + \ldots + \sum_{j=1}^{\mu} w_j \\
=& (l+1) r_{f,\mu} + l h_{f,\mu} + \sum_{j=l+1}^{\mu} a_j + \sum_{j=l+2}^{\mu} w_j + \sum_{j=l+1}^{\mu} w_j + \ldots + \sum_{j=1}^{\mu} w_j \\
& + r_{f,\mu} + h_{f,\mu} - a_{l+1}
\end{aligned}
$$

$$
\begin{aligned}
&\overset{(3.65)}{=} && \tilde{r}_{f,l} + r_{f,\mu} + h_{f,\mu} - a_{l+1} + \sum_{j=l+2}^{\mu} w_j \\
&= && \tilde{r}_{f,l} + r_{f,\mu} + h_{f,\mu} - a_{l+1} + \sum_{j=1}^{\mu} w_j - \sum_{j=1}^{l+1} w_j \\
&\overset{(3.19h)}{=} && \tilde{r}_{f,l} + r_{f,\mu} + h_{f,\mu} - a_{l+1} + (v_\mu - v_0) - (v_{l+1} - v_0) \\
&= && \tilde{r}_{f,l} - a_{l+1} - v_{l+1} + \underbrace{r_{f,\mu} + h_{f,\mu} + v_\mu}_{\overset{(3.18g)}{=} m},
\end{aligned}
$$

which, with help of the inductive assumption, yields (3.62).
Let Z be a basis of corange $\left(M^{(l)}\right)$. We want to determine rank $\left(Z^H N^{(l)}\right)$ in order to prove the statement about $\tilde{h}_{f,l}$. Since only the first (three) block rows of $N^{(l)}$ contain non-zero entries, we only need to consider the first (three) block columns of Z^H, i.e., the first (three) block rows of Z. Let us denote the first (three) block rows by the matrix $\begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \end{bmatrix}$. Then due to the structure of (3.64) we have $Z_1 = 0$. Further, we have that

$$
\begin{aligned}
\operatorname{rank}(Z_2) &= v_\mu - \sum_{j=l+1}^{\mu} w_j, \\
\operatorname{rank}(Z_3) &= h_{f,\mu} - \sum_{j=l+1}^{\mu} a_j,
\end{aligned}
$$

because of Lemma 3.23. Therefore, we know that

$$
\begin{aligned}
\tilde{h}_{f,l} = \operatorname{rank}\left(Z^H N^{(l)}\right) &= && \operatorname{rank}\left(\begin{bmatrix} 0 & Z_2^H & Z_3^H \end{bmatrix} \begin{bmatrix} * & * & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_{h_{f,\mu}} \end{bmatrix} \right) \\
&= && \operatorname{rank}\left(Z_3^H\right) = h_{f,\mu} - \sum_{j=l+1}^{\mu} a_j \\
&\overset{(3.18b)}{=} && h_{f,\mu} - (h_{f,\mu} - h_{f,\mu-1}) - \ldots - (h_{f,l+1} - h_{f,l}) \\
&= && h_{f,l} = \sum_{j=0}^{l} a_j.
\end{aligned}
$$

This proves the claim. □

Obviously, the formulas (3.62) and (3.63) correspond to the formulas in [8] when one adds up the algebraic part and the strangeness to one quantity. Also, as in [8], we get the converse of the result, i.e., the knowledge of the sequence $\{(\tilde{r}_{f,l}, \tilde{h}_{f,l})\}$ allows for the determination of the sequence $\{(r_{f,i}, h_{f,i})\}$ of the global characteristic values of the sequence of matrix pairs $\{(E_k, A_k)\}_{k\in\mathbb{K}}$.

Corollary 3.31. *Let the strangeness index μ of $\{(E_k, A_k)\}_{k\in\mathbb{Z}}$ as in (3.25) be well defined and let $(\tilde{r}_{f,l}, \tilde{h}_{f,l})$, $l = 0, \ldots, \mu$ be the local characteristic values of $(M_{\hat{k}}^{(l)}, N_{\hat{k}}^{(l)})$ for some $\hat{k} \in \mathbb{Z}$ (as in Theorem 3.30). Then the sequence $\{(r_{f,i}, h_{f,i})\}_{i\in\mathbb{N}_0}$ of the global characteristic values of the sequence of matrix pairs $\{(E_k, A_k)\}_{k\in\mathbb{Z}}$ can be obtained from*

$$h_{f,l} = \tilde{h}_{f,l}, \tag{3.66a}$$

$$r_{f,l} = \tilde{r}_{f,l} - \tilde{r}_{f,l-1} - \tilde{h}_{f,l-1}, \tag{3.66b}$$

where we have used $\tilde{h}_{f,-1} = 0$ and $\tilde{r}_{f,-1} = 0$.

Proof. Using the formulas from Theorem 3.30 we immediately get (3.66a). Also, from Theorem 3.30 we see that

$$\tilde{r}_{f,l} - \tilde{r}_{f,l-1} = \sum_{i=0}^{l} r_{f,i} + \sum_{i=0}^{l-1} h_{f,i} - \sum_{i=0}^{l-1} r_{f,i} - \sum_{i=0}^{l-2} h_{f,i} = r_{f,l} + h_{f,l-1},$$

which proves (3.66b). $\square$

3.2.2 Existence and uniqueness of solutions

Concerning existence and uniqueness of sequences of matrix pairs with well defined strangeness index we get similar results as in [8].

Theorem 3.32. *Let the strangeness index μ of the sequence $\{(E_k, A_k)\}_{k\in\mathbb{Z}}$ as in (3.25) be well defined. Then the discrete descriptor system (1.5) is equivalent (in the sense that there is a one-to-one correspondence between the solution/sequence spaces) to a discrete descriptor system of the form*

$$x_1^{k+1} = A_k^{(1)} x_1^k + A_k^{(3)} x_3^k + f_1^k, \qquad r_{f,\mu} \tag{3.67a}$$

$$0 = x_2^k + f_2^k, \qquad h_{f,\mu} \tag{3.67b}$$

$$0 = f_3^k, \qquad v_\mu \tag{3.67c}$$

where with $u_\mu := n - r_{f,\mu} - h_{f,\mu}$ we have $x_3^k \in \mathbb{C}^{u_\mu}$ and each of the inhomogeneities f_1^k, f_2^k, f_3^k is determined by the original right hand sides $f^k, \ldots, f^{k+\mu+1}$ as in (1.5) for all $k \in \mathbb{Z}$. For the associated forward problem

$$E_k x^{k+1} = A_k x^k + f^k, \text{ for all } k \in \mathbb{N}_0, \tag{3.68}$$

we also have the following results:

1. *(3.68) is solvable if and only if the v_μ consistency conditions conditions*

$$f_3^k = 0$$

are fulfilled for all $k \in \mathbb{N}_0$.

2. *An initial condition $x^0 = \hat{x}$ together with (3.68) is consistent if and only if in addition the $h_{f,\mu}$ conditions*

$$x_2^0 = \hat{x}_2 = -f_2^0$$

are satisfied.

3. *The corresponding initial value problem is uniquely solvable if and only if in addition*

$$u_\mu = 0$$

holds.

Proof. Under the assumptions of the Theorem the original sequence $\{(E_k, A_k)\}_{k\in\mathbb{Z}}$ can be transformed to the form (3.26) by $\mu + 1$ reduction steps and proper global equivalence transformations. Both of these operations generate a one-to-one correspondence of solutions. □

3.3 Backward global canonical form

Looking back at Theorem 2.20 we recall that one can look at three different cases of descriptor systems. The first case is where one starts at a point in time k_0, and calculates into the future, i.e., one calculates a solution $\{x^k\}_{k\geq k_0}$. This case has been considered in the previous section.

The second case is where one starts at a point in time k_0, and calculates into the past, i.e., one calculates a solution $\{x^k\}_{k\leq k_0}$. This case is closely related to the first case. To see this suppose that a descriptor system of the form

$$\begin{aligned} E_k x^{k+1} &= A_k x^k + f^k,\ k \in \mathbb{Z}, \\ x^{k_0} &= \hat{x}, \end{aligned} \tag{3.69}$$

is given and we are looking for a solution for all $k \leq k_0$. Then, by defining $y^k := x^{-k+1}$ and $g^k := f^{-k}$, (3.69) is equivalent to

$$\begin{aligned} & & E_{-k}\, x^{-k+1} &= A_{-k}\, x^{-k} + f^{-k}, & k &\in \mathbb{Z} \\ &\Leftrightarrow & E_{-k}\, y^k &= A_{-k}\, y^{k+1} - g^k, & k &\in \mathbb{Z} \\ &\Leftrightarrow & A_{-k}\, y^{k+1} &= E_{-k}\, y^k + g^k, & k &\in \mathbb{Z}. \end{aligned}$$

By calculating the solution of the very last system into the future with the initial condition $y^{-k_0+1} = \hat{x}$, i.e., by calculating $\{y^k\}_{k\geq -k_0+1}$, we see through resubstitution, that we got a solution $\{y^k\}_{k\geq -k_0+1} = \{x^{-k+1}\}_{k\geq -k_0+1} = \{x^{k+1}\}_{-k\geq -k_0+1} = \{x^{k+1}\}_{k\leq k_0-1} = \{x^k\}_{k-1\leq k_0-1} = \{x^k\}_{k\leq k_0}$, i.e., a solution corresponding to the second case of Theorem 2.20. Thus, we do not have to consider the second case separately. We make the following definition.

Definition 3.33. Let $\{(E_k, A_k)\}_{k\in\mathbb{Z}}$ be a sequence of matrix pairs. Then

$$\{(A_{-k}, E_{-k})\}_{k\in\mathbb{Z}} \tag{3.70}$$

is called the *reversed sequence of matrix pairs*. Analogously, the descriptor system corresponding to (3.70) is called the *reversed descriptor system*. Also, the strangeness index of (3.70) is call *reversed strangeness index* and is denoted by μ_b (for backwards). In contrast to this, the strangeness index of the original sequence is also called *ordinary strangeness index* or *forward strangeness index* and denoted by μ_f.

Clearly the reversed system is very similar to the original system and obviously the following Lemma may be derived.

Lemma 3.34. *Let $\{(E_k, A_k)\}_{k\in\mathbb{Z}}$ and $\{(\tilde{E}_k, \tilde{A}_k)\}_{k\in\mathbb{Z}}$ be two globally equivalent sequences of matrix pairs. Then the reversed sequences are also globally equivalent.*

Proof. By assumption we know that there exist invertible matrices P_k, Q_k such that

$$\begin{aligned} E_k &= P_k \tilde{E}_k Q_{k+1}, \\ A_k &= P_k \tilde{A}_k Q_k, \end{aligned}$$

for all $k \in \mathbb{Z}$. Substituting k by $-k$ then yields

$$\begin{aligned} E_{-k} &= P_{-k}\, \tilde{E}_{-k}\, Q_{-k+1}, \\ A_{-k} &= P_{-k}\, \tilde{A}_{-k}\, Q_{-k}, \end{aligned}$$

for all $k \in \mathbb{Z}$. Setting $R_k := P_{-k}$ and $S_k := Q_{-k+1}$, this condition then is finally equivalent to

$$\begin{aligned} A_{-k} &= R_k\, \tilde{A}_{-k}\, S_{k+1}, \\ E_{-k} &= R_k\, \tilde{E}_{-k}\, S_k, \end{aligned}$$

for all $k \in \mathbb{Z}$, which proves the claim with the transformation matrix sequences $\{R_k\}$ and $\{S_k\}$. $\square$

Chapter 4

Two-way global canonical form for linear descriptor systems

In the previous chapter we generalized the first and second case of Theorem 2.20 to linear discrete-time descriptor systems with variable coefficients. This leaves the third case of Theorem 2.20, i.e., the case where one wants to get a solution on the whole $\mathbb{Z}$. This case is really different from the first two cases and it will therefore be studied in this chapter. To see that the third case of Theorem 2.20 really is different from the first two cases consider the form (3.26). The problem is that in this (strangeness-free) form (3.26) the $A_k^{(1)}$ are allowed to be arbitrary. Consider a descriptor system which only consists of the (1,1) block in (3.26). For such a system one can easily compute the unique value of x^{k_0+1} once the value of x^{k_0} is given. In contrast, if the value for x^{k_0} is given there may be many choices of appropriate x^{k_0-1} values (e.g., $x^{k_0} = 0x^{k_0-1}$, $x^{k_0-1} = x^{k_0-2}$, $x^{k_0-2} = x^{k_0-3}$, ...) or even no possible choice of an appropriate x^{k_0-1} value (e.g., $x^{k_0} = x^{k_0-1}$, $x^{k_0-1} = 0x^{k_0-2}$, given that $x^{k_0} \neq 0$), depending on the sequence of the $A_k^{(1)}$ matrices. Also, the solvability may vary from iterate to iterate. This shows that additional rank assumptions are necessary to generalize the third case of Theorem 2.20 to linear discrete-time descriptor systems with variable coefficients.
One approach that suggests itself is to not only demand the system itself to have well defined strangeness index but to also demand the reversed system to have well defined strangeness index.

Lemma 4.1. *For $k \in \mathbb{Z}$ let $E_k, A_k \in \mathbb{C}^{m,n}$ be such matrices, that the strangeness index μ_f and the reversed strangeness index μ_b of $\{(E_k, A_k)\}_{k\in\mathbb{Z}}$ are both well defined. Perform one step of index reduction from (3.13) to (3.14) on the reversed sequence $\{(A_{-k}, E_{-k})\}_{k\in\mathbb{Z}}$ and denote the so obtained sequence by $\{(\tilde{A}_{-k}, \tilde{E}_{-k})\}_{k\in\mathbb{Z}}$. Then, not only the reversed strangeness index $\tilde{\mu}_b$ (i.e., the strangeness index of $\{(\tilde{A}_{-k}, \tilde{E}_{-k})\}_{k\in\mathbb{Z}}$) but also the strangeness index $\tilde{\mu}_f$ of $\{(\tilde{E}_k, \tilde{A}_k)\}_{k\in\mathbb{Z}}$ is well defined. We have $\tilde{\mu}_f \leq \mu_f$ and $\tilde{\mu}_b \leq \mu_b$.*

Proof. Since the strangeness index of $\{(E_k, A_k)\}_{k\in\mathbb{Z}}$ is well defined and all A_k have the same constant rank (since the strangeness index of $\{(A_{-k}, E_{-k})\}_{k\in\mathbb{Z}}$ is also well defined) we know from Corollary 3.21 and from Theorem 3.16 part 2. that with $p \in \{1, \ldots, r_{f,0}\}$, the definitions from (3.18) and with $b_i := a_i - w_{i+1}$ for $i \in \mathbb{N}_0$ the sequence $\{(E_k, A_k)\}_{k\in\mathbb{Z}}$ is globally equivalent to a sequence of matrix pairs of the form

$$
\left\{ \left(\begin{bmatrix}
I_p & 0 & 0 & * & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & * \\
0 & I_{r_{f,0}-p} & 0 & * & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & * \\
0 & \cdots & 0 & 0 & 0 & I_{w_{\mu_f}} & 0 & & & & & \\
\vdots & & \vdots & & & & & I_{w_{\mu_f-1}} & 0 & & & \\
\vdots & & \vdots & & & & & & & \ddots & & \\
\vdots & & \vdots & & & & & & & & I_{w_1} & 0 \\
0 & \cdots & 0 & & & & & & & & & 0 \\
0 & \cdots & 0 & 0_{w_{\mu_f+1}} & 0 & 0 & D_k^{(\mu_f)} & 0 & * & \cdots & 0 & * \\
\vdots & & \vdots & 0 & 0_{b_{\mu_f}} & 0 & E_k^{(\mu_f)} & 0 & * & \cdots & 0 & * \\
\vdots & & \vdots & & & 0_{w_{\mu_f}} & 0 & 0 & D_k^{(\mu_f-1)} & \cdots & 0 & * \\
\vdots & & \vdots & & & 0 & 0_{b_{\mu_f-1}} & 0 & E_k^{(\mu_f-1)} & \cdots & 0 & * \\
\vdots & & \vdots & & & & & & & \ddots & \vdots & \vdots \\
\vdots & & \vdots & & & & & & & & 0 & D_k^{(1)} \\
\vdots & & \vdots & & & & & & & & 0 & E_k^{(1)} \\
\vdots & & \vdots & & & & & & & & 0_{w_1} & 0 \\
0 & \cdots & 0 & & & & & & & & 0 & 0_{b_0}
\end{bmatrix}, \right. \right.
$$

$$
\left. \left. \begin{bmatrix}
A_k^{(1)} & A_k^{(2)} & A_k^{(3)} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & 0 & 0 & \vdots & & & & & & & & \vdots \\
0 & \cdots & 0 & \vdots & & & & & & & & \vdots \\
\vdots & & \vdots & \vdots & & & & & & & & \vdots \\
\vdots & & \vdots & \vdots & & & & & & & & \vdots \\
\vdots & & \vdots & \vdots & & & & & & & & \vdots \\
0 & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & \cdots & 0 & I_{w_{\mu_f+1}} & & & & & & & & \\
\vdots & & \vdots & & I_{b_{\mu_f}} & & & & & & & \\
\vdots & & \vdots & & & I_{w_{\mu_f}} & & & & & & \\
\vdots & & \vdots & & & & I_{b_{\mu_f-1}} & & & & & \\
\vdots & & \vdots & & & & & I_{w_{\mu_f-1}} & & & & \\
\vdots & & \vdots & & & & & & I_{b_{\mu_f-2}} & & & \\
\vdots & & \vdots & & & & & & & \ddots & & \\
\vdots & & \vdots & & & & & & & & I_{w_1} & \\
0 & \cdots & 0 & & & & & & & & & I_{b_0}
\end{bmatrix} \right) \right\}_{k\in\mathbb{Z}},
$$

where all $\begin{bmatrix} A_k^{(1)} & A_k^{(2)} & A_k^{(3)} \end{bmatrix}$ have full row rank. Then, with $I_{r_{f,0}-p}$ all $*$ blocks in the same row can be eliminated. This may yield $*$ blocks in the first row of the left matrices (next to $\begin{bmatrix} A_k^{(1)} & A_k^{(2)} & A_k^{(3)} \end{bmatrix}$), but these $*$ blocks can again be eliminated by the identity matrices below. Thus, the system is also equivalent to

$$
\left\{ \left(\begin{bmatrix}
I_p & 0 & 0 & * & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & * \\
0 & I_{r_{f,0}-p} & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & \cdots & 0 & 0 & 0 & I_{w_{\mu_f}} & 0 & & & & & \\
\vdots & & \vdots & & & & & I_{w_{\mu_f-1}} & 0 & & & \\
\vdots & & \vdots & & & & & & & \ddots & & \\
\vdots & & \vdots & & & & & & & & I_{w_1} & 0 \\
0 & \cdots & 0 & & & & & & & & & 0 \\
0 & \cdots & 0 & 0_{w_{\mu_f+1}} & 0 & 0 & D_k^{(\mu_f)} & 0 & * & \cdots & 0 & * \\
\vdots & & \vdots & 0 & 0_{b_{\mu_f}} & 0 & E_k^{(\mu_f)} & 0 & * & \cdots & 0 & * \\
\vdots & & \vdots & & & 0_{w_{\mu_f}} & 0 & 0 & D_k^{(\mu_f-1)} & \cdots & 0 & * \\
\vdots & & \vdots & & & 0 & 0_{b_{\mu_f-1}} & 0 & E_k^{(\mu_f-1)} & \cdots & 0 & * \\
\vdots & & \vdots & & & & & & & \ddots & \vdots & \vdots \\
\vdots & & \vdots & & & & & & & & 0 & D_k^{(1)} \\
\vdots & & \vdots & & & & & & & & 0 & E_k^{(1)} \\
\vdots & & \vdots & & & & & & & & 0_{w_1} & 0 \\
0 & \cdots & 0 & & & & & & & & 0 & 0_{b_0}
\end{bmatrix}, \right. \right. \tag{4.1}
$$

$$
\left. \left. \begin{bmatrix}
A_k^{(1)} & A_k^{(2)} & A_k^{(3)} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & 0 & 0 & \vdots & & & & & & & & \vdots \\
0 & \cdots & 0 & \vdots & & & & & & & & \vdots \\
\vdots & & \vdots & \vdots & & & & & & & & \vdots \\
\vdots & & \vdots & \vdots & & & & & & & & \vdots \\
\vdots & & \vdots & \vdots & & & & & & & & \vdots \\
0 & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & \cdots & 0 & I_{w_{\mu_f+1}} & & & & & & & & \\
\vdots & & \vdots & & I_{b_{\mu_f}} & & & & & & & \\
\vdots & & \vdots & & & I_{w_{\mu_f}} & & & & & & \\
\vdots & & \vdots & & & & I_{b_{\mu_f-1}} & & & & & \\
\vdots & & \vdots & & & & & I_{w_{\mu_f-1}} & & & & \\
\vdots & & \vdots & & & & & & I_{b_{\mu_f-2}} & & & \\
\vdots & & \vdots & & & & & & & \ddots & & \\
\vdots & & \vdots & & & & & & & & I_{w_1} & \\
0 & \cdots & 0 & & & & & & & & & I_{b_0}
\end{bmatrix} \right) \right\}_{k\in\mathbb{Z}} .
$$

Performing one reduction step on the reversed of system (4.1) and reversing the reduced system back shows that $\{(\tilde{E}_k, \tilde{A}_k)\}_{k\in\mathbb{Z}}$ is globally equivalent to

$$
\left\{ \left(\begin{bmatrix}
I_p & 0 & 0 & * & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & * \\
0 & I_{r_{f,0}-p} & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & \cdots & 0 & 0 & 0 & I_{w_{\mu_f}} & 0 & & & & & \\
\vdots & & \vdots & & & & & I_{w_{\mu_f-1}} & 0 & & & \\
\vdots & & \vdots & & & & & & & \ddots & & \\
\vdots & & \vdots & & & & & & & & I_{w_1} & 0 \\
0 & \cdots & 0 & & & & & & & & & 0 \\
0 & \cdots & 0 & 0_{w_{\mu_f+1}} & 0 & 0 & D_k^{(\mu_f)} & 0 & * & \cdots & 0 & * \\
\vdots & & \vdots & 0 & 0_{b_{\mu_f}} & 0 & E_k^{(\mu_f)} & 0 & * & \cdots & 0 & * \\
\vdots & & \vdots & & & 0_{w_{\mu_f}} & 0 & 0 & D_k^{(\mu_f-1)} & \cdots & 0 & * \\
\vdots & & \vdots & & & 0 & 0_{b_{\mu_f-1}} & 0 & E_k^{(\mu_f-1)} & \cdots & 0 & * \\
\vdots & & \vdots & & & & & & & \ddots & \vdots & \vdots \\
\vdots & & \vdots & & & & & & & & 0 & D_k^{(1)} \\
\vdots & & \vdots & & & & & & & & 0 & E_k^{(1)} \\
\vdots & & \vdots & & & & & & & & 0_{w_1} & 0 \\
0 & \cdots & 0 & & & & & & & & 0 & 0_{b_0}
\end{bmatrix}, \right. \right.
$$

$$
\left. \left. \begin{bmatrix}
A_k^{(1)} & 0 & A_k^{(3)} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & 0 & 0 & \vdots & & & & & & & & \vdots \\
0 & \cdots & 0 & \vdots & & & & & & & & \vdots \\
\vdots & & \vdots & \vdots & & & & & & & & \vdots \\
\vdots & & \vdots & \vdots & & & & & & & & \vdots \\
\vdots & & \vdots & \vdots & & & & & & & & \vdots \\
0 & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & \cdots & 0 & I_{w_{\mu_f+1}} & & & & & & & & \\
\vdots & & \vdots & & I_{b_{\mu_f}} & & & & & & & \\
\vdots & & \vdots & & & 0_{w_{\mu_f}} & & & & & & \\
\vdots & & \vdots & & & & I_{b_{\mu_f-1}} & & & & & \\
\vdots & & \vdots & & & & & 0_{w_{\mu_f-1}} & & & & \\
\vdots & & \vdots & & & & & & I_{b_{\mu_f-2}} & & & \\
\vdots & & \vdots & & & & & & & \ddots & & \\
\vdots & & \vdots & & & & & & & & 0_{w_1} & \\
0 & \cdots & 0 & & & & & & & & & I_{b_0}
\end{bmatrix} \right) \right\}_{k\in\mathbb{Z}} .
$$

Reordering of rows shows that $\{(\tilde{E}_k, \tilde{A}_k)\}_{k\in\mathbb{Z}}$ is globally equivalent to

$$
\left\{\left(
\begin{bmatrix}
I_p & 0 & 0 & * & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & * \\
0 & I_{r_{f,0}-p} & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & \cdots & 0 & 0 & 0 & I_{w_{\mu_f}} & 0 & & & & & \\
\vdots & & \vdots & & & & & I_{w_{\mu_f-1}} & 0 & & & \\
\vdots & & \vdots & & & & & & & \ddots & & \\
\vdots & & \vdots & & & & & & & & I_{w_1} & 0 \\
0 & \cdots & 0 & & & & & & & & & 0 \\
0 & \cdots & 0 & & & 0_{w_{\mu_f}} & 0 & 0 & D_k^{(\mu_f-1)} & \cdots & 0 & * \\
\vdots & & \vdots & & & & & & & \ddots & \vdots & \vdots \\
\vdots & & \vdots & & & & & & & & 0 & D_k^{(1)} \\
\vdots & & \vdots & & & & & & & & 0_{w_1} & 0 \\
\vdots & & \vdots & 0_{w_{\mu_f+1}} & 0 & 0 & D_k^{(\mu_f)} & 0 & * & \cdots & 0 & * \\
\vdots & & \vdots & 0 & 0_{b_{\mu_f}} & 0 & E_k^{(\mu_f)} & 0 & * & \cdots & 0 & * \\
\vdots & & \vdots & & & 0 & 0_{b_{\mu_f-1}} & 0 & E_k^{(\mu_f-1)} & \cdots & 0 & * \\
\vdots & & \vdots & & & & & & & \ddots & 0 & E_k^{(1)} \\
0 & \cdots & 0 & & & & & & & & 0 & 0_{b_0}
\end{bmatrix},\right.\right.
$$
$$
\left.\left.
\begin{bmatrix}
A_k^{(1)} & 0 & A_k^{(3)} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & 0 & 0 & \vdots & & & & & & & & \vdots \\
0 & \cdots & 0 & \vdots & & & & & & & & \vdots \\
\vdots & & \vdots & \vdots & & & & & & & & \vdots \\
\vdots & & \vdots & \vdots & & & & & & & & \vdots \\
\vdots & & \vdots & \vdots & & & & & & & & \vdots \\
0 & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & \cdots & 0 & & & 0_{w_{\mu_f}} & & & & & & \\
\vdots & & \vdots & & & & & 0_{w_{\mu_f-1}} & & & & \\
\vdots & & \vdots & & & & & & & \ddots & & \\
\vdots & & \vdots & & & & & & & & 0_{w_1} & \\
\vdots & & \vdots & I_{w_{\mu_f+1}} & & & & & & & & \\
\vdots & & \vdots & & I_{b_{\mu_f}} & & & & & & & \\
\vdots & & \vdots & & & & I_{b_{\mu_f-1}} & & & & & \\
\vdots & & \vdots & & & & & & \ddots & & & \\
0 & \cdots & 0 & & & & & & & & & I_{b_0}
\end{bmatrix}\right)\right\}_{k\in\mathbb{Z}}.
$$

Reordering of columns shows that $\{(\tilde{E}_k, \tilde{A}_k)\}_{k\in\mathbb{Z}}$ is globally equivalent to

$$
\left\{\left(\begin{bmatrix}
I_p & 0 & * & \cdots & * & 0 & * & \cdots & \cdots & \cdots & \cdots & * \\
0 & I_{r_{f,0}-p} & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
 & & I_{w_{\mu_f}} & & & & & & & & & \\
 & & & \ddots & & & & & & & & \\
 & & & & I_{w_1} & & & & & & & \\
 & & & & 0 & \cdots & \cdots & \cdots & \cdots & 0 & & \\
 & & & & & & & & & D_k^{(\mu_f-1)} & & \\
 & & & & & & & & & & \ddots & \\
 & & & & & & & & & & & D_k^{(1)} \\
 & & & & & & & & & & & 0 \\
 & & & & & & 0_{w_{\mu_f+1}} & 0 & D_k^{(\mu_f)} & * & \cdots & * \\
 & & & & & & 0 & 0_{b_{\mu_f}} & E_k^{(\mu_f)} & * & \ddots & \vdots \\
 & & & & & & & & 0_{b_{\mu_f-1}} & E_k^{(\mu_f-1)} & & * \\
 & & & & & & & & & & \ddots & E_k^{(1)} \\
 & & & & & & & & & & & 0_{b_0}
\end{bmatrix},\right.\right.
$$

$$
\left.\left.\begin{bmatrix}
A_k^{(1)} & 0 & \cdots & \cdots & 0 & A_k^{(3)} & & & & & & \\
 & & & & & & & & & & & \\
 & & & & & & & & & & & \\
 & & & & & & I_{w_{\mu_f+1}} & & & & & \\
 & & & & & & & I_{b_{\mu_f}} & & & & \\
 & & & & & & & & I_{b_{\mu_f-1}} & & & \\
 & & & & & & & & & I_{b_{\mu_f-2}} & & \\
 & & & & & & & & & & \ddots & \\
 & & & & & & & & & & & I_{b_0}
\end{bmatrix}\right)\right\}_{k\in\mathbb{Z}}
$$

$$
\sim\left\{\left(\begin{bmatrix} I_{\hat{r}_f} & 0 & * \\ 0 & 0 & \tilde{D}_k \\ 0 & 0 & \tilde{E}_k \end{bmatrix}, \begin{bmatrix} \tilde{A}_k^{(1)} & A_k^{(3)} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_{\hat{h}_f} \end{bmatrix}\right)\right\}_{k\in\mathbb{Z}},
$$

by setting $\hat{h}_f = b_{\mu_f} + \ldots + b_0$, $\hat{r}_f = r_{f,0} + w_{\mu_f} + \ldots + w_1$ and

$$
\tilde{D}_k = \begin{bmatrix} 0 & D_k^{(\mu_f)} & & 0 \\ & \ddots & \ddots & \\ & & \ddots & D_k^{(1)} \\ & & & 0 \end{bmatrix},
$$

$$\tilde{E}_k = \begin{bmatrix} 0_{b_{\mu_f}} & E_k^{(\mu_f)} & & 0 \\ & \ddots & \ddots & \\ & & \ddots & E_k^{(1)} \\ & & & 0_{b_0} \end{bmatrix},$$

since $w_{\mu_f+1} = 0$ (which can be seen from (3.24), (3.18g), (3.18c) and (3.20)).
By setting $\hat{a}_i := b_i$ and $\hat{w}_i := w_{i+1}$ for $i \in \mathbb{N}_0$ we finally see, that the sequence $\{\hat{a}_i\}_{i\in\mathbb{N}_0}$ is non-increasing due to (3.19l). Also it is clear that

$$\hat{w}_i + \hat{a}_i = w_{i+1} + b_i = w_{i+1} + a_i - w_{i+1} = a_i \overset{(3.34)}{\leq} a_{i-1} - w_i = b_i = \hat{a}_{i-1}.$$

It may happen that some $\hat{a}_{\mu_f} = \ldots = \hat{a}_{\tilde{\mu}_f+1} = 0$ in which case the strangeness index $\tilde{\mu}_f$ of $\{(\tilde{E}_k, \tilde{A}_k)\}_{k\in\mathbb{Z}}$ has been decreased. Anyway, with (3.44), all assumptions of Lemma 3.22 are fulfilled, which shows that the strangeness index $\tilde{\mu}_f$ really is well defined.
That the reversed strangeness index $\tilde{\mu}_b$ is still well defined follows from the fact that we have performed the reduction step on the reversed system. To understand this, recall that Definition 3.15 demands the constant rank assumptions (3.12) to hold for every step of the reduction procedure. Actually performing one reduction step uses only the constant rank assumptions (3.12) of the first step and the so obtained system will still satisfy the constant rank assumptions (3.12) in every further reduction step. □

The index reduction performed in the previous Lemma 3.22 will be used frequently in the following, which is why we introduce the following Definition.

Definition 4.2. Let $\{(E_k, A_k)\}_{k\in\mathbb{Z}}$ be a sequence of matrix pairs. Then performing one step of index reduction from form (3.13) to (3.14) on the reversed sequence $\{(A_{-k}, E_{-k})\}_{k\in\mathbb{Z}}$ and re-reversing the so obtained sequence is called one step of *reversed index reduction*. In contrast to this, the index reduction from form (3.13) to (3.14) on the original sequence $\{(E_k, A_k)\}_{k\in\mathbb{Z}}$ is also called *ordinary index reduction* or *forward index reduction*.

Lemma 4.1 shows that under the assumption that both the strangeness index and the reversed strangeness index are well-defined one can perform ordinary and reversed index reduction steps at will. One may conjecture, that the order in which one performs the index reduction steps is of no meaning, as long as one performs the same number of reduction steps, but this is false as the following example shows.

Example 4.3. Consider the constant sequence of matrix pairs

$$\left\{ \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) \right\}_{k\in\mathbb{Z}}. \tag{4.2}$$

First performing one ordinary step of index reduction on (4.2) yields the sequence

$$\left\{ \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) \right\}_{k\in\mathbb{Z}}. \tag{4.3}$$

Performing one step of reversed index reduction on this sequence does not alter the sequence any more. First performing one reversed step of index reduction on (4.2), however, yields the sequence

$$\left\{ \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right) \right\}_{k \in \mathbb{Z}}, \tag{4.4}$$

which again is not altered anymore by one further step of ordinary index reduction. Comparing (4.3) with (4.4) clearly shows that these two sequences are not globally equivalent, since (corresponding to Lemma 3.10) those matrix pairs do not have the same characteristic values.

Remark 4.4. Example 4.3 also shows that a step of reversed index reduction may really change the ordinary index. Therefore note that (4.2) has an ordinary strangeness index of 1. Nonetheless, performing one step of reversed index reduction on (4.2) yields (4.4), which surely has an ordinary strangeness index of 0.

Let us derive the canonical form under the assumption that both the strangeness index and the reversed strangeness index are well defined.

Theorem 4.5. *For $k \in \mathbb{Z}$ let $E_k, A_k \in \mathbb{C}^{m,n}$ be matrices, such that the strangeness index and the reversed strangeness index of $\{(E_k, A_k)\}_{k \in \mathbb{Z}}$ are both well defined. Define the matrices*

$$\begin{array}{lll} Z_k & \textit{as a basis of} & \operatorname{corange}(E_k) \ \textit{for } k \in \mathbb{Z}, \\ Y_k & \textit{as a basis of} & \operatorname{corange}(A_k) \ \textit{for } k \in \mathbb{Z}. \end{array}$$

Then, there exist $h_f, h_b, q \in \mathbb{N}_0$ such that for all $k \in \mathbb{Z}$ we have

$$\begin{aligned} h_f &= \operatorname{rank}\left(Z_k^H A_k\right), && \textit{(forward direction)} \\ h_b &= \operatorname{rank}\left(Y_k^H E_k\right), && \textit{(backward direction)} \\ q &= h_f + h_b - \operatorname{rank}\left(\begin{bmatrix} Y_k^H E_k \\ Z_{k+1}^H A_{k+1} \end{bmatrix} \right). \end{aligned} \tag{4.5}$$

These quantities in (4.5) are invariant under global equivalence and we have

$$\{(E_k, A_k)\}_{k \in \mathbb{Z}} \sim \left\{ \left(\begin{bmatrix} E_k^{(1)} & 0 & 0 & E_k^{(2)} \\ 0 & I_{h_b - q} & 0 & 0 \\ 0 & 0 & I_q & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} A_k^{(1)} & A_k^{(2)} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I_q & 0 \\ 0 & 0 & 0 & I_{h_f - q} \\ 0 & 0 & 0 & 0 \end{bmatrix} \right) \right\}, \tag{4.6}$$

where for all $k \in \mathbb{Z}$ the matrices $\begin{bmatrix} E_k^{(1)} & E_k^{(2)} \end{bmatrix}$ and $\begin{bmatrix} A_k^{(1)} & A_k^{(2)} \end{bmatrix}$ have full row rank.

Proof. That rank $\left(Z_k^H A_k\right)$ and rank $\left(Y_k^H E_k\right)$ are invariant under global equivalence follows from Lemma 3.9. That they are constant for all $k \in \mathbb{Z}$ follows from the fact that the strangeness index and the reversed strangeness index are both well defined.
To show that q is independent of the choice of the bases, let Y_k and $\tilde{Y}_k$ be bases of corange (A_k) and let Z_k and $\tilde{Z}_k$ be bases of corange (E_k) for all $k \in \mathbb{Z}$. Then for all $k \in \mathbb{Z}$ there exists invertible matrices M_{Y_k} and M_{Z_k} such that $Y_k = \tilde{Y}_k M_{Y_k}$ and $Z_k = \tilde{Z}_k M_{Z_k}$. This shows that

$$\operatorname{rank}\left(\begin{bmatrix} Y_k^H E_k \\ Z_{k+1}^H A_{k+1} \end{bmatrix}\right) = \operatorname{rank}\left(\begin{bmatrix} M_{Y_k}^{-H} & 0 \\ 0 & M_{Z_{k+1}}^{-H} \end{bmatrix} \begin{bmatrix} Y_k^H E_k \\ Z_{k+1}^H A_{k+1} \end{bmatrix}\right) = \operatorname{rank}\left(\begin{bmatrix} \tilde{Y}_k^H E_k \\ \tilde{Z}_{k+1}^H A_{k+1} \end{bmatrix}\right),$$

and thus, that q is independent of the choice of the bases. To show the invariance under global equivalence, let $\{(\tilde{E}_k, \tilde{A}_k)\}_{k\in\mathbb{Z}}$ be globally equivalent to $\{(E_k, A_k)\}_{k\in\mathbb{Z}}$, i.e., let Q_k and P_k be invertible matrices, such that for all $k \in \mathbb{Z}$ we have

$$\begin{aligned} E_k &= P_k \tilde{E}_k Q_{k+1}, \\ A_k &= P_k \tilde{A}_k Q_k. \end{aligned}$$

Since

$$\begin{aligned} 0 = Y_k^H A_k &= Y_k^H P_k \tilde{A}_k Q_k = \left(P_k^H Y_k\right)^H \tilde{A}_k Q_k, \\ 0 = Z_k^H E_k &= Z_k^H P_k \tilde{E}_k Q_{k+1} = \left(P_k^H Z_k\right)^H \tilde{E}_k Q_{k+1}, \end{aligned}$$

it is clear that $\hat{Y}_k := P_k^H Y_k$ is a basis of corange $\left(\tilde{A}_k\right)$ and that $\hat{Z}_k := P_k^H Z_k$ is a basis of corange $\left(\tilde{E}_k\right)$. With this we see that

$$\begin{aligned} \operatorname{rank}\left(\begin{bmatrix} \hat{Y}_k^H \tilde{E}_k \\ \hat{Z}_{k+1}^H \tilde{A}_{k+1} \end{bmatrix}\right) &= \operatorname{rank}\left(\begin{bmatrix} Y_k^H P_k \tilde{E}_k \\ Z_{k+1}^H P_{k+1} \tilde{A}_{k+1} \end{bmatrix}\right) \\ &= \operatorname{rank}\left(\begin{bmatrix} Y_k^H P_k \tilde{E}_k Q_{k+1} \\ Z_{k+1}^H P_{k+1} \tilde{A}_{k+1} Q_{k+1} \end{bmatrix}\right) \\ &= \operatorname{rank}\left(\begin{bmatrix} Y_k^H E_k \\ Z_{k+1}^H A_{k+1} \end{bmatrix}\right), \end{aligned}$$

which means that q does only depend on the equivalence class.
Since the strangeness index of $\{(E_k, A_k)\}_{k\in\mathbb{Z}}$ is well defined, it is clear that the sequence can be transformed to the form (3.13). Since the reversed strangeness index is also well defined, we also know that all A_k have constant rank. Thus, in (3.13) all $A_k^{(1)}$ matrices also have to have constant rank. Thus, by transforming the first block row of (3.13) from the left we have that $\{(E_k, A_k)\}_{k\in\mathbb{Z}}$ is equivalent to

$$\left\{\left(\begin{bmatrix} E_k^{(1,1)} & E_k^{(1,2)} \\ E_k^{(2,1)} & E_k^{(2,2)} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A_k^{(1,1)} & 0 \\ 0 & 0 \\ 0 & I_{h_f} \\ 0 & 0 \end{bmatrix}\right)\right\}_{k\in\mathbb{Z}}, \tag{4.7}$$

with all $A_k^{(1,1)}$ having full (constant) row rank. Performing one (ordinary) reduction step from (3.13) to (3.14) on (4.7) yields the sequence

$$\left\{\left(\begin{bmatrix} E_k^{(1,1)} & 0 \\ E_k^{(2,1)} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A_k^{(1,1)} & 0 \\ 0 & 0 \\ 0 & I_{h_f} \\ 0 & 0 \end{bmatrix}\right)\right\}_{k\in\mathbb{Z}}. \tag{4.8}$$

Then it follows from Lemma 4.1 that (4.8) still has well-defined reversed strangeness index. Let $\tilde{Y}_k$ be bases of corange $\left(\begin{bmatrix} A_k^{(1,1)} & 0 \\ 0 & 0 \\ 0 & I_{h_f} \\ 0 & 0 \end{bmatrix}\right)$. Then clearly $\tilde{Y}_k = \begin{bmatrix} 0 & 0 \\ I & 0 \\ 0 & 0 \\ 0 & I \end{bmatrix}$, since all $A_k^{(1,1)}$ have full row rank. Thus, since the reversed strangeness index of (4.8) is well defined, we know that for every $k \in \mathbb{Z}$ the matrix

$$\tilde{Y}_k^H \begin{bmatrix} E_k^{(1,1)} & 0 \\ E_k^{(2,1)} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} E_k^{(2,1)} & 0 \\ 0 & 0 \end{bmatrix}$$

has to have constant rank, which means that all $E_k^{(2,1)}$ have to have constant rank. Let us say all $E_k^{(2,1)}$ matrices have constant rank $\hat{g}$. By reducing all $E_k^{(2,1)}$ in (4.7) to echelon form and adapting the indexing we then see that $\{(E_k, A_k)\}_{k\in\mathbb{Z}}$ is globally equivalent to

$$\left\{\left(\begin{bmatrix} E_k^{(1,1)} & E_k^{(1,2)} & E_k^{(1,3)} \\ 0 & I_{\hat{g}} & E_k^{(2,3)} \\ 0 & 0 & E_k^{(3,3)} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} A_k^{(1,1)} & A_k^{(1,2)} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_{h_f} \\ 0 & 0 & 0 \end{bmatrix}\right)\right\}_{k\in\mathbb{Z}}$$
$$\sim \left\{\left(\begin{bmatrix} E_k^{(1,1)} & 0 & \tilde{E}_k^{(1,3)} \\ 0 & I_{\hat{g}} & E_k^{(2,3)} \\ 0 & 0 & E_k^{(3,3)} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} A_k^{(1,1)} & A_k^{(1,2)} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_{h_f} \\ 0 & 0 & 0 \end{bmatrix}\right)\right\}_{k\in\mathbb{Z}}$$
$$\sim \left\{\left(\begin{bmatrix} E_k^{(1,1)} & 0 & \tilde{E}_k^{(1,3)} \\ 0 & I_{\hat{g}} & 0 \\ 0 & 0 & E_k^{(3,3)} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} A_k^{(1,1)} & A_k^{(1,2)} & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_{h_f} \\ 0 & 0 & 0 \end{bmatrix}\right)\right\}_{k\in\mathbb{Z}}$$

$$\sim\left\{\left(\begin{bmatrix} E_k^{(1,1)} & 0 & \tilde{E}_k^{(1,3)} \\ 0 & I_{\hat{g}} & 0 \\ 0 & 0 & E_k^{(3,3)} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} A_k^{(1,1)} & A_k^{(1,2)} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_{h_f} \\ 0 & 0 & 0 \end{bmatrix}\right)\right\}_{k\in\mathbb{Z}},$$

where all $\begin{bmatrix} A_k^{(1,1)} & A_k^{(1,2)} \end{bmatrix}$ have full row rank. Also, all $\begin{bmatrix} E_k^{(1,1)} & 0 & \tilde{E}_k^{(1,3)} \\ 0 & I_{\hat{g}} & 0 \\ 0 & 0 & E_k^{(3,3)} \end{bmatrix}$ have full row rank, since those matrices are equivalent to the matrices $\begin{bmatrix} E_k^{(1,1)} & E_k^{(1,2)} \\ E_k^{(2,1)} & E_k^{(2,2)} \end{bmatrix}$ as in (4.7), which have full row rank. So all $E_k^{(3,3)}$ also have full row rank. Reducing all $E_k^{(3,3)}$ to echelon form then finally shows that $\{(E_k, A_k)\}_{k\in\mathbb{Z}}$ is globally equivalent to

$$\left\{\left(\begin{bmatrix} E_k^{(1,1)} & 0 & E_k^{(1,3)} & E_k^{(1,4)} \\ 0 & I_{\hat{g}} & 0 & 0 \\ 0 & 0 & I_{\hat{q}} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} A_k^{(1,1)} & A_k^{(1,2)} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I_{\hat{q}} & 0 \\ 0 & 0 & 0 & I_{h_f-\hat{q}} \\ 0 & 0 & 0 & 0 \end{bmatrix}\right)\right\}_{k\in\mathbb{Z}}$$

$$\sim\left\{\left(\begin{bmatrix} E_k^{(1,1)} & 0 & 0 & E_k^{(1,4)} \\ 0 & I_{h_b-\hat{q}} & 0 & 0 \\ 0 & 0 & I_{\hat{q}} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} A_k^{(1,1)} & A_k^{(1,2)} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I_{\hat{q}} & 0 \\ 0 & 0 & 0 & I_{h_f-\hat{q}} \\ 0 & 0 & 0 & 0 \end{bmatrix}\right)\right\}_{k\in\mathbb{Z}},$$

where $h_b := \hat{g} + \hat{q}$ has been used. Finally, we have $q = \hat{q}$, since (as shown above) the quantity defined in (4.5) is invariant under global equivalence and q can directly be calculated from the last sequence of matrix pairs. □

From the form (4.6) one may conjecture that it is also possible to show Theorem 4.5 by defining

$$q = h_f + h_b - \operatorname{rank}\left(\begin{bmatrix} Y_k^H E_k \\ Z_k^H A_k \end{bmatrix}\right), \tag{4.9}$$

instead of (4.5). This is not the case. If one would do so, q would not be invariant under global equivalence any more as shown by the following example.

Example 4.6. Define the (constant) sequence of matrix pairs

$$\{(E_k, A_k)\}_{k\in\mathbb{Z}} := \left\{\left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\right)\right\}_{k\in\mathbb{Z}},$$

which has $h_f = 1$, $h_b = 1$ and with both (4.9) or (4.5) $q = 1$. Transforming this sequence from the right by the sequence $\{Q_k\}_{k\in\mathbb{Z}}$ defined through

$$Q_{2k} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } Q_{2k+1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ for all } k \in \mathbb{Z},$$

will yield a sequence $\{(\tilde{E}_k, \tilde{A}_k)\}_{k\in\mathbb{Z}} = \{(E_k Q_{k+1}, A_k Q_k)\}_{k\in\mathbb{Z}}$ which satisfies

$$\begin{aligned} (\tilde{E}_{2k}, \tilde{A}_{2k}) &= \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \right) \text{ and} \\ (\tilde{E}_{2k+1}, \tilde{A}_{2k+1}) &= \left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \right) \text{ for all } k \in \mathbb{Z}. \end{aligned}$$

This sequence would have $q = 0$ if one would apply definition (4.9).

The same result as in Theorem 4.5 can be obtained under a weaker assumption.

Corollary 4.7. *Let $\{(E_k, A_k)\}_{k\in\mathbb{Z}}$ be a sequence and define the matrices*

$$\begin{aligned} &Z_k \quad \textit{as a basis of} \quad \operatorname{corange}(E_k) \ \textit{for } k \in \mathbb{Z}, \\ &Y_k \quad \textit{as a basis of} \quad \operatorname{corange}(A_k) \ \textit{for } k \in \mathbb{Z}. \end{aligned}$$

Assume that the quantities

$$r_f = r_{f,k} \equiv \operatorname{rank}(E_k), \tag{4.10a}$$

$$h_f = h_{f,k} \equiv \operatorname{rank}\left(Z_k^H A_k\right), \tag{4.10b}$$

$$h_b = h_{b,k} \equiv \operatorname{rank}\left(Y_k^H E_k\right), \tag{4.10c}$$

$$q = q_k \equiv h_{f,k} + h_{b,k} - \operatorname{rank}\left(\begin{bmatrix} Y_k^H E_k \\ Z_{k+1}^H A_{k+1} \end{bmatrix} \right), \tag{4.10d}$$

(which are invariant under global equivalence as shown in Theorem 4.5) are constant for all $k \in \mathbb{Z}$. Then we also have the relation (4.6), where for all $k \in \mathbb{Z}$ the matrices $\begin{bmatrix} E_k^{(1)} & E_k^{(2)} \end{bmatrix}$ and $\begin{bmatrix} A_k^{(1)} & A_k^{(2)} \end{bmatrix}$ have full row rank.

Proof. First we note that under the given assumptions the sequence $\{(E_k, A_k)\}_{k\in\mathbb{Z}}$ is globally equivalent to a sequence of the form

$$\left\{ \left(\begin{bmatrix} E_k^{(1,1)} & E_k^{(1,2)} \\ E_k^{(2,1)} & E_k^{(2,2)} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A_k^{(1,1)} & 0 \\ 0 & 0 \\ 0 & I_{h_f} \\ 0 & 0 \end{bmatrix} \right) \right\}_{k\in\mathbb{Z}},$$

with all $\begin{bmatrix} E_k^{(1,1)} & E_k^{(1,2)} \\ E_k^{(2,1)} & E_k^{(2,2)} \end{bmatrix}$ and all $A_k^{(1,1)}$ having full row rank. Since q is invariant under global equivalence, it is clear that

$$\operatorname{rank}\left(\begin{bmatrix} E_k^{(2,1)} & E_k^{(2,2)} \\ 0 & I_{h_f} \end{bmatrix}\right) = \operatorname{rank}\left(\begin{bmatrix} E_k^{(2,1)} & 0 \\ 0 & I_{h_f} \end{bmatrix}\right)$$

has to be constant for all $k \in \mathbb{Z}$. Thus, also all $E_k^{(2,1)}$ have to have constant rank. The remainder of the proof can then be carried out analogously to the proof of Theorem 4.5. □

Applying one step of ordinary and one step of reversed index reduction to the form (4.6) yields the form

$$\left\{\left(\begin{bmatrix} E_k^{(1,1)} & 0 & 0 & 0 \\ 0 & I_{h_b-q} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} A_k^{(1,1)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I_q & 0 \\ 0 & 0 & 0 & I_{h_f-q} \\ 0 & 0 & 0 & 0 \end{bmatrix}\right)\right\}_{k\in\mathbb{Z}}. \tag{4.11}$$

It is clear that first applying one step of reversed and then one step of ordinary index reduction will yield another form (i.e., the I_s block then stays in the left matrices and is therefore missing in the right matrices), as in Example 4.3.

Remark 4.8. The preceding results allow for an inductive procedure similar to Remark 3.13. For an original sequence of matrix pairs $\{(E_k, A_k)\}_{k\in\mathbb{Z}} =: \{(E_{k,0}, A_{k,0})\}_{k\in\mathbb{Z}}$ we define a sequence (of matrix pair sequences) $\{\{(E_{k,i}, A_{k,i})\}_{k\in\mathbb{Z}}\}_{i\in\mathbb{Z}}$ by the following procedure. First we reduce $\{(E_{k,i}, A_{k,i})\}_{k\in\mathbb{Z}}$ by Corollary 4.7 to the from (4.6) assuming that the local invariants $r_f =: r_{f,i}$, $h_f =: h_{f,i}$, $h_b =: h_{b,i}$ and $q =: q_i$ are constant for all matrix pairs on the whole interval $\mathbb{Z}$. Then we reduce the so obtained sequence of matrix pairs by one step of ordinary and one step of reversed index reduction to the form (4.11), which yields the next sequence of matrix pairs $\{(E_{k,i+1}, A_{k,i+1})\}_{k\in\mathbb{Z}}$. The so obtained sequence of values $\{(r_{f,i}, h_{f,i}, h_{b,i}, q_i)\}_{i\in\mathbb{N}_0}$ is characteristic for a given equivalence class of sequences of matrix pairs, due to Corollary 3.12, Corollary 4.7 and Lemma 3.34.

Remark 4.9. Under the assumption that the strangeness index and the reversed strangeness index of the sequence $\{(E_k, A_k)\}_{k\in\mathbb{Z}}$ are both well defined, all constant rank assumptions which are required in Remark 4.8 are satisfied, because of Lemma 4.1 and Theorem 4.5.

To define a strangeness index under the assumptions of Remark 4.8 we need a Lemma similar to Lemma 3.14.

Lemma 4.10. *Let the sequences $\{(r_{f,i}, h_{f,i}, h_{b,i}, q_i)\}_{i\in\mathbb{N}_0}$ and $\{\{(E_{k,i}, A_{k,i})\}_{k\in\mathbb{Z}}\}_{i\in\mathbb{N}_0}$ be defined as in Remark 4.8. In particular, let the constant rank assumptions (4.10) hold for every step* *of the reduction process in Remark 4.8. Defining the quantities*

$$
\begin{aligned}
r_{b,i} &:= r_{f,i} - h_{b,i} + h_{f,i} && \forall i \in \mathbb{N}_0, && \text{(4.12a)}\\
s_{E,i} &:= r_{f,i} - r_{f,i+1} && \forall i \in \mathbb{N}_0, && \text{(4.12b)}\\
s_{A,i} &:= r_{b,i} - r_{b,i+1} && \forall i \in \mathbb{N}_0, && \text{(4.12c)}\\
s_i &:= s_{E,i} + s_{A,i} && \forall i \in \mathbb{N}_0, && \text{(4.12d)}
\end{aligned}
$$

there exists a $\mu \in \mathbb{N}_0$ *so that*

$$
\begin{aligned}
r_{b,i} &= \operatorname{rank}(A_{k,i}) && \forall i \in \mathbb{N}_0,\ k \in \mathbb{Z}, && \text{(4.13a)}\\
r_{f,i+1} &\leq r_{f,i} && \forall i \in \mathbb{N}_0, && \text{(4.13b)}\\
r_{b,i+1} &\leq r_{b,i} && \forall i \in \mathbb{N}_0, && \text{(4.13c)}\\
s_{E,i} = s_{A,i} = s_i &= 0 && \forall i \geq \mu. && \text{(4.13d)}
\end{aligned}
$$

Proof. (4.13a) follows directly from (3.6d). Let $i \in \mathbb{N}_0$ be any non-negative integer. Then we know from Corollary 4.7 that

$$
\{(E_{k,i}, A_{k,i})\}_{k\in\mathbb{Z}} \sim \left\{\left(\begin{bmatrix} E_{k,i}^{(1)} & 0 & 0 & E_{k,i}^{(2)} \\ 0 & I_{h_{b,i}-q_i} & 0 & 0 \\ 0 & 0 & I_{q_i} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} A_{k,i}^{(1)} & A_{k,i}^{(2)} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I_{q_i} & 0 \\ 0 & 0 & 0 & I_{h_{f,i}-q_i} \\ 0 & 0 & 0 & 0 \end{bmatrix}\right)\right\}_{k\in\mathbb{Z}}
$$

$$
\Rightarrow \{(E_{k,i+1}, A_{k,i+1})\}_{k\in\mathbb{Z}} \sim \left\{\left(\begin{bmatrix} E_{k,i}^{(1)} & 0 & 0 & 0 \\ 0 & I_{h_{b,i}-q_i} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} A_{k,i}^{(1)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I_{q_i} & 0 \\ 0 & 0 & 0 & I_{h_{f,i}-q_i} \\ 0 & 0 & 0 & 0 \end{bmatrix}\right)\right\}_{k\in\mathbb{Z}}.
$$

This clearly shows that $\operatorname{rank}(E_{k,i+1}) \leq \operatorname{rank}(E_{k,i})$ and $\operatorname{rank}(A_{k,i+1}) \leq \operatorname{rank}(A_{k,i})$, which implies (4.13b) and (4.13c). Since we know that both of the sequences $\{r_{f,i}\}_{i\in\mathbb{N}_0}$ and $\{r_{b,i}\}_{i\in\mathbb{N}_0}$ are non-increasing and bounded by zero, they have to become stationary a some point μ, which shows (4.13d). $\square$

The previous Lemma 4.10 leads to the following Definition.

Definition 4.11. Let $\{(E_k, A_k)\}_{k\in\mathbb{Z}}$ be a sequence of matrix pairs. Let the sequence $\{(r_{f,i}, h_{f,i}, h_{b,i}, q_i)\}_{i\in\mathbb{N}_0}$ (as described in Remark 4.8) be well defined. In particular, let (4.10) hold for every entry $\{(E_{k,i}, A_{k,i})\}_{k\in\mathbb{Z}}$ of the sequence (of sequences of matrix pairs) in Remark 4.8. Then, with the definitions (4.12) we call

$$
\mu = \min\{i \in \mathbb{N}_0 \mid s_i = 0\} \tag{4.14}
$$

the *two-way strangeness index* of the sequence of matrix pairs $\{(E_k, A_k)\}_{k\in\mathbb{Z}}$ and of the associated descriptor system (1.5). In the case that $\mu = 0$ we call $\{(E_k, A_k)\}_{k\in\mathbb{Z}}$ and (1.5) *two-way strangeness-free.*

Since one step of the iterative procedure described in Remark 4.8 involves one step of ordinary and one step of reversed index reduction it may happen that the two-way strangeness index is smaller than the ordinary strangeness index, as shown in the following Example.

Example 4.12. Consider the sequence of (constant) matrix pairs

$$\{(E_k, A_k)\}_{k\in\mathbb{Z}} = \left\{\left(\begin{bmatrix}1 & 0\\0 & 1\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\1 & 0\\0 & 1\end{bmatrix}\right)\right\}_{k\in\mathbb{Z}}.$$

With Definition 3.15 we get the sequences

$$\begin{aligned}
(r_{f,0}, h_{f,0}, a_0, s_0) &= (2,1,1,1),\\
(r_{f,1}, h_{f,1}, a_1, s_1) &= (1,2,1,1),\\
(r_{f,2}, h_{f,2}, a_2, s_2) &= (0,3,1,0),\\
(r_{f,3}, h_{f,3}, a_3, s_3) &= (0,3,0,0),\\
(r_{f,4}, h_{f,4}, a_4, s_4) &= (0,3,0,0),\\
&\vdots
\end{aligned}$$

and thus an ordinary strangeness index of 2. With Definition 4.11, however, we face the reduction process

$$\begin{aligned}
&\left\{\left(\begin{bmatrix}1 & 0\\0 & 1\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\1 & 0\\0 & 1\end{bmatrix}\right)\right\}_{k\in\mathbb{Z}} \sim \left\{\left(\begin{bmatrix}0 & 1\\1 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}1 & 0\\0 & 0\\0 & 1\end{bmatrix}\right)\right\}_{k\in\mathbb{Z}}\\
\overset{\text{reduction}}{\sim}&\left\{\left(\begin{bmatrix}0 & 0\\1 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\0 & 0\\0 & 1\end{bmatrix}\right)\right\}_{k\in\mathbb{Z}}
\end{aligned}$$

and thus the sequences

$$\begin{aligned}
(r_{f,0}, h_{f,0}, h_{b,0}, q_0, r_{b,0}, s_{E,0}, s_{A,0}, s_0) &= (2,1,1,0,2,1,1,2),\\
(r_{f,1}, h_{f,1}, h_{b,1}, q_1, r_{b,1}, s_{E,1}, s_{A,1}, s_1) &= (1,1,1,0,1,0,0,0),\\
(r_{f,2}, h_{f,2}, h_{b,2}, q_2, r_{b,2}, s_{E,2}, s_{A,2}, s_2) &= (1,1,1,0,1,0,0,0),\\
&\vdots
\end{aligned}$$

which shows that the two-way strangeness index is 1.

On the other hand, the ordinary strangeness index and the two-way strangeness index can also coincide as shown in the following example.

Example 4.13. Consider the sequence of (constant) matrix pairs

$$\{(E_k, A_k)\}_{k\in\mathbb{Z}} = \left\{ \left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \right\}_{k\in\mathbb{Z}}. \tag{4.15}$$

With Definition 3.15 we get the sequences

$$\begin{aligned}
(r_{f,0}, h_{f,0}, a_0, s_0) &= (2, 1, 1, 1), \\
(r_{f,1}, h_{f,1}, a_1, s_1) &= (1, 2, 1, 1), \\
(r_{f,2}, h_{f,2}, a_2, s_2) &= (0, 3, 1, 0), \\
(r_{f,3}, h_{f,3}, a_3, s_3) &= (0, 3, 0, 0), \\
(r_{f,4}, h_{f,4}, a_4, s_4) &= (0, 3, 0, 0), \\
&\vdots
\end{aligned}$$

and thus an ordinary strangeness index of 2 as shown in Example 3.17. With Definition 4.11 however we face the same reduction process. This is due to the fact that the second matrix in the matrix pair (4.15) has full row rank, and thus every step of reversed index reduction has no effect. Thus we get the sequences

$$\begin{aligned}
(r_{f,0}, h_{f,0}, h_{b,0}, q_0, r_{b,0}, s_{E,0}, s_{A,0}, s_0) &= (2, 1, 0, 0, 3, 1, 0, 1), \\
(r_{f,1}, h_{f,1}, h_{b,1}, q_1, r_{b,1}, s_{E,1}, s_{A,1}, s_1) &= (1, 2, 0, 0, 3, 1, 0, 1), \\
(r_{f,2}, h_{f,2}, h_{b,2}, q_2, r_{b,2}, s_{E,2}, s_{A,2}, s_2) &= (0, 3, 0, 0, 3, 0, 0, 0), \\
(r_{f,3}, h_{f,3}, h_{b,3}, q_3, r_{b,3}, s_{E,3}, s_{A,3}, s_3) &= (0, 3, 0, 0, 3, 0, 0, 0), \\
&\vdots
\end{aligned}$$

which shows that the two-way strangeness index is also 2.

In Example 4.13 we have seen that the ordinary strangeness index can be equal to the two-way strangeness index. However, in Example 4.12 we have seen that the ordinary strangeness index can be bigger as the two-way strangeness index. Also, we observe that one step of two-way index reduction involves one step of ordinary index reduction. Thus, it should be possible to show the following Conjecture.

Conjecture 4.14. *1. Every sequence of matrix pairs with well defined two-way strangeness index has well defined ordinary strangeness index.*

2. Let the strangeness index μ_f, the reversed strangeness index μ_b and the two-way strangeness index μ all be well defined. Then we have $2\mu \geq \max(\mu_f, \mu_b) \geq \mu$.

4.1 Existence and uniqueness of solutions

With the notation of Remark 4.8 we know that for the two-way strangeness index μ we have

$$\{(E_{k,\mu}, A_{k,\mu})\} \sim \left\{ \left(\begin{bmatrix} E_{k,\mu}^{(1)} & 0 & 0 & E_{k,\mu}^{(2)} \\ 0 & I_{h_{b,\mu}-q_\mu} & 0 & 0 \\ 0 & 0 & I_{q_\mu} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} A_{k,\mu}^{(1)} & A_{k,\mu}^{(2)} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I_{q_\mu} & 0 \\ 0 & 0 & 0 & I_{h_{f,\mu}-q_\mu} \\ 0 & 0 & 0 & 0 \end{bmatrix} \right) \right\}_{k\in\mathbb{Z}}$$

$$\Rightarrow \{(E_{k,\mu+1}, A_{k,\mu+1})\} \sim \left\{ \left(\begin{bmatrix} E_{k,\mu}^{(1)} & 0 & 0 & 0 \\ 0 & I_{h_{b,\mu}-q_\mu} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} A_{k,\mu}^{(1)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I_{q_\mu} & 0 \\ 0 & 0 & 0 & I_{h_{f,\mu}-q_\mu} \\ 0 & 0 & 0 & 0 \end{bmatrix} \right) \right\}_{k\in\mathbb{Z}}.$$

But we also know from the definitions (4.12) that $\operatorname{rank}(E_{k,\mu}) = \operatorname{rank}(E_{k,\mu+1})$ from which we see that $q_\mu = 0$ and that $E_{k,\mu}^{(1)}$ is a matrix with full row rank for all $k \in \mathbb{Z}$. From $\operatorname{rank}(A_{k,\mu}) = \operatorname{rank}(A_{k,\mu+1})$, we analogously see that all $A_{k,\mu}^{(1)}$ already have full row rank. Thus, every sequence with well defined two-way strangeness index can be transformed by $\mu+1$ reduction steps and appropriate global equivalence transformations to a two-way strangeness-free sequence of the form

$$\{(E_{k,\mu+1}, A_{k,\mu+1})\} \sim \left\{ \left(\begin{bmatrix} E_{k,\mu}^{(1)} & 0 & 0 \\ 0 & I_{h_{b,\mu}} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} A_{k,\mu}^{(1)} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_{h_{f,\mu}} \\ 0 & 0 & 0 \end{bmatrix} \right) \right\}_{k\in\mathbb{Z}}, \tag{4.16}$$

where all $A_{k,\mu}^{(1)}$ and all $E_{k,\mu}^{(1)}$ have full row rank. By transformations of the (1,1)-block in (4.16) one can also achieve that

$$\begin{gathered} E_{k,\mu}^{(1)} = \begin{bmatrix} I & 0 \end{bmatrix} \text{ for all } k \in \mathbb{N}_0 \\ \text{and} \\ A_{k,\mu}^{(1)} = \begin{bmatrix} I & 0 \end{bmatrix} \text{ for all } k \leq -1. \end{gathered} \tag{4.17}$$

From (4.16) with (4.17) one can derive a statement similar to Theorem 3.32 for the case where one wants to get a solution for all $k \in \mathbb{Z}$.

Theorem 4.15. *Let the two-way strangeness index μ of the sequence $\{(E_k, A_k)\}_{k\in\mathbb{Z}}$ as in (4.14) be well defined. Then the discrete descriptor system (1.5) is equivalent (in the sense that there is a one-to-one correspondence between the solution/sequence spaces) to a discrete* *descriptor system of the form*

$$
\begin{aligned}
x_1^{k+1} &= A_k^{(1)} x_1^k + A_k^{(2)} x_4^k + f_1^k,\ k \geq 0, && r_{f,\mu} - h_{b,\mu} && (4.18a)\\
x_1^{k-1} &= E_{k-1}^{(1)} x_1^k + E_{k-1}^{(2)} x_4^k - f_1^{k-1},\ k \leq 0, && r_{b,\mu} - h_{f,\mu} && (4.18b)\\
x_2^{k+1} &= f_2^k, && h_{b,\mu} && (4.18c)\\
0 &= x_3^k + f_3^k, && h_{f,\mu} && (4.18d)\\
0 &= f_4^k, && m - r_{f,\mu} - h_{f,\mu} && (4.18e)
\end{aligned}
$$

where with $u_\mu := n - r_{f,\mu} - h_{f,\mu}$ we have $x_4^k \in \mathbb{C}^{u_\mu}$ and each of the inhomogeneities f_1^k, f_2^k, f_3^k, f_4^k is determined by the original right hand sides $f^{k-\mu-1}, \ldots, f^k, \ldots, f^{k+\mu+1}$ as in (1.5) for all $k \in \mathbb{Z}$. For the problem

$$E_k x^{k+1} = A_k x^k + f^k,\ k \in \mathbb{Z}, \tag{4.19}$$

we also have the following results:

1. *(4.19) is solvable if and only if the $v_\mu := m - r_{f,\mu} - h_{f,\mu}$ consistency conditions conditions*

   $$f_4^k = 0$$

   are fulfilled for all $k \in \mathbb{Z}$.

2. *An initial condition $x^0 = \hat{x}$ together with (4.19) is consistent if and only if in addition the $h_{f,\mu} + h_{b,\mu}$ conditions*

   $$
   \begin{aligned}
   x_2^0 = \hat{x}_2 &= f_2^{-1},\\
   x_3^0 = \hat{x}_3 &= -f_3^0,
   \end{aligned}
   $$

   are satisfied.

3. *The corresponding initial value problem is uniquely solvable if and only if in addition*

   $$u_\mu = 0$$

   holds.

Proof. The proof can be carried out as the proof of Theorem 3.32 by using (4.16) and (4.17). □

Chapter 5

Algorithms for linear discrete-time descriptor systems

In the following some algorithms are proposed, which may be used to solve linear discrete-time descriptor systems (as described in the Chapters 2, 3 and 4). Some of the algorithms are only suited for the constant coefficient case, while others may also be used in the variable coefficient case.

5.1 A method with the Drazin inverse

Theorem 2.20 can be used to compute the solution of a regular linear discrete-time descriptor system $(E, A) \in \mathbb{C}^{n,n} \times \mathbb{C}^{n,n}$ with constant coefficients. There are two problems that arise in this case.

1. Typically E and A will not commute. If they do not commute, one first has to determine an

$$\text{invertible matrix } P \text{ such that } PEPA = PAPE, \tag{5.1}$$

   i.e., that PE and PA commute. One of the following approaches can be chosen to determine such a matrix P:

   (a) Lemma 2.23 can be used to compute such a matrix. The problem with this approach is to determine a proper $\tilde{\lambda}$ (as in Lemma 2.23), such that $\tilde{\lambda}E - A$ is good conditioned. It is a good idea to choose a $\tilde{\lambda}$ which is nowhere close to any eigenvalue of $\lambda E - A$, so that $\tilde{\lambda}E - A$ is well conditioned.

   (b) One may also employ the Kronecker product to determine such a matrix. For this note that (5.1) is equivalent to solving the Sylvester equation

$$EPA = APE, \tag{5.2}$$

   for an invertible P. (5.2) can be rewritten as a system of linear equations, $Gy = 0$, using the Kronecker product (see [11]). By calculating a singular value decomposition of G we can then determine a basis of kernel (G). Each of the basis vectors can then be reshaped to a matrix P, which solves (5.2). However, we still have to check if that matrix is invertible. Note that a SVD of G can be obtained by computing an SVD of E and A separately (see [11]), which makes the computation considerably faster.

(c) The GUPTRI-Algorithm (see [4]) together with the solution of a generalized Sylvester equation (to decouple the eigenvalue infinity from the finite eigenvalues; compare [19]) could be employed to calculate a matrix P with (5.1). To see this in the real case, let U and V be orthogonal matrices such that

$$E = V \begin{bmatrix} E_f & E_u \\ 0 & E_\infty \end{bmatrix} U^T, \qquad\qquad A = V \begin{bmatrix} A_f & A_u \\ 0 & A_\infty \end{bmatrix} U^T \ ,$$

is in GUPTRI form, i.e., E_f is regular and triangular, A_f is quasi-triangular, E_∞ is strict upper triangular and A_∞ is regular and triangular. Following the approach in [19] we can solve the generalized Sylvester equation

$$\begin{aligned} E_f Y - Z E_\infty &= -E_u, \\ A_f Y - Z A_\infty &= -A_u. \end{aligned}$$

This equation can be solved uniquely and we obtain matrices Y and Z such that with

$$W := V \begin{bmatrix} I & Z \\ 0 & I \end{bmatrix}, \qquad\qquad T := \begin{bmatrix} I & -Y \\ 0 & I \end{bmatrix} U^T,$$

we get

$$\lambda E - A = W \begin{bmatrix} \lambda E_f - A_f & 0 \\ 0 & \lambda E_\infty - A_\infty \end{bmatrix} T.$$

Thus, we can rewrite equation (5.2) as

$$W^{-1} E T^{-1} \underbrace{TPW}_{:=\tilde{P}} W^{-1} A T^{-1} = W^{-1} A T^{-1} T P W W^{-1} E T^{-1},$$

which is equivalent to

$$\begin{bmatrix} E_f & 0 \\ 0 & E_\infty \end{bmatrix} \tilde{P} \begin{bmatrix} A_f & 0 \\ 0 & A_\infty \end{bmatrix} = \begin{bmatrix} A_f & 0 \\ 0 & A_\infty \end{bmatrix} \tilde{P} \begin{bmatrix} E_f & 0 \\ 0 & E_\infty \end{bmatrix}. \tag{5.3}$$

For example, choosing

$$\tilde{P} = \begin{bmatrix} E_f^{-1} & 0 \\ 0 & A_\infty^{-1} \end{bmatrix},$$

solves equation (5.3), which means that $T^{-1}\tilde{P}W^{-1}$ solves (5.2). Anyway, the solution is not unique as one can see by assuming that $A_f = 0$ and $E_\infty = 0$ (this means that we only have the infinite and the zero eigenvalue) in which case every

$$\tilde{P} = \operatorname{diag}(P_1, P_2)$$

solves (5.3). Thus, the question arises whether there is a solution which is better conditioned than $T^{-1}\tilde{P}W^{-1}$.

2. To compute the solution with Theorem 2.20 one needs to compute the products $A^D x$ and $E^D x$ for arbitrary x and also the index of E and A is needed.
   Some effort has been made to numerically compute the Drazin inverse [5, 18, 20, 21]. One could also compute the Schur form of E or A, respectively and then solve a Sylvester equation (to decouple the zero eigenvalues from the other non-zero eigenvalues, similar to the approach in [19]) to calculate the Drazin inverse. To see this, let

$$E = U \begin{bmatrix} S_n & S_u \\ 0 & S_z \end{bmatrix} U^H,$$

   be the Schur decomposition of a matrix E in such a way that S_z only has zero eigenvalues and S_n only has non-zero eigenvalues. Following the approach in [19] we can solve the Sylvester equation

$$ZS_z - S_n Z = -S_u.$$

   This equation can be solved uniquely, since S_z and S_n have no common eigenvalues and we obtain a matrix Z such that with

$$W := U \begin{bmatrix} I & -Z \\ 0 & I \end{bmatrix},$$

   we get

$$E = W \begin{bmatrix} S_n & 0 \\ 0 & S_z \end{bmatrix} W^{-1}.$$

   Computing the Jordan form of the blocks $S_n = V_n J_n V_n^{-1}$ and $S_z = V_z J_z V_z^{-1}$ shows that

$$E = W \underbrace{\begin{bmatrix} V_n & 0 \\ 0 & V_z \end{bmatrix}}_{:=V} \underbrace{\begin{bmatrix} J_n & 0 \\ 0 & J_z \end{bmatrix}}_{:=J} V^{-1} W^{-1},$$

   which proves that J is the Jordan canonical form of E. Thus, we know that

$$\begin{aligned} E^D &= WVJ^D V^{-1} W^{-1} = WV \begin{bmatrix} J_n^{-1} & 0 \\ 0 & 0 \end{bmatrix} V^{-1} W^{-1} \\ &= WV \begin{bmatrix} V_n^{-1} S_n^{-1} V_n & 0 \\ 0 & 0 \end{bmatrix} V^{-1} W^{-1} = W \begin{bmatrix} S_n^{-1} & 0 \\ 0 & 0 \end{bmatrix} W^{-1}, \end{aligned}$$

   which means that we can compute the Drazin inverse of E without computing the Jordan canonical form.
   Anyway, one should remember, that (like with the ordinary inverse) computing $E^D x$ without explicitly computing E^D has better numerical properties.

Looking at Theorem 2.20 we see that for one element of the solution sequence there are many calculations of the form $E^D x$ necessary, which makes the whole procedure very expensive. Also this method (in general) involves the transformation with (regular but) non-unitary matrices and it can only be used for regular matrix pencils.
All this shows that this method will be of no practical use.

5.2 A method employing the GUPTRI-Algorithm

The GUPTRI-Algorithm [4] can also be used directly to compute the solution of a regular linear discrete-time descriptor system $(E, A) \in \mathbb{C}^{n,n} \times \mathbb{C}^{n,n}$ with constant coefficients. Once we have computed the GUPTRI form we can determine the solution by backward substitution. Let us therefore assume that

$$(E, A) = \left(\begin{bmatrix} E_f & E_u \\ 0 & E_\infty \end{bmatrix}, \begin{bmatrix} A_f & A_u \\ 0 & A_\infty \end{bmatrix} \right) , \tag{5.4}$$

already is in GUPTRI form. Also assume for simplicity that (E, A) has no complex eigenvalues, i.e., E_f is regular and triangular, A_f is triangular, E_∞ is strict upper triangular and A_∞ is regular and triangular. Let us further denote the entries of E and A by $[e_{i,j}]_{i,j=1,\dots,n}$ and $[a_{i,j}]_{i,j=1,\dots,n}$, respectively, and let r be the dimension of the E_f and A_f blocks. Then we know that $e_{i,j} = a_{i,j} = 0$ for all $i > j$. Additionally, we know that $e_{i,i} = 0$ for all $i = r + 1, \dots, n$ and that $a_{i,i} \neq 0$ for all $i = r + 1, \dots, n$. The equations of the descriptor system (1.4) belonging to (5.4) can thus be written as

$$\sum_{j=i}^{n} e_{i,j} x_j^{k+1} = \sum_{j=i}^{n} a_{i,j} x_j^k + f_i^k, \qquad \text{for } i = 1, \dots, r, \tag{5.5}$$

$$\sum_{j=i+1}^{n} e_{i,j} x_j^{k+1} = \sum_{j=i}^{n} a_{i,j} x_j^k + f_i^k, \qquad \text{for } i = r + 1, \dots, n. \tag{5.6}$$

To solve the equations (5.6) we start with the last equation (i.e., for $i = n$). This equation can be written as $0 = a_{n,n} x_n^k + f_n^k$, which allows for the determination of all x_n^k, once all right hand sides are available. More precisely, we have $x_n^k = -\frac{f_n^k}{a_{n,n}}$. We further observe that the last but one of the equations (5.6) can be written as

$$\begin{aligned} e_{n-1,n} x_n^{k+1} &= a_{n-1,n-1} x_{n-1}^k + a_{n-1,n} x_n^k + f_{n-1}^k \\ \Leftrightarrow x_{n-1}^k &= \frac{1}{a_{n-1,n-1}} \left(e_{n-1,n} x_n^{k+1} - a_{n-1,n} x_n^k - f_{n-1}^k \right) \\ &= \frac{1}{a_{n-1,n-1}} \left(a_{n-1,n} \frac{f_n^k}{a_{n,n}} - e_{n-1,n} \frac{f_n^{k+1}}{a_{n,n}} - f_{n-1}^k \right). \end{aligned}$$

Proceeding in this way we see that we can obtain all x_i^k for $i = r + 1, \dots, n$, once all right hand sides are available. Note, that by doing so we need the right hand side $f^{k+n-r-1}$ to

compute x_{r+1}^k. Thus, there may be more right hand sides necessary than demanded by the index, because the $\lambda E_\infty - A_\infty$ block does not reveal the index.
Given an initial condition for the regular part (5.5) we can also compute the forward solution, since E_f is regular. A backward solution may also be computed, if we divide the $\lambda E_f - A_f$ block further into a part belonging to the non-zero eigenvalues and a part belonging to the zero eigenvalues, which is automatically done by the GUPTRI algorithm. The backward solution can then be found by performing the very same kind of process described above (for the equations (5.6)) for the zero eigenvalues of $\lambda E_f - A_f$.
Finally, note that the GUPTRI form can also be computed for singular pencils. One could examine if this GUPTRI form also allows for an easy determination of the solution of such descriptor systems.
This method will be considerably faster than the one introduced in section 5.1 since we only have to compute the GUPTRI form and perform backward substitution. Nevertheless, this method has the drawback that there may have to be $n - r - 1$ future right hand sides available, where $n - r - 1$ is the number of infinite eigenvalues.

5.3 A reduction method

In this section an algorithm is presented, which can also be applied to discrete-time descriptor systems with variable coefficients, as long as the system has a well defined forward strangeness index and/or a well defined backward strangeness index. The matlab code of the algorithm can be found in Appendix A and is called `solve_dds`. We will use the line numbers on the left side of the algorithm to refer to specific parts of the code and we will use true-type fonts to reference to parameters, variables and other interna of the code `solve_dds` (e.g., `Efun` means the first parameter of the matlab function in Appendix A).
The code starts in lines 91-101 by first setting up a few variables and then determining if a forward computation is necessary (result saved in `do_forward`) and if a backward computation is necessary (result saved in `do_backward`).
If the forward computation is necessary, `solve_dds` first computes the quantities `r_f(i+1)` $= r_{f,i}$ and `h_f(i+1)` $= h_{f,i}$ as in Remark 3.13 and `mu_f` $= \max(\mu_f, 1)$, where μ_f is the forward strangeness index (in lines 103-128). If the backward computation is necessary `solve_dds` also computes the quantities `r_b` and `h_b` as in Remark 3.13 for the reversed descriptor system and `mu_b` $= \max(\mu_b, 1)$, where μ_b is the reversed strangeness index (in lines 130-155). `mu_f` and `mu_b` are then later updated by the real strangeness indices in lines 225 and 269.
In lines 158-184 all constraints on the initial condition are determined and the initial condition `xhat` is altered so that new initial condition is the consistent initial condition which is closest to the old initial condition. As shown in Corollary 2.21, there are more constraints on the initial condition if one is looking for a two-way solution. This is represented by lines 160-169 of the code. Later, in lines 186-272 the actual forward and backward solutions are determined. Finally, in lines 274-281 some terminal computations are done.
The actual core of the algorithm is the function `advance_inflated_system`, which is needed for the index determination as well as the computation of the solution. In the following we

will shortly illustrate the functioning of `advance_inflated_system` in the forward case. Note that in every call to `advance_inflated_system` the matlab cell arrays `Ek`, `Ak` and `fk` are inserted into the function and are also returned by the function. These cell arrays are assumed to have $\mu :=$`mu` (parameter to the function `advance_inflated_system`) entries on entry and they have $\mu+1$ entries on exit. These cell arrays are interpreted as the entries of the matrix

$$D^{(1)} := \begin{bmatrix} -A_1 & E_1 & & & & -f^1 \\ & \ddots & \ddots & & & \vdots \\ & & -A_{\mu-1} & E_{\mu-1} & & -f^{\mu-1} \\ & & & -A_\mu & E_\mu & -f^\mu \end{bmatrix},$$

where $A_i :=$ `Ak{i}`, $E_i :=$ `Ek{i}` and $f^i :=$ `fk{i}`. Note that this matrix can be generated with the help of the function `dumparray` (in line 415). Clearly, any vector

$$\tilde{x}_\mu = \begin{bmatrix} x^1 \\ \vdots \\ x^\mu \\ x^{\mu+1} \\ 1 \end{bmatrix} \tag{5.7}$$

with $D^{(1)}\tilde{x}_\mu = 0$ gives a sequence $\{x^i\}_{i=1}^{\mu+1}$, which solves the equations $E_i x^{i+1} = A_i x^i + f^i$ for $i \in \{1, \dots, \mu\}$. In lines 294-297 matrix $D^{(1)}$ is first extended to the matrix

$$D^{(2)} := \begin{bmatrix} -A_1 & E_1 & & & & -f^1 \\ & \ddots & \ddots & & & \vdots \\ & & -A_{\mu} & E_{\mu} & & -f^{\mu} \\ & & & -A_{\mu+1} & E_{\mu+1} & -f^{\mu+1} \end{bmatrix},$$

where $A_{\mu+1} :=$ `feval(Afun,kact+mu)`, $E_{\mu+1} :=$ `feval(Efun,kact+mu)` and $f^{\mu+1} :=$ `feval ( ffun,kact+mu)`, since we only consider the forward case here. Then `advance_inflated _system` performs a singular value decomposition of $E_{\mu+1}$ to discover corange $(E_{\mu+1})$ (in line 301). This yields a matrix $U_{\mu+1}$ with

$$U_{\mu+1}E_{\mu+1} = \begin{bmatrix} E_{\mu+1}^{(1)} \\ 0 \end{bmatrix},$$

where $E_{\mu+1}^{(1)}$ has full row rank $r_{f,0}$. Setting $D^{(3)} := \operatorname{diag}(I_{m\mu}, U_{\mu+1})\, D^{(2)}$ shows that

$$D^{(2)}\tilde{x}_{\mu+1} = 0 \;\Leftrightarrow\; D^{(3)}\tilde{x}_{\mu+1} = 0,$$

where $\tilde{x}_{\mu+1}$ is as in (5.7). $D^{(3)}$ has the form

$$D^{(3)} = \begin{bmatrix} -A_1 & E_1 & & & & -f^1 \\ & \ddots & \ddots & & & \vdots \\ & & -A_\mu & E_\mu & & -f^\mu \\ & & & -A_{\mu+1}^{(1)} & E_{\mu+1}^{(1)} & -f_1^{\mu+1} \\ & & & -A_{\mu+1}^{(2)} & 0 & -f_2^{\mu+1} \end{bmatrix}.$$

In line 319 we then perform a singular value decomposition of $A_{\mu+1}^{(2)} = U_{\mu+1}^{(2)} S_{\mu+1}^{(2)} V_{\mu+1}^{(2)\,T}$. Setting $D^{(4)} := \mathrm{diag}\left(I_{m\mu+r_{f,0}}, U_{\mu+1}^{(2)}\right)^T D^{(3)} \mathrm{diag}\left(I_{n\mu}, V_{\mu+1}^{(2)}, I_{n+1}\right)$ shows that the solution sets of $D^{(3)}\tilde{x}_{\mu+1} = 0$ and $D^{(4)}\tilde{x}_{\mu+1} = 0$ differ, but there is a one-to-one correspondence between the sets through a linear mapping, which can be represented by $V_{\mu+1}^{(2)}$. To keep track of this linear mapping this transformation is stored in line 335, so that we can compute the original solution later. $D^{(4)}$ has the form

$$D^{(4)} = \begin{bmatrix} -A_1 & E_1 & & & & & -f^1 \\ & \ddots & \ddots & & & & \vdots \\ & & -A_\mu & E_\mu^{(3)} & E_\mu^{(4)} & & -f^\mu \\ & & & -A_{\mu+1}^{(3)} & -A_{\mu+1}^{(4)} & E_{\mu+1}^{(1)} & -f_1^{\mu+1} \\ & & & -\tilde{S}_{\mu+1}^{(2)} & 0 & 0 & -\tilde{f}_2^{\mu+1} \\ & & & 0 & 0 & 0 & -\tilde{f}_3^{\mu+1} \end{bmatrix},$$

where $\tilde{S}_{\mu+1}^{(2)}$ has full row and full column rank $h_{f,0}$ and $E_\mu V_{\mu+1}^{(2)} = \begin{bmatrix} E_\mu^{(3)} & E_\mu^{(4)} \end{bmatrix}$. Thus, we can eliminate the entries above $-\tilde{S}_{\mu+1}^{(2)}$ which yields the matrix

$$D^{(5)} = \begin{bmatrix} -A_1 & E_1 & & & & & -f^1 \\ & \ddots & \ddots & & & & \vdots \\ & & -A_\mu & 0 & E_\mu^{(4)} & & -\tilde{f}^\mu \\ & & & 0 & -A_{\mu+1}^{(4)} & E_{\mu+1}^{(1)} & -\tilde{f}_1^{\mu+1} \\ & & & -\tilde{S}_{\mu+1}^{(2)} & 0 & 0 & -\tilde{f}_2^{\mu+1} \\ & & & 0 & 0 & 0 & -\tilde{f}_3^{\mu+1} \end{bmatrix}.$$

This is done in lines 337-345. One can see that we have computed the form (3.13) for the single matrix pair $(E_{\mu+1}, A_{\mu+1})$. This process, which leads from $D^{(2)}$ to $D^{(5)}$, is then repeated for the next matrix pair $(\begin{bmatrix} 0 & E_\mu^{(4)} \end{bmatrix}, A_\mu)$. This is done over and over again until the final upper left matrix pair is reached. This final matrix pair is then handled in the lines 347 - 396. Call this final matrix $D^{(6)}$.

We see that due to the structure of the main program all matrix pairs in $D^{(1)}$ already have been reduced at least once. The process leading from $D^{(1)}$ to $D^{(6)}$ reduces all matrix pairs once again. Thus, the upper left matrix pair in $D^{(6)}$ has been reduced $\mu + 1$ times. As we know from (3.26) $\mu + 1$ reductions suffice to obtain a strangeness-free from, once μ is the forward strangeness index. Thus, the upper left matrix pair in $D^{(6)}$ can be used to compute one iterate of the solution, which is done in lines 237 - 252. In these lines the consistency conditions on the inhomogeneity are ignored and the undetermined components of the solution are set to zero.

5.3.1 Numerical results

To present numerical results, we consider a linear differential-algebraic equation from [8], namely

$$\begin{bmatrix} 0 & 0 \\ 1 & -t \end{bmatrix} \dot{x}(t) = \begin{bmatrix} -1 & t \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} t\sin(t) \\ t+\cos(t) \end{bmatrix}, \text{ for } t \in [-7; 7]. \tag{5.8}$$

As shown in [8] on page 57,

$$x(t) = \begin{bmatrix} t^2 + t\cos(t) - t^2\cos(t) \\ t + \cos(t) - \sin(t) - t\cos(t) \end{bmatrix} \tag{5.9}$$

is the only solution of (5.8). This means that there is also only one consistent initial condition. Note that (5.8) is a special case of (1.7). Thus, we know that by discretizing (5.8) with the explicit Euler method and the equidistant grid

$$\ldots < -2h < -h < 0 < h < 2h < \ldots, \tag{5.10}$$

for some step size $h \in \mathbb{R}^+$ yields the discrete-time descriptor system

$$\begin{bmatrix} 0 & 0 \\ 1/h & -k \end{bmatrix} x^{k+1} = \begin{bmatrix} -1 & kh \\ 1/h & -k \end{bmatrix} x^k + \begin{bmatrix} kh\sin(kh) \\ kh+\cos(kh) \end{bmatrix}, \ k \in \mathbb{Z}. \tag{5.11}$$

Solving this discrete-time descriptor system with the algorithm from Appendix A for a given h on the discrete interval $\mathbb{Z} \cap [\frac{-7}{h}; \frac{7}{h}]$ yields an approximation to the actual solution (5.9). Figures 5.1 and 5.2 show the exact solution (5.9) and the approximation for several step sizes h. One can clearly see how the approximation gets better as the step sizes gets smaller.
In any case only one solution was found and although an inconsistent initial condition was given to the algorithm from Appendix A (parameter `xhat`) the algorithm found the only consistent initial condition.
In table 5.1 the runtime and the approximation errors of the algorithm form Appendix A are shown for different step sizes h. The *runtime in seconds* relates to a 64bit processor from Intel with 2.13GHz. The ∅ *solution difference* value is the average error between the iterates of the solution computed from (5.11) with the help of the algorithm from Appendix A and the exact solution of (5.8), i.e., (5.9). The *max. solution difference* value is the maximum error between the iterates of the solution computed from (5.11) with the help of the algorithm from Appendix A and the exact solution of (5.8), i.e., (5.9).
One clearly sees how the errors are proportional to the step size h. Also the runtimes seem to be inverse proportional to the step size h. Thus, the explicit Euler method is convergent of order one, when applied to the differential-algebraic equation (5.8).

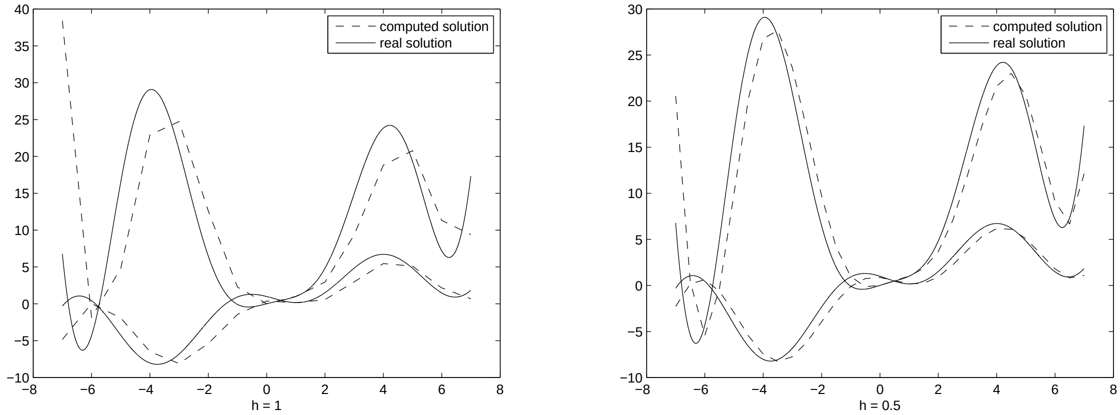

Figure 5.1: Discretization with h = 1 and h = 0.5

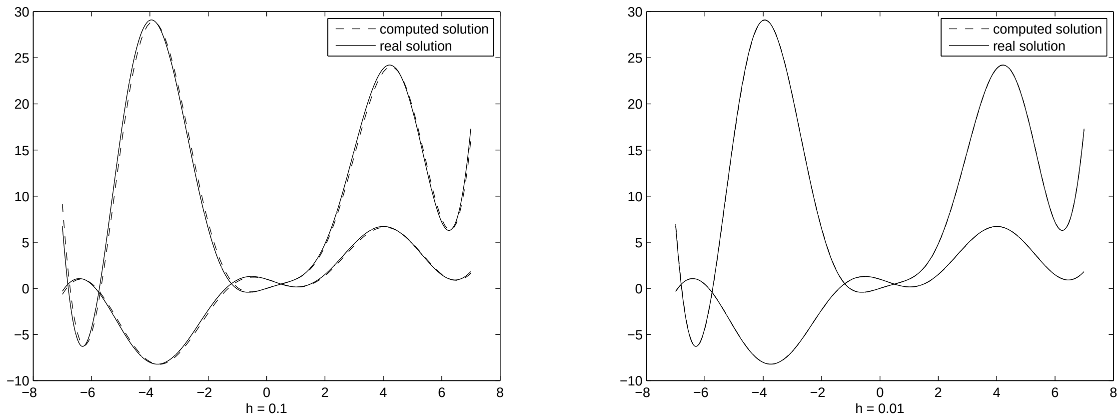

Figure 5.2: Discretization with h = 0.1 and h = 0.01

h	runtime in seconds	∅ solution difference	max. solution difference
1	0.01305	6.7431	31.945
0.5	0.02230	2.9053	13.926
0.1	0.09910	0.51967	2.3753
0.05	0.1944	0.2565	1.1601
0.01	0.9677	0.050795	0.22757
0.001	10.96	0.0050684	0.022657
0.0001	362.75	0.00050673	0.002265

Table 5.1: Runtime and approximation errors for different step sizes h

Chapter 6

Summary

In this diploma thesis some concepts from the book [8] are transferred to the discrete-time case. In chapter 2 we discuss linear discrete-time descriptor systems with constant coefficients. This is done in terms of the Kronecker canonical form and afterwards in explicit form in analogy to [8]. In chapter 2 we already note that one may distinguish three cases of (linear) discrete time descriptor systems. The first case is the forward case, i.e., the case where one has an initial condition given at point $k_0 \in \mathbb{Z}$ and only wants to get a solution for indices $k \geq k_0$. The second case is the backward case, i.e., the case where one has an initial condition given at point $k_0 \in \mathbb{Z}$ and only wants to get a solution for indices $k \leq k_0$. These first two cases are closely related, since the first case can be transferred into the second one by a variable substitution. The third case is the two-way case, i.e., the case where an initial condition is given at some point $k_0 \in \mathbb{Z}$ but one is looking for a solution for indices $k \geq k_0$ as well as for indices $k \leq k_0$. This case puts stronger restrictions on the initial condition, i.e., the set of consistent initial conditions in the third case is smaller than in the first or second case.

In chapter 3 we then move on to descriptor systems with variable coefficients. We first identify the local characteristics of such systems. Then, with the help of some constant rank assumptions, we derive a canonical form, which allows for statements about the existence and uniqueness of solutions for variable coefficient descriptor systems in the forward case. To achieve this canonical form one has to employ global equivalence transformations and shifts of equations. The shift is the discrete-time analogon to the differentiation in the continuous-time case. A strangeness index is defined, which counts the number of shifts necessary to obtain the canonical form. In section 3.2.1 the inflated descriptor system is introduced and it is shown (analogous to [8]) that, under some constant rank assumptions, one can determine the characteristics of the original systems once one knows the local invariants of the inflated systems. Section 3.3 discusses how to carry over the results from the forward case to the backward case.

Chapter 4 is devoted to the combination of the forward and backward base. Here even stronger constant rank assumptions are introduced, which make it possible to obtain a two-way canonical form. With this two-way canonical form one can make statements about the existence and uniqueness of solutions in the two-way case. A new strangeness index is

defined for systems, which satisfy this stronger constant rank assumptions.
Finally, in chapter 5 two algorithms for descriptor systems with constant coefficients and one algorithm for descriptor systems with variable coefficients are introduced. The first algorithms will never be used. The second one may be fast enough but it does not respect the actual index of the descriptor system. The third algorithm seems to be the appropriate one.

6.1 Outlook

We have only considered the general linear discrete-time descriptor system, but other special cases might also be interesting, for example the periodic time-variant case, which is considered in [17]. Another interesting topic would be polynomial descriptor systems, i.e., systems of the form

$$A_k^{(l)} x^{k+l} + \ldots + A_k^{(1)} x^{k+1} + A_k^{(0)} x^k = 0 \text{ for } k \in \mathbb{Z},$$

although the theory might get very complicated for such systems. For such systems one could also investigate the associated control problem, which has already been done for non-polynomial descriptor systems (e.g., [10, 15, 22]).
The constant rank assumptions (3.12) make it possible to obtain the canonical form (3.26). One can relax these assumptions, similar to Hypothesis 3.48 in [8]. Then, it should be possible to create an algorithm that computes the solution of a linear discrete-time descriptor system under these relaxed assumptions.
It would be interesting to investigate the discretization of linear continuous-time descriptor systems (i.e., differential-algebraic equations) and connections between properties of the original continuous-time descriptor system and the discretized descriptor system. Another thing one could do is the derivation of a canonical form similar to the one in Theorem 3.20 for the two-way reduction process. Having obtained such a canonical form one could try to prove Conjecture 4.14.
Other topics, which are commonly examined for linear systems of the form $x^{k+1} = A_k x^k + f^k$, also make sense for discrete-time descriptor systems. Such topics include Stability and Stochastic Systems (see [7, 9]).
Finally, of course, non-linear descriptor systems also have to be considered.

Acknowledgements
I want to thank Volker Mehrmann for numerous revisions of the manuscript and continuous support during its creation and Peter Kunkel for the discussion in which he noticed that the form (3.26) only allows for statements about the existence and uniqueness of forward solutions.

Appendix A

Matlab Code

Note that although the following matlab code works, there are major improvements necessary. For example, it is not always necessary to compute the singular value decompositions of the whole matrix in the function *advance_inflated_system*. Also, there may occur an *division by zero* error in line 243 and line 199 for underdetermined systems, which can be avoided. The file can be downloaded from `http://www.math.tu-berlin.de/~bruell/matlab/`.

```
1  % solve_dds Compute a solution of a discrete-time descriptor system.
2  %      solve_dds(Efun,Afun,ffun,kb,k0,kf,xhat,tol) tries to compute the
3  %      iterates x_kb , ... , x_kf of a solution of one of the discrete-time
4  %      descriptor systems
5  %
6  %       (1)  E_k x_{k+1} = A_k x_{k} + f_k     for       kb <= k <= infty
7  %       (2)  E_k x_{k+1} = A_k x_{k} + f_k     for  -infty <= k <= kf
8  %       (3)  E_k x_{k+1} = A_k x_{k} + f_k     for  -infty <= k <= infty
9  %
10 %      along with          xhat = x_{k0},
11 %      where the matrices E_k and A_k and the vector f_k are obtained by
12 %      evaluation the function Efun, Afun and ffun, respectively, i.e.
13 %          E_k = feval( Efun, k ), A_k = feval( Afun, k ),
14 %          f_k = feval( ffun, k ).
15 %      If no such solution exists, the right hand side f_k is changed so
16 %      that a solution exists. If there are multiple solutions, one
17 %      solution is selected. If the initial value 'xhat' is inconsistent,
18 %      the consistent initial value is chosen which is closest to
19 %      'xhat' (in the 2-norm).
20 %      If the constant rank assumptions form [1] are not satisfied an
21 %      error message is issued and the algorithm is aborted, although
22 %      there might still exist a solution.
23 %      If kb == k0 and k0  < kf equation (1) is considered.
24 %      If kb  < k0 and k0 == kf equation (2) is considered.
25 %      If kb  < k0 and k0  < kf equation (3) is considered.
26 %      If kb == k0 and k0 == kf no equation  is considered and no real
27 %      computations are performed.
28 %
29 %      Input parameters:
30 %
```

```
%         Efun   :  A function handle or name of a function that takes
%                   one input parameter k and returns the matrix E_k.
%
%         Afun   :  A function handle or name of a function that takes
%                   one input parameter k and returns the matrix A_k.
%
%         ffun   :  A function handle or name of a function that takes
%                   one input parameter k and returns the vector f_k.
%
%         kb     :  The index of the first iterate to compute.
%         k0     :  The index, where to apply the initial condition.
%                   k0 has to satisfy kb <= k0 <= kf.
%         kf     :  The index of the last iterate to compute.
%
%         xhat   :  The desired value of the solution at iterate x_k0.
%
%         tol    :  Tolerance, below which a singular value is considered
%                   zero.
%
%      Output parameters
%
%         x      :  The solution iterates x_kb , ... , x_kf side by side
%                   in a matrix. Thus, x(:,k-kb+1) represents the x_k
%                   iterate of the solution. x(:,1) represents the x_kb
%                   iterate and x(kf-kb+1) represents the x_kf iterate.
%
%         isunique : This is set to 1 if there is only one unique
%                   solution and to 0 otherwise.
%
%         r_f    :  The sequence of the r_f as in [1]. Note that r_f(i)
%                   corresponds to r_{f,i-1} in the forward-only
%                   reduction process as described in [1].
%
%         h_f    :  The sequence of the h_f as in [1]. Note that h_f(i)
%                   corresponds to h_{f,i-1} in the forward-only
%                   reduction process as described in [1].
%
%         mu_f   :  The forward strangeness index of the system as
%                   defined in [1].
%
%         r_b    :  The sequence of the r_b as in [1]. Note that r_b(i)
%                   corresponds to r_{b,i-1} in the backward-only
%                   reduction process as described in [1].
%
%         h_b    :  The sequence of the h_b as in [1]. Note that h_b(i)
%                   corresponds to h_{b,i-1} in the backward-only
%                   reduction process as described in [1].
%
%         mu_b   :  The backward strangeness index of the system as
%                   defined in [1].
%
%      Reference:
```

```
%         [1]     Bruell, T.
%                 Linear discrete-time descriptor systems;
%                 Diploma-Thesis (2007);
%                 http://www.math.tu-berlin.de/~bruell
%
function [x,isunique,r_f,h_f,mu_f,r_b,h_b,mu_b]=...
            solve_dds(Efun,Afun,ffun,kb,k0,kf,xhat,tol)

r_f=[]; h_f=[];
mu_f = 0;
r_b=[]; h_b=[];
mu_b = 0;

dimmat = feval( Efun, k0 );
[m,n] = size(dimmat);
clear dimmat;

do_forward  = ( k0 < kf );
do_backward = ( k0 > kb );

if( do_forward )
   Ek_f = {};
   Ak_f = {};
   Qk_f = {};
   fk_f = {};

   kact = k0;

   % determine the forward index
   haveindex = 0;
   while( ~haveindex )
      [Ek_f,Ak_f,Qk_f,fk_f,r_f,h_f]=advance_inflated_system...
      (Efun,Afun,ffun,m,n,Ek_f,Ak_f,Qk_f,fk_f,kact,mu_f,r_f,h_f,tol,0);

      if( ( mu_f >= 1 && r_f(mu_f+1) == r_f(mu_f) ) )
         haveindex = 1;
      else
         mu_f = mu_f + 1;
      end
   end

   r_mu_f = r_f(mu_f+1);
   h_mu_f = h_f(mu_f+1);
else
   r_mu_f = -1; h_mu_f = -1;
end

if( do_backward )
   Ek_b = {};
   Ak_b = {};
   Qk_b = {};
   fk_b = {};
```

```
    kact = k0-1;

    % determine the backward index
    haveindex = 0;
    while( ~haveindex )
       [Ek_b,Ak_b,Qk_b,fk_b,r_b,h_b]=advance_inflated_system...
       (Afun,Efun,ffun,m,n,Ek_b,Ak_b,Qk_b,fk_b,kact,mu_b,r_b,h_b,tol,1);

       if( ( mu_b >= 1 && r_b(mu_b+1) == r_b(mu_b) ) )
          haveindex = 1;
       else
          mu_b = mu_b + 1;
       end
    end

    r_mu_b = r_b(mu_b+1);
    h_mu_b = h_b(mu_b+1);
else
    r_mu_b = -1; h_mu_b = -1;
end

% find all constraints that the initial condition has to fulfill
constraintA = zeros(0,n);
constraintf = zeros(0,1);
if( do_forward )
    constraintA = [constraintA;-Ak_f{1}(r_mu_f+1:r_mu_f+h_mu_f,:)...
                                                          *Qk_f{1}'];
    constraintf = [constraintf; fk_f{1}(r_mu_f+1:r_mu_f+h_mu_f)];
end
if( do_backward )
    constraintA = [constraintA; Ak_b{1}(r_mu_b+1:r_mu_b+h_mu_b,:)...
                                                          *Qk_b{1}'];
    constraintf = [constraintf; fk_b{1}(r_mu_b+1:r_mu_b+h_mu_b)];
end

% find the consistent initial condition that is closest to the given
% one
[U,S,V] = svd(constraintA);
cnum = rank(S); % compute the number of constraints
cnum

ytilde = zeros(n,1);
xtilde = V' * xhat;
ftilde = U' * constraintf;

ytilde(1:cnum)   = S(1:cnum,1:cnum) \ ftilde(1:cnum);
ytilde(cnum+1:n) = xtilde(cnum+1:n);

xhat = V * ytilde;

if( do_forward )
```

```
    kact = k0;

    xtt_old = (Qk_f{1})' * xhat;

    % start forward solver
    for i=1:(kf-k0)
       xtt = zeros(n,1);

       xtt(1:h_mu_f) = -Ak_f{2}(r_mu_f+1:r_mu_f+h_mu_f,1:h_mu_f)\...
                          fk_f{2}( r_mu_f+1 : (r_mu_f+h_mu_f) );

       xtt(h_mu_f+1:h_mu_f+r_mu_f) = ...
             Ek_f{1}(1:r_mu_f,h_mu_f+1:h_mu_f+r_mu_f)\(...
                        Ak_f{1}(1:r_mu_f,:) * xtt_old ...
                      + fk_f{1}(1:r_mu_f) ...
                      - Ek_f{1}(1:r_mu_f,:) * xtt );

       % choose a solution
       xtt(h_mu_f+r_mu_f+1:n) = zeros(n-r_mu_f-h_mu_f,1);

       x(1:n, k0-kb+i+1) = (Qk_f{2}) * xtt;
       xtt_old = xtt;

       % proceed one step
       kact = kact+1;
       for i=1:prod(size(Ek_f))-1
          Ek_f{i}=Ek_f{i+1};
          Ak_f{i}=Ak_f{i+1};
          fk_f{i}=fk_f{i+1};
          Qk_f{i}=Qk_f{i+1};
       end
       [Ek_f,Ak_f,Qk_f,fk_f,r_f,h_f]=advance_inflated_system...
       (Efun,Afun,ffun,m,n,Ek_f,Ak_f,Qk_f,fk_f,kact,mu_f,r_f,h_f,tol,0);
    end

    % compute the _real_ forward strangeness index (i.e. as in [1])
    for i=length(r_f)-1:-1:1
       if( r_f(i) == r_f(i+1) )
          mu_f = i-1;
       end
    end
end

if( do_backward )
    kact = k0-1;

    xtt_old = (Qk_b{1})' * xhat;

    % start backward solver
    for i=0:(k0-kb-1)
       xtt = zeros(n,1);

```

```
        xtt(1:h_mu_b) =  Ak_b{2}(r_mu_b+1:r_mu_b+h_mu_b,1:h_mu_b)\...
                         fk_b{2}( r_mu_b+1 : (r_mu_b+h_mu_b) );

        xtt(h_mu_b+1:h_mu_b+r_mu_b) = ...
                Ek_b{1}(1:r_mu_b,h_mu_b+1:h_mu_b+r_mu_b)\( ...
                      Ak_b{1}(1:r_mu_b,:) * xtt_old ...
                    - fk_b{1}(1:r_mu_b) ...
                    - Ek_b{1}(1:r_mu_b,:) * xtt );

        % choose a solution
        xtt(h_mu_b+r_mu_b+1:n) = zeros(n-r_mu_b-h_mu_b,1);

        x(1:n, k0-kb-i) =  (Qk_b{2}) * xtt;
        xtt_old = xtt;

        % proceed one step
        kact = kact-1;
        for i=1:prod(size(Ek_b))-1
           Ek_b{i}=Ek_b{i+1};
           Ak_b{i}=Ak_b{i+1};
           fk_b{i}=fk_b{i+1};
           Qk_b{i}=Qk_b{i+1};
        end
        [Ek_b,Ak_b,Qk_b,fk_b,r_b,h_b]=advance_inflated_system...
        (Afun,Efun,ffun,m,n,Ek_b,Ak_b,Qk_b,fk_b,kact,mu_b,r_b,h_b,tol,1);
    end

    % compute the _real_ backward strangeness index (i.e. as in [1])
    for i=length(r_b)-1:-1:1
       if( r_b(i) == r_b(i+1) )
          mu_b = i-1;
       end
    end
end

if( ( ~do_forward  || r_mu_f + h_mu_f == n ) && ...
     ( ~do_backward || r_mu_b + h_mu_b == n ) )
    isunique = 1;
else
    isunique = 0;
end

x(:,k0-kb+1) = xhat;



function [Ek,Ak,Qk,fk,r,h]=advance_inflated_system...
               (Efun,Afun,ffun,m,n,Ek,Ak,Qk,fk,kact,mu,r,h,tol,backward)

if( backward )
    signed_mu = -mu;
else
```

```
    signed_mu =  mu;
end

Ek{mu+1} = feval( Efun, kact+signed_mu );
Ak{mu+1} = feval( Afun, kact+signed_mu );
Qk{mu+1} = eye(n);
fk{mu+1} = feval( ffun, kact+signed_mu );

for j=mu:-1:1
    % calculate a svd of E
    [U,S,V] = svd(Ek{1+j}); % Ek == U * S * V'

    % determine the rank (only for sure)
    loc_r = sum( getdiag(S) > tol );
    if( loc_r ~= r(mu-j+1) )
       error(['r␣not␣invariant!␣␣(k=',num2str(kact),',loc_r=',...
              num2str(loc_r),',r(',num2str(mu-j+1),')=',...
              num2str(r(mu-j+1)),')']);
    end

    % apply transformation
    S( (loc_r+1):m, : ) = zeros(m-loc_r,n);
    Ek{1+j} = S * V'; % = U'* Ekact
    Ak{1+j} = U'* Ak{1+j};
    fk{1+j} = U'* fk{1+j};

    % calculate a svd of A
    A2 = Ak{1+j}( loc_r+1:m, : );
    [U,S,V] = svd(A2); % Ak == U * S * V'

    % determine the rank (only for sure)
    loc_h = sum( getdiag(S) > tol );
    if( loc_h ~= h(mu-j+1) )
       error(['h␣not␣invariant!␣␣(k=',num2str(kact),',loc_h=',...
              num2str(loc_h),',h(',num2str(mu-j+1),')=',...
              num2str(h(mu-j+1)),')']);
    end

    % apply transformation
    S( loc_h+1:m-loc_r, : ) = zeros(m-loc_r-loc_h,n);
    Ek{j} = Ek{j}*V;
    Ak{1+j}( loc_r+1:m, : ) = S;
    Ak{1+j}( 1:loc_r, : )   = Ak{1+j}( 1:loc_r, : ) * V;
    fk{1+j}( loc_r+1:m, : ) = U'* fk{1+j}( loc_r+1:m, : );
    Qk{1+j} = Qk{1+j}*V;

    % do block elimination (in A)
    fk{1+j}(1:loc_r) = fk{1+j}(1:loc_r) - Ak{1+j}(1:loc_r,1:loc_h)*...
       (S(1:loc_h,1:loc_h)\fk{1+j}((loc_r+1):(loc_r+loc_h)));
    Ak{1+j}(1:loc_r,1:loc_h) = zeros(loc_r,loc_h);

    % do block elimination (in E)
```

```
   fk{j}(1:loc_r) = fk{j}(1:loc_r) + Ek{j}(1:loc_r,1:loc_h)*...
      (S(1:loc_h,1:loc_h)\fk{1+j}((loc_r+1):(loc_r+loc_h)));
   Ek{j}(1:loc_r,1:loc_h) = zeros(loc_r,loc_h);

end

[U,S,V] = svd(Ek{1}); % Ek == U * S * V'

% determine the rank
loc_r = sum( getdiag(S) > tol );
if( length(r) >= mu+1 )
   if( loc_r ~= r(mu+1) )
      error(['r not invariant!  (k=',num2str(kact),',loc_r=',...
            num2str(loc_r),',r(',num2str(mu+1),')=',...
            num2str(r(mu+1)),')']);
   end
else
   r(mu+1) = loc_r;
end

% apply transformation
S( r(mu+1)+1:m, : ) = zeros(m-r(mu+1),n);
Ek{1} = S * V'; % = U'* Ekact
Ak{1} = U'* Ak{1};
fk{1} = U'* fk{1};

A2 = Ak{1}( r(mu+1)+1:m, : );
[U,S,V] = svd(A2); % Ak == U * S * V'

% determine the rank
loc_h = sum( getdiag(S) > tol );
if( length(h) >= mu+1 )
   if( loc_h ~= h(mu+1) )
      error(['h not invariant!  (k=',num2str(kact),',loc_h=',...
            num2str(loc_h),',h(',num2str(mu+1),')=',...
            num2str(h(mu+1)),')']);
   end
else
   h(mu+1) = loc_h;
end

% apply transformation
S( h(mu+1)+1:m-r(mu+1), : ) = zeros(m-r(mu+1)-h(mu+1),n);
Ek{1} = Ek{1};
Ak{1}( r(mu+1)+1:m, : ) = S;
Ak{1}( 1:r(mu+1), : ) = Ak{1}( 1:r(mu+1), : ) * V;
fk{1}( r(mu+1)+1:m, : ) = U'* fk{1}( r(mu+1)+1:m, : );
Qk{1} = Qk{1} * V;

% eliminate in A
fk{1}(1:r(mu+1)) = fk{1}(1:r(mu+1)) - Ak{1}(1:r(mu+1),1:h(mu+1))*...
   (S(1:h(mu+1),1:h(mu+1))\fk{1}((r(mu+1)+1):(r(mu+1)+h(mu+1))));
```

```
Ak{1}(1:r(mu+1),1:h(mu+1)) = zeros(r(mu+1),h(mu+1));



% Returns the diagonal elements of a matrix in a vector.
function x=getdiag(A)

if( size(A,1) > 1 && size(A,2) > 1 )
   x = diag(A);
else
   if( prod(size(A)) > 0 )
      x = A(1,1);
   else
      x = [];
   end
end


% Dumps the inflated descriptor system represented by the arrays
% Ek, Ak and fk.
function dumparray(Ek,Ak,fk,m,n)
outsys = zeros(0,n);
outf   = [];
for i=1:prod(size(Ek))
   outsys = [ outsys , zeros((i-1)*m,n); ...
              zeros(m,(i-1)*n) , -Ak{i}, Ek{i} ];
   outf   = [outf;-fk{i}];
end
[outsys, outf]
```

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