Arithmetical Foundations

Recursion. Evaluation. Consistency

Excerpt

Michael Pfender and Students

TU Berlin

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Preface

Recursive maps, nowadays called *primitive recursive maps,* PR maps, have been introduced by GÖDEL in his 1931 article for the arithmetisation, *gödelisation,* of metamathematics.

For construction of his *undecidable formula* he introduces a non-constructive, non-recursive predicate *beweisbar,* prov*able.*

Staying within the area of categorical free-variables theory **PR** of primitive recursion or appropriate extensions opens the chance to avoid the two (original) Gödel's incompleteness theorems: these are stated for *Principia Mathematica und verwandte Systeme,* "related systems" such as in particular Zermelo-Fraenkel **set** theory **ZF** and v. Neumann Gödel Bernays **set** theory **NGB**.

On the basis of primitive recursion we consider μ-recursive maps as *partial p. r. maps.* Special *terminating* general recursive maps considered are *complexity controlled* iterations. *Map code evaluation* is then given in terms of such an iteration.

We discuss iterative p. r. map code evaluation versus *termination conditioned soundness* and based on this decidability of primitive recursive predicates. This leads to consistency provability and soundness for classical, quantified arithmetical and **set** theories as well as for the PR *descent* theory $\pi\mathbf{R}$, with unexpected consequences:

We show *inconsistency provability* for the quantified theories as well as *consistency provability* and logical *soundness* for the theory $\pi\mathbf{R}$ of primitive recursion, strengthened by an axiom scheme of *non-infinite descent of complexity controlled iterations* like (iterative) map-code evaluation.

Berlin, December 2013 M. Pfender.

The Student coworkers of present version are

Alistair Cloete, Julian Fagir, Christian Fischer, Joseph Helfer, Chi-Than C. Nguyen, Anke Stüber, and Vanessa Vallet.

P. S. I am obviously not an English native speaker. As Joseph Helfer puts it, my mathematical thinking and speach is somewhat special, it is *Germish.*

Contents

Introduction

We fix *constructive foundations* for arithmetic on a *map* theoretical, *algorithmical* level. In contrast to *elementhood* and *quantification* based traditional foundations such as Principia Mathematica **PM** or Zermelo-Fraenkel set theory **ZF**, our *fundamental primitive recursive theory* **PR** has as its "undefined" terms just terms for objects and maps. On that language level it is *variable free,* and it is free from formal quantification on individuals like numbers or number pairs.

This theory **PR** is a formal, *combinatorial category* with cartesian i. e. universal *product* and a natural numbers object (NNO) $\mathbb{N}$, a *PR cartesian category,* cf. Romàn 1989.

The NNO $\mathbb{N}$ admits *iteration of endo maps* and the *full scheme of primitive recursion.* Such NNO has been introduced in categorical terms by Freyd 1972, on the basis of the NNO of Lawvere 1964, and named later (e.g. by Maietti 2010) *parametrised NNO.*

We will remain on the purely **syntactical** level of this categorical theory, and later **extensions:** *no formal semantics* necessary into an outside, non-combinatorial world. Cf. Hilbert's formalistic program.

We then introduce into our *variable-free* setting *free variables,* which are introduced by *interpretation* of these variables as *names* for *projections.* As a consequence, we have in the present context '*free variable*' as a *defined* notion. We have object and map *constants* such as *terminal object,* NNO, zero etc. and use free metavariables for objects and for maps.

Fundamental arithmetic is further developed along Goodstein's 1971 *free variables Arithmetic* whose *uniqueness rules* are derived as theorems of categorical theory **PR**, with its "eliminable" notion of a *free variable.* This gives the expected *structure theorem for the algebra and order* on NNO $\mathbb{N}$. "On the way", via Goodstein's *truncated subtraction,* and "his" *commutativity of maximum function,* we obtain the *Equality Definability theorem:* If predicative equality of two p. r. maps is derivably true, then map equality between these maps is derivable. It follows a section on the derivation of the Peano axioms as theorems.

The subsequent chapter brings into the game an embedding theory extension of **PR** by *abstraction* of *predicates* into "virtual" new *objects.* This enrichment makes emerging *basic* theory $\mathbf{PRa} = \mathbf{PR} + (\mathrm{abstr})$ more comfortable, in direction to **set** theories, with their *sets* and *subsets.*

Chapter 3 introduces the general concept of *partial* maps, proves a structure theorem on the theory $\widehat{\mathbf{PRa}}$ of these maps and shows that μ-recursive maps and while-loop programs are just partial p. r. maps; in particular our evaluations will be such (formally) partial maps.

Categories of *partial maps* are introduced in the literature via idempotent monos taken as domains, see Robinson & Rosolini 1988, and Cockett & Lack 2002.

Partial maps are introduced here as map pairs consisting of a *domain-of-definition enumeration* (in general not mono) and of a *rule* to throw an enumeration index of a *defined argument* into the *value* of that argument. *Equality* of partial maps is by availability of extension maps between the enumeration domains of the two partial maps under consideration, in both directions.

These partial maps form a primitive recursive diagonal-monoidal half-cartesian theory $\widehat{\mathbf{PRa}}$ (cf. Budach & Hoehncke 1975) which contains theory **PRa** embedded as theory of this type, composition being defined via composition of pullbacks: Structure theorem for partials. theory extension by partiality is a *Closure* operator: *partial* partial maps are just partial maps.

Chapter 4 then exhibits within theory **PRa** a *universal object,* $\mathbb{X}$, of all *numerals* and nested pairs of numerals, and constructs by means of that object *universe theories* $\mathbf{PR}\mathbb{X}$ and $\mathbf{PR}\mathbb{X}\mathbf{a}$: theory $\mathbf{PR}\mathbb{X}$ is good for a one-object map-code evaluation, $\mathbf{PR}\mathbb{X}\mathbf{a}$ contains **PRa** as a cartesian PR embedded theory with predicate extensions.

Chapter 5 on *evaluation* strengthens p. r. theory $\mathbf{PR}\mathbb{X}\mathbf{a}$ into *descent theory* $\pi\mathbf{R}$, by an axiom of *non-infinite iterative descent* with order values in polynomial semiring $\mathbb{N}[\omega]$ ordered lexicographically.

This theory is shown to derive the—free variable PR—consistency formula for p. r. theories $\mathbf{PR}\mathbb{X}\mathbf{a}$ (and **PR**). The proof relies on constructive, complexity controlled code evaluation, which is extended to evaluation of argumented deduction trees:

theorem on p. r. soundness within **set** theory as frame (chapter 6), and *termination conditioned soundness of* $\mathbf{PRa} \subset \mathbf{PR}\mathbb{X}\mathbf{a}$ *within theory* $\pi\mathbf{R}$ *taken as frame* (chapter 7).

The consequence is decidability of p. r. predicates within both theories. Since consistency formulae Con of both theories can be expressed as (free variable) p. r. predicates, this leads to

1. *Inconsistency provability* of **set** theory by Gödel's second incompleteness theorem, and to

2. *Consistency provability* and soundness of descent theory $\pi\mathbf{R}$, under ***assumption*** of μ-consistency.

[The latter is a (**set** theoretically) equivalent variant of ω-consistency, expressible in $\mathbf{P}\widehat{\mathbf{R}}\mathbf{a}, \pi\widehat{\mathbf{R}}$.]

Notes to the literature are inserted which are based mainly on Remarks of the **Referee** to Pfender 2012.

1 Primitive Recursion

1.1 The fundamental theory PR of primitive recursion

We fix here **terms** and **axioms** for the *fundamental* categorical (formally variable-free) cartesian theory **PR** of primitive recursion.

The fundamental objects of the theory **PR** are the *natural numbers object* ('NNO') $\mathbb{N}$ and the *terminal* object $\mathbb{1}$.

Composed objects of **PR** come in as *"cartesian" products* $(A \times B)$ of objects already enumerated. Formally:

$$(\mathrm{Obj}_{\mathrm{Cart}}) \quad \frac{A, B \text{ objects}}{(A \times B) \text{ object}}$$

[Here outmost brackets may be dropped]

Maps: *Basic maps* ("map constants") of the theory **PR** are

the *zero map* $0 : \mathbb{1} \to \mathbb{N},$ and

the *successor map* $s : \mathbb{N} \to \mathbb{N}$

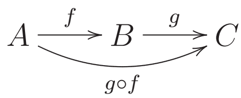
Structure of PR as a category:

- generation—enumeration—of *identity maps*

$$(\text{id generation}) \quad \frac{A \text{ an object}}{\mathrm{id}_A : A \to A \text{ map}}$$

- Composition:

$$(\circ) \quad \frac{f : A \to B,\ g : B \to C \text{ maps}}{(g \circ f) : A \to C \text{ map, diagram:}}$$



Here are the **axioms** making **PR** into a category:

- **Associativity** of **composition:**

$$(\circ_{\text{ass}}) \quad \frac{f : A \to B,\ g : B \to C,\ h : C \to D \text{ maps}}{h \circ (g \circ f) = (h \circ g) \circ f : A \to D}$$

- **Neutrality** of **identities**

$$(\text{neutr}_{\text{id}}) \quad \frac{f : A \to B \text{ map}}{\begin{array}{l}(f \circ \text{id}_A) = f : A \to A \to B \quad \text{and} \\ (\text{id}_B \circ f) = f : A \to B \to B.\end{array}}$$

map equality $f = g : A \to B$ satisfies the **axioms** of reflexivity, symmetry, and transitivity:

$$\text{(refl)}\quad \frac{f : A \to B \text{ map}}{f = f : A \to B}$$

$$\text{(sym)}\quad \frac{f = g : A \to B \text{ map}}{g = f : A \to B}$$

$$\text{(trans)}\quad \frac{f = g,\ g = h : A \to B \text{ maps}}{f = h : A \to B}$$

Composition is compatible with equality:

$$(\circ_=)\quad \frac{f = f' : A \to B,\ g = g' : B \to C}{(g \circ f) = (g' \circ f') : A \to B \to C}$$

Because of technical simplicity in later code evaluation, we split this **axiom** into the following two ones:

$$(\circ_= \text{1st})\quad \frac{f = f' : A \to B,\ g : B \to C}{(g \circ f) = (g \circ f') : A \to B \to C}$$

$$(\circ_= \text{2nd}) \quad \frac{f : A \to B,\ \ g = g' : B \to C}{(g \circ f) = (g' \circ f) : A \to B \to C}$$

Cartesian map structure:

- enumeration of *terminal maps*

$$\frac{A \text{ object}}{\Pi = \Pi_A : A \to \mathbb{1} \text{ map}}$$

[In Eilenberg & Elgot's notation. Lawvere designates this projection $! : A \to \mathbb{1}$.]

- uniqueness **axiom** for terminal map family:

$$(\Pi) \quad \frac{A \text{ object},\ f : A \to \mathbb{1} \text{ map}}{f = \ \Pi_A : A \to \mathbb{1}}$$

Π-naturality **Lemma:** $\Pi = [\Pi : A \to \mathbb{1}]_A$ is natural, i. e.

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle \Pi_A}\downarrow & = & \downarrow{\scriptstyle \Pi_B} \\ \mathbb{1} & \overset{\text{id}}{=\!=\!=} & \mathbb{1} \end{array}$$

- generation of left and right *projections:*

$$(\text{proj})\quad \frac{A,\ B \text{ objects}}{\begin{array}{l}\ell = \ell_{A,B} : A \times B \to A \ \ \textit{left projection,} \\ r = r_{A,B} : A \times B \to B \ \ \textit{right projection}\end{array}}$$

- generation of *induced maps* into products:

$$(\text{ind})\quad \frac{f : C \to A,\ \ g : C \to B \text{ maps}}{\begin{array}{l}(f, g) : C \to A \times B \text{ map,} \\ \text{the map } \textit{induced} \text{ by } f \text{ and } g\end{array}}$$

- compatibility of induced map formation with equality:

$$(\text{ind}_=)\quad \frac{f = f' : C \to A,\ g = g' : C \to B \qquad \text{maps}}{(f, g) = (f', g') : C \to A \times B}$$

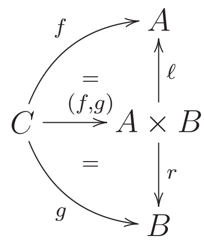
- characteristic (Godement) equations

$$(\text{GODE}_\ell)\quad \frac{f : C \to A,\ g : C \to B}{\ell \circ (f, g) = f : C \to A}$$

as well as

$$(\mathrm{GODE}_r) \quad \frac{f : C \to A,\ g : C \to B}{r \circ (f, g) = g : C \to B}$$

in *commutative* diagram form:



- uniqueness of induced map (Godement):

$$(\mathrm{ind!}) \quad \frac{\begin{array}{l} f : C \to A,\ g : C \to B,\ h : C \to A \times B \text{ maps,} \\ \ell \circ h = f : C \to A \text{ and } r \circ h = g : C \to B \end{array}}{h = (f, g) : C \to A \times B}$$

SP Lemma: In presence of the other axioms, this *uniqueness of the induced map* is equivalent to the following equational **axiom** of *Surjective Pairing*, see Lambek-Scott 1986:

$$(\mathrm{SP}) \quad \frac{h : C \to A \times B}{(\ell \circ h\,,\ r \circ h) = h : C \to A \times B}$$

Proof as an **exercise:** Use compatibility of forming the induced map with equality.

We will formally **rely** on **this** equation as an **axiom.** It replaces uniqueness of forming the induced map.

We eventually replace equivalently, given the other **axioms,** inferential axiom ($\mathrm{ind}_=$) by *distributivity* **equation**

$$(\mathrm{distr}_\circ)\quad \frac{h : D \to C,\ \ f : C \to A,\ \ g : C \to B}{(f, g) \circ h = (f \circ h, g \circ h) : D \to A \times B}$$

taken from Lambek-Scott. Equivalence **proof** as an **exercise,** proof of *uniqueness of the induced* in op. cit. Draw the diagram.

Definition: we define, for a map $g : B \to B'$, *cylindrification*

$$A \times g \ =_{\mathrm{def}}\ \mathrm{id}_A \times g \ =_{\mathrm{def}}\ (\mathrm{id}_A \circ \ell, g \circ r) : A \times B \to A \times B'.$$

Diagram:

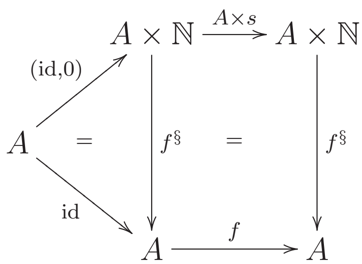
$$\begin{array}{ccc}
A & \xrightarrow{\mathrm{id}} & A \\
\uparrow{\scriptstyle \ell} & = & \uparrow{\scriptstyle \ell} \\
A \times B & \overset{A \times g}{\dashrightarrow} & A \times B' \\
\downarrow{\scriptstyle r} & = & \downarrow{\scriptstyle r} \\
B & \xrightarrow{g} & B'
\end{array}$$

This **ends** the list of **axioms** for the **cartesian structure** of the theory **PR**.

Axioms for the iteration of endo maps:

$$(\S)\quad \frac{f : A \to A \text{ (endo) map}}{\begin{array}{l} f^{\S} : A \times \mathbb{N} \to A \textit{ iterated of } f\text{, satisfies} \\ f^{\S} \circ (\mathrm{id}_A, 0) = \mathrm{id}_A : A \to A \quad [0 := 0\,\Pi] \text{ (anchor)}, \\ f^{\S} \circ (A \times s) = f \circ f^{\S} : A \times \mathbb{N} \to A \to A \text{ (step)}. \end{array}}$$

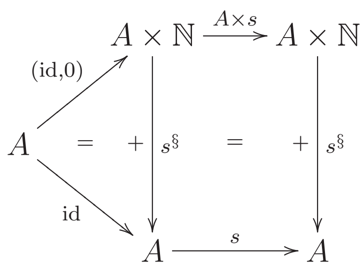
"Pentagonal" diagram:



(it)

basic iteration DIAGRAM

As a first **example** for an iterated endo map take *addition* $+ : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, having properties



(+)

i. e. satisfying the free-variables equations

$$a + 0 = a : \mathbb{N} \to \mathbb{N} \times \mathbb{N} \to \mathbb{N},$$
$$a + s\,n = s(a + n) = (a + n) + 1 : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \xrightarrow{s} \mathbb{N},$$
$$\text{where } 1 \ =_{\text{def}}\ s \circ 0 : \mathbb{1} \to \mathbb{N} \to \mathbb{N}.$$

[A formal introduction of free variables as projections see below.]

uniqueness **axiom** for the iterated:

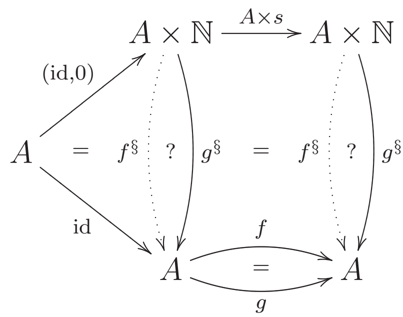
$$(\S!)\ \frac{\begin{array}{l} f : A \to A \text{ (endo map)} \\ h : A \times \mathbb{N} \to A, \\ h \circ (\mathrm{id}_A, 0) = \mathrm{id}_A \text{ and} \\ h \circ (A \times s) = f \circ h \text{ ``as well''} \end{array}}{h = f^{\S} : A \times \mathbb{N} \to A}$$

By this uniqueness **axiom,** the iterated map is **characterised** by the commutative pentagonal diagram above.

Theorem (compatibility of iteration with equality): uniqueness **axiom** (§!) infers

$$(\S_=)\ \frac{f = g : A \to A}{f^{\S} = g^{\S} : A \times \mathbb{N} \to A}$$

Proof: Consider the diagram



Since $f^{\S}$ is the *unique* commutative fill-in into this pentagonal diagram over endomorphism f, it is sufficient to show that $g^{\S} : A \times \mathbb{N} \to A$ equally is such a commutative fill in.

For the triangle (anchor) this is trivial: $g^{\S}(\mathrm{id}, 0) = \mathrm{id} : A \to A$ by definition of the null-fold iterated.

For the square (step) we have

$$g^{\S} \circ (A \times s) = g \circ g^{\S} \ (\textit{definition of } g^{\S})$$
$$= f \circ g^{\S} : A \times \mathbb{N} \to A,$$

by assumption $f = g$ and by compatibility of $\circ$ with $=$ in first composition factor, **axiom** ($\circ_{=}$1st).

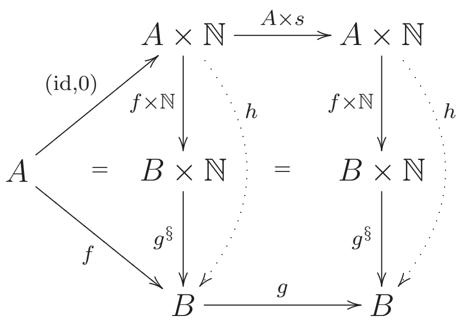
So $g^{\S}$ turns out to be another iterated of endo f, whence in fact $g^{\S} = f^{\S}$ by uniqueness of the iterated **q.e.d.**

These **axioms** give all objects and maps of theory **PR**.

Freyd's **uniqueness scheme** which completes the **axioms** constituting theory **PR**, reads

$$(\mathrm{FR!})\quad \frac{\begin{array}{l} f : A \to B,\ g : B \to B,\ h : A \times \mathbb{N} \to B, \\ h \circ (\mathrm{id}_A, 0 \,\circ\, \Pi_A) = f : A \to B,\ (\mathrm{init}) \\ h \circ (A \times s) = g \circ h : A \times \mathbb{N} \to B,\ (\mathrm{step}) \end{array}}{h = g^{\S} \circ (f \times \mathbb{N}) : A \times \mathbb{N} \to B \times \mathbb{N} \to B,}$$

in form of FREYD's pentagonal diagram:

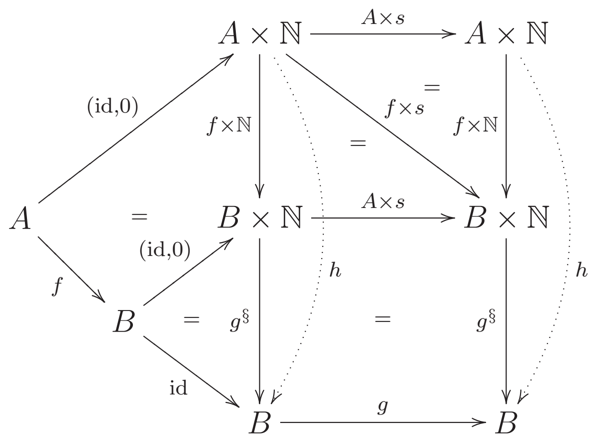


FREYD's uniqueness DIAGRAM (FR!)

Remark: This uniqueness of the *initialised iterated* obviously specialises to **axiom** (§!) of uniqueness of "simple" iterated $f^{\S} : A \times \mathbb{N} \to A$ and so makes that uniqueness axiom redundant.

Problem: Is, conversely, stronger Freyd's uniqueness **axiom** already covered by uniqueness (§!) of "simply" iterated $f^{\S} : A \times \mathbb{N} \to A$? My guess is "no".

Freyd's existence and uniqueness of the *initialised iterated* is displayed as the following commutative diagram:



Freyd's uniqueness diagram (FR!)

Existence of $g^{\S}$ and commutativity of lower triangle and square follow directly from **axiom** (§). Upper right commutativity is splitting a cartesian product $f \times s$ in the two ways into compositions of right and left cylindrified maps.

Remaining equation

$$(\mathrm{id}_B, 0 \circ \Pi_B) \circ f = (f \times \mathbb{N}) \circ (\mathrm{id}_A, 0 \circ \Pi_A) : A \to B \times \mathbb{N}$$

is given by uniqueness of the induced map into the cartesian product $B \times \mathbb{N}$, in detail:

$$\ell \circ (\mathrm{id}_B, 0) \circ f = \mathrm{id}_B \circ f = f \quad \text{and}$$
$$\ell \circ (f \times \mathbb{N}) \circ (\mathrm{id}_A, 0) = f \circ \ell \circ (\mathrm{id}_A, 0) = f \circ \mathrm{id}_A = f,$$
$$r \circ (\mathrm{id}_B, 0) \circ f = 0 \circ f = 0 \circ \Pi_A \quad \text{and}$$
$$r \circ (f \times \mathbb{N}) \circ (\mathrm{id}_A, 0 \circ \Pi_A) = r \circ (\mathrm{id}_A, 0) = 0 \circ \Pi_A.$$

Together this shows constructive *availability* of wanted *initialised iterated* $h : A \times \mathbb{N} \to B$.

uniqueness of h, namely

$$\text{(FR!)} \quad \frac{\begin{array}{l} f : A \to B,\ g : B \to B,\ h : A \times \mathbb{N} \to B \\ h \circ (\mathrm{id}_A, 0) = f \\ h \circ (A \times s) = g \circ h \end{array}}{h = g^{\S} \circ (f \times \mathbb{N}).}$$

is just required as an **axiom,** final axiom of theory **PR**.

From (FR!) we get trivially, with data

$$A \xrightarrow{\mathrm{id}_A} A \xrightarrow{f} A \ \text{ specializing data } \ A \xrightarrow{f} B \xrightarrow{g} B$$

uniqueness (§!) of *iterated* map $f^{\S} : A \times \mathbb{N} \to A$.

1.2 The full scheme of primitive recursion

Already for **definition** and characterisation of *multiplication* and moreover for **proof** of "the" laws of arithmetic, the following *full scheme* (pr) of primitive recursion is needed:[1]

Theorem (Full scheme of PR): PR admits scheme

[1] in pure categorical form see Freyd 1972, and (then) Pfender, Kröplin, and Pape 1994, not to forget its uniqueness clause

$g : A \to B$ (*init map*)

$h : (A \times \mathbb{N}) \times B \to B$ (*step map*)

(pr) $\overline{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}$

$\mathrm{pr}[g, h] := f : A \times \mathbb{N} \to B$

is given such that

$f(\mathrm{id}_A, 0) = g : A \to B$ (init), and

$f\,(\mathrm{id}_A \times s) = h\,(\mathrm{id}_{A \times \mathbb{N}}, f) :$

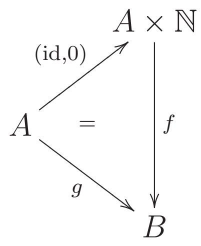
$(A \times \mathbb{N}) \to (A \times \mathbb{N}) \times B \to B,$ (step)

as well as

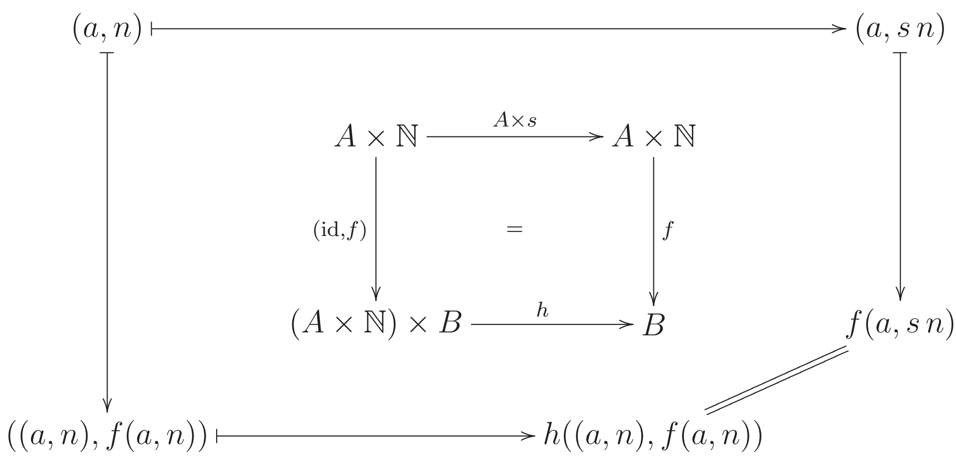
(pr!) : f is *unique* with these properties.

Proof: *construction* of the map $f = \mathrm{pr}\,[\,g, h\,] : A \times \mathbb{N} \to B$ out of data $g : A \to B$ (*initialisation*) and $h : (A \times \mathbb{N}) \times B \to B$ (*iteration step*):

Wanted $f : A \times \mathbb{N} \to B$ is to satisfy (init) und (step) given as the two commuting DIAGRAMS



(init)

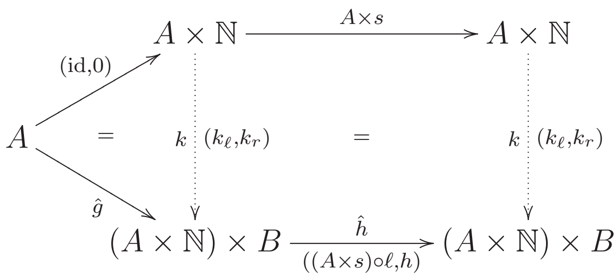


(step)

With $\hat{g} := ((\mathrm{id}_A, 0), g)$ and $\hat{h} := ((A \times s) \circ l, h)$ we get by (FR!) a uniquely determined map

$$k = (k_\ell, k_r) : A \times \mathbb{N} \to (A \times \mathbb{N}) \times B$$

satisfying

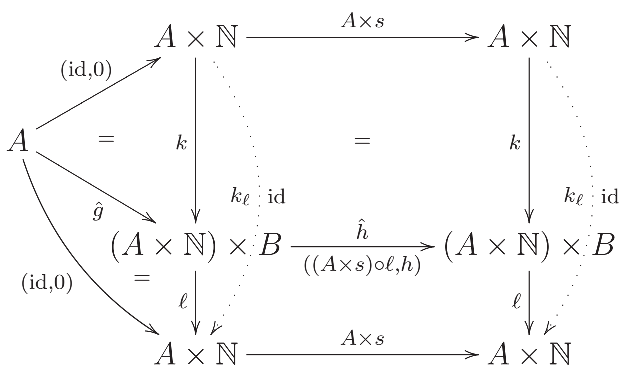


i. e.

$$k \circ (\mathrm{id}_A, 0) = \hat{g} \quad \text{and}$$
$$k \circ (A \times s) = \hat{h} \circ k.$$

[It will turn out that $k = (\mathrm{id}_{A\times\mathbb{N}}, f)$ for wanted map $f : A \times \mathbb{N} \to B$.]

For our unique k, consider first its left component $k_\ell = \ell \circ k : A \times \mathbb{N} \to A \times \mathbb{N}$, unique—by (FR!)—in

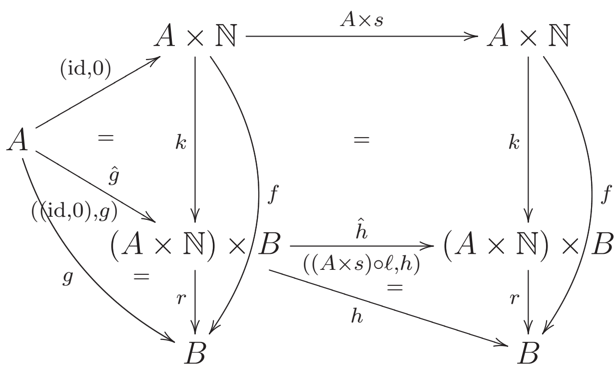


We have

$$\ell \circ k \circ (\mathrm{id}_A, 0) = \ell \circ \hat{g} = (\mathrm{id}_A, 0) \quad \text{and}$$
$$\ell \circ k \circ (A \times s) = l \circ \hat{h} \circ k = (A \times s) \circ \ell \circ k$$

Since these two equations hold likewise for $\mathrm{id}_{A\times\mathbb{N}}$ instead of $\ell \circ k$, it follows by uniqueness (FR!) of such a map $\ell \circ k = \mathrm{id}_{A\times\mathbb{N}}$.

Taking now $f := r \circ k : A \times \mathbb{N} \to B$, we have the following diagram for this (unique) right component of $k : A \times \mathbb{N} \to (A \times \mathbb{N}) \times B$:



obtain

$$
\begin{aligned}
k &= (\ell \circ k, r \circ k) = (\mathrm{id}_{A\times\mathbb{N}}, f), \\
f \circ (\mathrm{id}_A, 0) &= r \circ k \circ (\mathrm{id}_A, 0) = r \circ \hat{g} = g \quad \text{and} \\
f \circ (A \times s) &= r \circ k \circ (A \times s) = r \circ \hat{h} \circ k \\
&= h \circ k = h \circ (\mathrm{id}_{A\times\mathbb{N}}, f)
\end{aligned}
$$

So this map $f : A \times \mathbb{N} \to B$ is *available,* to fullfill the requirements of $\mathrm{pr}\,[\,g, h\,] : A \times \mathbb{N} \to B$.

uniqueness proof for such map f: Let f' be a map assumed likewise to satisfy equations (init) and (step).

Then take $k' := (\mathrm{id}_{A\times\mathbb{N}}, f') : A \times \mathbb{N} \to (A \times \mathbb{N}) \to B$ and calculate:

$$\begin{aligned}
k' \circ (\mathrm{id}_A, 0) &= (\mathrm{id}_{A\times\mathbb{N}}, f') \circ (\mathrm{id}_A, 0) \\
&= ((\mathrm{id}_A, 0), f' \circ (\mathrm{id}_A, 0)) \\
&= ((\mathrm{id}_A, 0), g) = \hat{g} \quad \text{as well as} \\
k' \circ (A \times s) &= (\mathrm{id}_{A\times\mathbb{N}}, f') \circ (A \times s) \\
&= ((A \times s), f' \circ (A \times s)) \\
&= ((A \times s), h) = \hat{h} \circ k'.
\end{aligned}$$

Since by (FR!), k above is the *unique* map to satisfy the equations above, we have necessarily $k' = k$ and hence $f' = r \circ k' = r \circ k = f : A \times \mathbb{N} \to B$. **q.e.d.**

$CLOETEENDE$

1.3 Uniqueness of the NNO $\mathbb{N}$

Strictly speaking, this subsection is not needed for the sequel.

1.4 A monoidal presentation of theory PR

straightforward categorically, not needed strictly.

1.5 Introduction of free variables

We start with a ("generic") example of **Elimination** of free variables by their Interpretation *into (possibly nested)* **projections:**

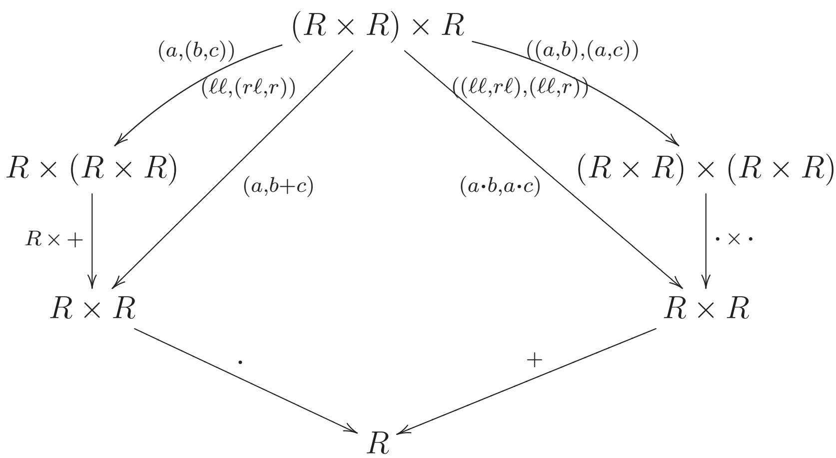
a distributive law $a \cdot (b+c) = a \cdot b + a \cdot c$ gets the map interpretation

$$a \cdot (b+c) = (a \cdot b) + (a \cdot c) :$$
$$R^3 \ =_{\text{by def}} \ R^2 \times R \ =_{\text{by def}} \ (R \times R) \times R \to R,$$
with *systematic* interpretation of variables:
$$a := \ell\ \ell\,,\ b := r\ \ell\,,\ c := r : R^3 = (R \times R) \times R \to R\,,$$

and infix writing of operations $op : R \times R \to R$ prefix interpreted as

$$\cdot \circ (a\,, + \circ (b, c)) = + \circ (\cdot \circ (a, b)\,, \cdot \circ (a, c)) : R^3 \to R.$$

In form of a commuting diagram:



An *iterated* $f^{\S} : A \times \mathbb{N}$ may be written in free-variables notation as

$$f^{\S} = f^{\S}(a, n) = f^n(a) : A \times \mathbb{N} \to A$$

with $a := \ell : A \times \mathbb{N} \to A$, and $n := r : A \times \mathbb{N} \to \mathbb{N}$.

Systematic map Interpretation of free-variables Equations:

1. extract the common codomain (domain of values), say B, of both sides of the equation (this codomain may be implicit);

2. "expand" operator priority into additional bracket pairs;

3. transform infix into prefix notation, on both sides of the equation;

4. order the (finitely many) variables appearing in the equation, e.g.lexically;

5. if these variables $a_1, a_2, \ldots, a_{\underline{m}}$ range over the objects $A_1, A_2, \ldots, A_{\underline{m}}$, then fix as common *domain object* (source of commuting diagram), the object

$$A = A_1 \times A_2 \times \ldots \times A_{\underline{m}} \ =_{\text{def}} \ (\ldots((A_1 \times A_2) \times \ldots) \times A_{\underline{m}});$$

6. interpret the variables as **identities** (possibly nested) **projections,** will say: **replace,** within the equation, all the occurences of a resp. **variable,** by the corresponding—in general *binary nested*—projection;

7. replace each symbol "0" by "$0\ \Pi_D$" where "D" is the (common) domain of (both sides) of the equation;

8. insert composition symbol $\circ$ between terms which are not bound together by an *induced map operator* as in (f_1, f_2);

9. By the above, we have the following two-maps-cartesian-Product rule, forth and back: For

$a := \ell_{A,B} : (A \times B) \to A$, $b := r_{A,B} : (A \times B) \to B$, and $f : A \to A'$ as well as $g : B \to B'$, the following identity holds:

$$\begin{aligned}
&(f \times g)(a, b) = (f \times g) \circ (\ell_{A,B}, r_{A,B}) \\
&= (f \times g) \circ \mathrm{id}_{(A \times B)} = (f \times g) \\
&= (f \circ \ell_{A,B}, g \circ r_{A,B}) \\
&= (f \circ a, g \circ b) = (f(a), g(b)) : A \times B \to A' \times B';
\end{aligned}$$

10. for free variables $a \in A$, $n \in \mathbb{N}$ interpret the term $f^n(a)$ as the map $f^{\S}(a, n) : A \times \mathbb{N} \to A$.

These 10 interpretation steps transform a (PR) free-variables equation into a variable-free, categorical equation of theory **PR** :

Elimination of (free) variables by Interpretation as *projections,* and vice versa: **Introduction of free variables** as *names* for projections. We allow for mixed notation too, all this, for the time being, only in the context of a cartesian (!) theory **T**.

All of our theories are free from classical, (axiomatic) formal quantification. free variables equations are understood naively as *universally quantified.* But a free variable ($a \in A$) occurring only in the premise of an *implication* takes (in suitable context, see below), the meaning

for any given $a \in A$: premise $(a, \ldots) \implies$ conclusion, i. e.

if exists $a \in A$ s. t. premise $(a, \ldots)$, *then* conclusion.

1.6 Goodstein FV arithmetic

In "Development of Mathematical Logic" (Logos Press 1971) R. L. Goodstein gives four basic uniqueness-rules for free-variable Arithmetics. We show here these rules for theory **PR**, and that these four rules are sufficient for proving the commutative and associative laws for multiplication and the distributive law, for addition as well as for truncated subtraction $a \dot{-} n$.

For our ***evaluation and consistency*** *considerations below we need from present section equality* ***predicate*** $[a \doteq b] : \mathbb{N} \times \mathbb{N} \to \mathbb{2}$, *and that this predicate* ***defines*** *map equality, see* ***equality definability scheme*** *in the middle of section. This scheme is a consequence of commutativity* $\max(a, b) \ =_{\text{def}} \ a + (b \dot{-} a) = b + (a \dot{-} b) \ =_{\text{by def}} \max(b, a)$ *which is difficult to show and which you may take on faith.*

Basic **GA** operations are *addition* '$+$', *predecessor* 'pre', *truncated subtraction* '$\dot{-}$', [in Goodstein *predecessor* written $\mathrm{pre} := (\_) \dot{-} 1$], as well as *multiplication* '$\cdot$'.

We include[2] into Goodstein's uniqueness rules a "passive parameter" a. These extended rules are derivable by use of Freyd's **uniqueness theorem** (pr!), part of *full scheme* (pr) of primitive recursion which he deduces from his uniqueness (FR!) of the *initialised iterated.*

Freyd 1972 deduces the latter from availability of a natural numbers object $\mathbb{N}$ in Lawvere's sense, *axiomatic* availability of higher order *internal* hom objects with, again axiomatic, *evaluation* map family for these objects, of form $\epsilon_{A,B} : B^A \times A \to B$ within the category considered.

Goodstein's rules with passive parameter:

Let $f, g : A \times \mathbb{N} \to \mathbb{N}$ be maps, $s : \mathbb{N} \to \mathbb{N}$ the successor map

[2] Sandra Andrasek and the author

$n \mapsto n+1$ and $\mathrm{pre} : \mathbb{N} \to \mathbb{N}$ the predecessor map, usually written as $n \mapsto n \dot{-} 1$.

Then Goodstein's rules read:

$$\mathrm{U}_1 \quad \frac{f(a, sn) = f(a,n) : A \times \mathbb{N} \to B}{f(a,n) = f(a,0) : A \times \mathbb{N} \to B}$$

no change by application of successor

infers equality *with value at zero* for f

$$\mathrm{U}_2 \quad \frac{f(a, s\,n) = s\,f(a,n) : A \times \mathbb{N} \to \mathbb{N}}{f(a,n) = f(a,0) + n : A \times \mathbb{N} \to \mathbb{N}}$$

accumulation of successors into $+n$

$$\mathrm{U}_3 \quad \frac{f(a, sn) = \mathrm{pre}\ f(a,n) : A \times \mathbb{N} \to \mathbb{N}}{f(a,n) = f(a,0) \dot{-} n : A \times \mathbb{N} \to \mathbb{N}}$$

accumulation of predecessors into $\dot{-}\, n$

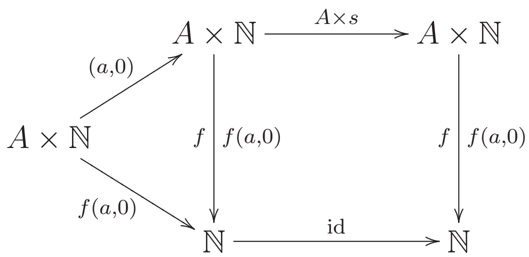
$$\mathrm{U}_4 \quad \frac{\begin{array}{l} f(a,0) = g(a,0) : A \to \mathbb{N} \\ f(a,sn) = g(a,sn) : A \times \mathbb{N} \to \mathbb{N} \end{array}}{f(a,n) = g(a,n) : A \times \mathbb{N} \to \mathbb{N}}$$

uniqueness of map definition by case-distinction

Rule U_4 is nothing else than *uniqueness* of the *induced map out of the sum* $A \times \mathbb{N} \cong (A \times \mathbb{1}) + (A \times \mathbb{N})$, this sum canonically realised via *injections* $\iota = (\mathrm{id}_A, 0) : A \to A \times \mathbb{N}$ as well as—right injection—$\kappa = \mathrm{id}_A \times s : A \times \mathbb{N} \to A \times \mathbb{N}$.

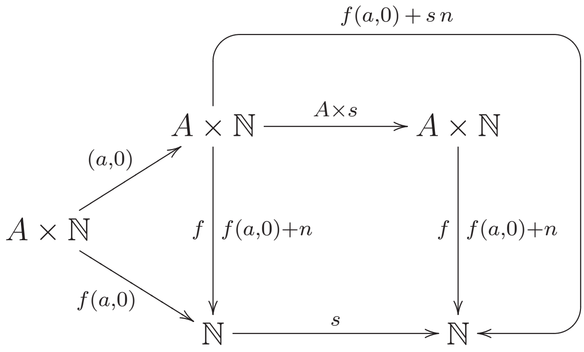
Proof of these four rules is straight forward for theory **PR**, using FREYD's uniqueness (FR!) and uniqueness clause (pr!) of the *full scheme of primitive recursion* respectively, as follows:

For scheme U_1 consider, with free variable $a := \ell : A \times \mathbb{N} \to A$,



$$(\mathrm{FR!}) \quad \frac{\qquad\qquad\qquad}{f(a,n) = f = f(a,0).}$$

Proof of U_2 of *"summing up successors":*



$$\text{(FR!)} \quad \frac{\text{pentagon commutative for both } f,\ f(a,0)+n}{f(a,n) = f(a,0)+n}$$

Proof of U_3 is exactly analogous to the above. Replace in **statement** of U_2 and its **proof** *stepwise augmentation* $f(a, sn) = s\, f(a,n)$ by *stepwise descent*

$$f(a, s\,n) = f(a,n) \dot{-} 1 \ =_{\text{by def}} \ \text{pre}\ f(a,n).$$

On right hand side replace *successor* $s : \mathbb{N} \to \mathbb{N}$ by *predecessor* pre : $\mathbb{N} \to \mathbb{N}$ which in turn is **defined** by the full scheme (pr) of primitive recursion. In **postcedent** replace *iterated successor* $a+n : \mathbb{N}\times\mathbb{N} \to \mathbb{N}$ by *iterated predecessor* $a \dot{-} n : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$.

[In GOODSTEIN's *original,* $\text{pre}(n) = n \dot{-} 1 : \mathbb{N} \to \mathbb{N}$ is a **basic,** "undefined" map constant]

We give a **Direct Proof** of U_4 :

We tailor first this scheme for convenient use of "full" uniqueness scheme (pr!), as follows:

$$U_4 \quad \frac{\begin{array}{l} f = f(a,n),\ f' = f'(a,n) : A \times \mathbb{N} \to B, \\ f(a,0) = f'(a,0) : A \to B, \\ f(a, s\,n) = f'(a, s\,n) : A \times \mathbb{N} \to A \times \mathbb{N} \to B \end{array}}{f = f' : A \times \mathbb{N} \to B.}$$

Choose the *anchor map*

$$g = g(a) := f(a,0) = f'(a,0) :$$
$$A \to A \times \mathbb{N} \to B$$

and the *step map*

$$h = h((a,n),b) := f(a, s\,n) = f'(a, s\,n) :$$
$$(A \times \mathbb{N}) \times B \xrightarrow{\ell} A \times \mathbb{N} \to B.$$

We obtain, via the *full* scheme (pr!) of PR:

$$(\mathrm{pr!}) \quad \frac{\begin{array}{l} f(a,0) = g(a) = f'(a,0), \ \text{(anchor hypothesis)} \\ f(a, s\,n) = h((a,n), f(a,n)) = f'(a, s\,n) \ \text{(step hypothesis)} \end{array}}{f = \mathrm{pr}[\,g, h\,] = f' : A \times \mathbb{N} \to B \quad \textbf{q.e.d.}}$$

Together with *reflexivity, symmetry,* and *transitivity* of *equality* $f = g : A \to B$: between maps as well as with the **defining equations** for the fundamental **operations** and $\mathrm{U}_1, \ldots, \mathrm{U}_4$ above, we **define** categorical Goodstein's **free-variables Arithmetic** which we name **Goodstein Arithmetic, GA**.

We now *quote,* with *passive parameters* made visible, GOODSTEIN's arithmetical equations together with his **proofs.**

The first equation is (Goodstein's statement numbers)

Lemma:

$$(a \dot{-} n) \dot{-} 1 =^{\mathbf{GA}} (a \dot{-} 1) \dot{-} n : \mathbb{N} \times \mathbb{N} \to \mathbb{N}, \qquad (1.)$$

$a \in \mathbb{N}$ free, "passive", $a := \ell : A \times \mathbb{N} \to A$,

$n \in \mathbb{N}$ free, *recursive,* $n := r : A \times \mathbb{N} \to \mathbb{N}$.

Proof:

$$\mathrm{U}_3 \ \frac{(a \dot{-} s\,n) \dot{-} 1 \ =_{\text{by def}} \ ((a \dot{-} n) \dot{-} 1) \dot{-} 1}{(a \dot{-} n) \dot{-} 1 = ((a \dot{-} 0) \dot{-} 1) \dot{-} n}$$

$$=_{\text{by def}} \ (a \dot{-} 1) \dot{-} n : \mathbb{N}^2 \to \mathbb{N} \quad \textbf{q.e.d.}$$

Next equation is

stepwise simplification rule for truncated subtraction:

$$s\ a \dot{-} s\ b = a \dot{-} b : \mathbb{N} \times \mathbb{N} \to \mathbb{N}, \qquad (1.1)$$

Proof:

$$\mathrm{U}_3 \ \frac{s\ a \dot{-} s\ s\ b \ =_{\text{by def}} \ (s\ a \dot{-} s\ b) \dot{-} 1}{s\ a \dot{-} s\ b = (s\ a \dot{-} s\ 0) \dot{-} b}$$

$$=_{\text{by def}} \ a \dot{-} b : \mathbb{N}^2 \to \mathbb{N},$$

the latter by **definition** of the *predecessor* "$\dot{-}\ 1$" **q.e.d.**

Lemma: $a \dot{-} a = 0 : \mathbb{N} \to \mathbb{N}.$ (1.2)

Proof:

$$
\mathrm{U}_1 \quad \frac{s\,a \dot{-} s\,a = a \dot{-} a \quad \text{(by stepwise simplification 1.1 above)}}{a \dot{-} a = 0 \dot{-} 0 \ =_{\text{by def}}\ 0 \quad \textbf{q.e.d.}}
$$

Lemma: $0 \dot{-} a = 0 : \mathbb{N} \to \mathbb{N}.$ (1.3)

Proof:

$$
\mathrm{U}_1 \quad \frac{\begin{array}{l} 0 \dot{-} s\,a \ =_{\text{by def}}\ (0 \dot{-} a) \dot{-} 1 \\ = (0 \dot{-} 1) \dot{-} a \quad \text{(by (1.) above)} \\ = 0 \dot{-} a : \mathbb{N} \to \mathbb{N} \end{array}}{0 \dot{-} a = 0 \dot{-} 0 = 0 : \mathbb{N} \to \mathbb{N} \quad \textbf{q.e.d.}}
$$

Proposition:

$a \dot{-} (b + c) = (a \dot{-} b) \dot{-} c : (\mathbb{N} \times \mathbb{N}) \times \mathbb{N} \to \mathbb{N}.$ (1.31)

Proof:

$$a := \ell_{\mathbb{N},\mathbb{N}} \circ \ell_{\mathbb{N}\times\mathbb{N},\mathbb{N}} : (\mathbb{N}\times\mathbb{N})\times\mathbb{N} \xrightarrow{\ell} \mathbb{N}\times\mathbb{N} \xrightarrow{\ell} \mathbb{N},$$
$$b := r \circ \ell : (\mathbb{N}\times\mathbb{N})\times\mathbb{N} \xrightarrow{\ell} \mathbb{N}\times\mathbb{N} \xrightarrow{r} \mathbb{N},$$
$$(a,b) = \ell_{\mathbb{N}\times\mathbb{N},\mathbb{N}} : A\times\mathbb{N} = \mathbb{N}^2\times\mathbb{N} \xrightarrow{\ell} A = \mathbb{N}^2,$$
$$c := r : A\times\mathbb{N} = \mathbb{N}^2\times\mathbb{N} \xrightarrow{r} \mathbb{N}.$$

$$a \dot{-} (b + s\,c) \;=_{\text{by def}}\; a \dot{-} s\,(b+c) \quad \text{(definition of +)},$$
$$=_{\text{by def}}\; (a \dot{-} (b+c)) \dot{-} 1 \quad \text{(definition of } \dot{-} \text{)}$$

(U$_3$) ____

$$a \dot{-} (b+c) = (a \dot{-} (b+0)) \dot{-} c \;=_{\text{by def}}\; (a \dot{-} b) \dot{-} c. \quad \textbf{q.e.d.}$$

Full Simplification:

$$(a+n) \dot{-} (b+n) = a \dot{-} b : \mathbb{N}^2\times\mathbb{N} \to \mathbb{N}. \tag{1.4}$$

Proof:

$$(a + s\,n) \dot{-} (b + s\,n)$$
$$=_{\text{by def}}\; s\,(a+n) \dot{-} s\,(b+n) = (a+n) \dot{-} (b+n),$$

by *substitution*—realised essentially as composition

—of $(a+n)$ into a, and $(a+n)$ into b within

stepwise simplification equation 1.1 above

(U$_1$) ____

$$(a+n) \dot{-} (b+n) = (a+0) \dot{-} (b+0) \;=_{\text{by def}}\; a \dot{-} b.$$

Lemma: $0 + n = n\ [\ =_{\text{by def}}\ \ n + 0\ ]\ : \mathbb{N} \to \mathbb{N},$ (2)

Proof:

$$\mathrm{U}_2 \quad \frac{\mathrm{id}_{\mathbb{N}}\, s\, a = s\, a}{\mathrm{id}_{\mathbb{N}}(a) = \mathrm{id}_{\mathbb{N}}(0) + a,}$$

and hence

$$a = \mathrm{id}_{\mathbb{N}}(a) = \mathrm{id}_{\mathbb{N}}(0) + a = 0 + a : \mathbb{N} \to \mathbb{N} \quad \textbf{q.e.d.}$$

Lemma: $a + s\, b = s\, a + b : \mathbb{N} \times \mathbb{N} \to B.$ (2.1)

Proof by U_2 as follows, with free variable $b := r : \mathbb{N}^2 \to \mathbb{N}$ as *recursion variable:*

For $f = f(a, b)\ =_{\text{def}}\ a + s\, b : \mathbb{N} \times \mathbb{N} \to N :$

$$U_2 \quad \frac{f(a, s\, b)\ =_{\text{by def}}\ a + s\, s\, b = s(a + s\, b) = s\, f(a, b) : \mathbb{N}^2 \to \mathbb{N}}{f(a, b) = a + s\, b = f(a, 0) + b}$$

$$=_{\text{by def}}\ (a + s\, 0) + b\ =_{\text{by def}}\ s\, a + b \quad \textbf{q.e.d.}$$

Theorem:

$a + b = b + a : \mathbb{N} \times \mathbb{N} \to \mathbb{N},$ (2.2),

$a := \ell : \mathbb{N}^2 \to \mathbb{N},$

$b := r : \mathbb{N}^2 \to \mathbb{N}.$

Proof:

$$a + 0 \ =_{\text{by def}} \ a = 0 + a \text{ by (2) above,}$$

$$U_4 \ \frac{a + s\,b = s\,a + b \text{ by (2.1) above (and symmetry of equality)}}{a + b \ =_{\text{by def}} \ f(a,b) = g(a,b)}$$

$$=_{\text{by def}} \ s\,a + b : \mathbb{N}^2 \to \mathbb{N} \quad \textbf{q.e.d.}$$

This **gives** also sort of

permutability for **truncated subtraction:**

$$(a \dot{-} b) \dot{-} c = (a \dot{-} c) \dot{-} b : (\mathbb{N} \times \mathbb{N}) \times \mathbb{N} \to \mathbb{N}.$$

Proof:

$(a \dot{-} b) \dot{-} c = a \dot{-} (b + c)$ by (1.31) above

$= a \dot{-} (c + b)$ by commutativity of addition above

$= (a \dot{-} c) \dot{-} b$ again by (1.31) **q.e.d.**

Lemma:

$$(a + n) \dot{-} n = (a + n) \dot{-} (0 + n) = a : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \ \textbf{q.e.d.} \quad (2.3)$$

Associativity of Addition

$(a + b) + c = a + (b + c) : (\mathbb{N} \times \mathbb{N}) \times \mathbb{N} \to \mathbb{N}$,

with free variables

$a := \ell \circ \ell : (\mathbb{N} \times \mathbb{N}) \times \mathbb{N} \to \mathbb{N} \times \mathbb{N} \to \mathbb{N}$,

$b := r \circ \ell : (\mathbb{N} \times \mathbb{N}) \times \mathbb{N} \to \mathbb{N} \times \mathbb{N} \to \mathbb{N}$,

$c := r : (\mathbb{N} \times \mathbb{N}) \times \mathbb{N}$.

Proof: for $f((a,b),c) \ =_{\text{def}} \ a+(b+c) : \mathbb{N}^2 \times \mathbb{N}:$

$$\begin{array}{ll} & f((a,b),s\,c) = a+(b+s\,c) = a+s(b+c) \\ & = s(a+(b+c)) = s\,f((a,b),c) \\ U_2 & \overline{} \\ & a+(b+c) = f((a,b),c) = f((a,b),0)+c \\ & =_{\text{by def}} \ (a+(b+0))+c = (a+b)+c : \mathbb{N}^2 \times \mathbb{N} \to \mathbb{N} \quad \textbf{q.e.d.} \end{array}$$

Recall p. r. **Definition** of **Multiplication:**

$$a \cdot 0 = 0 : \mathbb{N} \to \mathbb{N},$$
$$a \cdot (n+1) = (a \cdot n) + a.$$

For this operation, we have not only *annihilation by zero from the right,* but also

Left zero-Annihilation $0 \cdot n = 0 : \mathbb{N} \to \mathbb{N}$.

Proof:

$$U_1 \ \frac{0 \cdot s\,n = (0 \cdot n) + 0 = 0 \cdot n}{0 \cdot n = 0 \cdot 0 = 0} \quad \textbf{q.e.d.}$$

For **proving** the other equational laws making the natural numbers object $\mathbb{N}$ into a **unitary commutative semiring** with in addition truncated subtraction introduced above, GOODSTEIN's **derived** scheme V_4 below is helpfull.

For **proof** of that scheme, we rely on

Commutativity of maximum operation:[3]

$\max(a,b) =_{\text{def}} a + (b \dot{-} a) = b + (a \dot{-} b) =_{\text{by def}} \max(b,a) : \mathbb{N}\times\mathbb{N} \to \mathbb{N}$

Proof: As a first step, we **show**

Diagonal Reduction Lemma for maximum:

$$\max(a,b) = \max(a \dot{-} 1, b \dot{-} 1) + \text{sign}(a+b)$$
$$=_{\text{by def}} \max(a \dot{-} 1, b \dot{-} 1) + (1 \dot{-} (1 \dot{-} (a+b))) :$$
$$\mathbb{N} \times \mathbb{N} \to \mathbb{N},$$
$$\max(a, s\,b) = \max(a \dot{-} 1, s\,b \dot{-} 1) + \text{sign}(a + s\,b), \qquad (1)$$

(where $\text{sign}(0) = 0$, $\text{sign}(s\,n) = 1$), **as follows:**

$$\max(0, s\,b) = s\,b = \max(0,b) + 1 : \mathbb{N} \to \mathbb{N}, \qquad (2)$$
$$\max(s\,a, s\,b) = s\ \max(a,b) = \max(a,b) + 1$$
$$= \max(s\,a \dot{-} 1, s\,b \dot{-} 1) + \text{sign}(s\,a + s\,b) \qquad (3)$$

From (2) and (3) follows (1) by uniqueness U_4.

Furthermore

$$\max(a, 0) = a = (a \dot{-} 1) + \text{sign}(a)$$
$$= \max(a \dot{-} 1, 0 \dot{-} 1) + \text{sign}(a + 0). \qquad (4)$$

Together with (1) above, this **gives,** again by U_4, the **Diagonal Reduction Lemma.**

From this we get immediately by substitution

[3] in Goodstein 1957 this is taken as an axiom

Opposite Diagonal Reduction Lemma for maximum:

$$\max(b, a) = \max(b \dot{-} 1, a \dot{-} 1) + \text{sign}(b + a)$$
$$= \max(b \dot{-} 1, a \dot{-} 1) + \text{sign}(a + b) \quad \textbf{q.e.d.}$$

Now **let**

$$\phi = \phi(n, (a, b)) : \mathbb{N} \times (\mathbb{N} \times \mathbb{N}) \to \mathbb{N} \text{ by}$$
$$\phi(0, (a, b)) = 0 : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \text{ and}$$
$$\phi(s\,n, (a, b)) = \phi(n, (a, b)) + \text{sign}((a \dot{-} n) + (b \dot{-} n)) :$$
$$\mathbb{N} \times (\mathbb{N} \times \mathbb{N}) \to \mathbb{N}$$

We **show** for this *increment* map ϕ

$$\max(a \dot{-} n, b \dot{-} n) + \phi(n, (a, b))$$
$$= \max(a \dot{-} s\,n, b \dot{-} s\,n) + \phi(s\,n, (a, b)) \qquad (5)$$

as well as

$$\max(b \dot{-} n, a \dot{-} n) + \phi(n, (a, b))$$
$$= \max(a \dot{-} s\,n, b \dot{-} s\,n) + \phi(s\,n, (a, b)) \qquad (6)$$

(same increment).

First we **show** equation (5): Substitution of $(a \dot{-} n)$ for a and $(b \dot{-} n)$ for b within **Reduction Lemma** above gives

$$\max(a \dot{-} n, b \dot{-} n)$$
$$= \max((a \dot{-} n) \dot{-} 1, (b \dot{-} n) \dot{-} 1) + \text{sign}((a \dot{-} n) + (b \dot{-} n))$$

Adding $\phi(n,(a,b))$ to both sides of this equation gives

$$\max(a \dot{-} n, b \dot{-} n) + \phi(n,(a+b))$$
$$= \max((a \dot{-} n) \dot{-} 1, (b \dot{-} n) \dot{-} 1)$$
$$+ \operatorname{sign}((a \dot{-} n) + (b \dot{-} n)) + \phi(n,(a+b))$$
$$=_{\text{by def}} \ \max(a \dot{-} s\,n, b \dot{-} s\,n) + \phi(s\,n,(a,b)),$$

i. e. equation (5).

We **show** equation (6): By substitution of $(b \dot{-} n)$ for b and $(a \dot{-} n)$ for a in **Opposite Reduction Lemma** and addition of $\phi(n,(a,b))$ on both sides, we get

$$\max(b \dot{-} n, a \dot{-} n) + \phi(n,(a,b))$$
$$= \max((b \dot{-} n) \dot{-} 1, (a \dot{-} n) \dot{-} 1)$$
$$+ \operatorname{sign}((b \dot{-} n) + (a \dot{-} n)) + \phi(n,(a,b))$$
$$= \max((b \dot{-} n) \dot{-} 1, (a \dot{-} n) \dot{-} 1)$$
$$+ \operatorname{sign}((a \dot{-} n) + (b \dot{-} n)) + \phi(n,(a,b))$$
$$=_{\text{by def}} \ \max((b \dot{-} n) \dot{-} 1, (a \dot{-} n) \dot{-} 1) + \phi(s\,n,(a,b))$$
$$= \max(b \dot{-} s\,n, a \dot{-} s\,n) + \phi(s\,n,(a,b)),$$

i. e. equation (6).

From the two **Lemmata,** we get by uniqueness U_1

$$\max(a \dot{-} n, b \dot{-} n) + \phi(n,(a,b))$$
$$= \max(a \dot{-} 0, b \dot{-} 0) + \phi(0,(a,b)) = \max(a,b) + 0 = \max(a,b)$$

as well as

$$\max(b \dot{-} n, a \dot{-} n) + \phi(n,(a,b))$$
$$= \max(b \dot{-} 0, a \dot{-} 0) + \phi(0,(a,b)) = \max(b,a) + 0 = \max(b,a)$$

and hence

$$\max(a,b) = \max(a \dot{-} n, b \dot{-} n) + \phi(n,(a,b)) \text{ as well as}$$
$$\max(b,a) = \max(b \dot{-} n, a \dot{-} n) + \phi(n,(a,b)),$$

and so, by substitution of b into n :

$$\max(a,b) = \max(a \dot{-} b, b \dot{-} b) + \phi(b,a,b)$$
$$= (a \dot{-} b) + \phi(b,(a,b))$$
$$= \max(b \dot{-} b, a \dot{-} b) + \phi(b,(a,b))$$
$$= \max(b,a) : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$$

q.e.d.

This given, we can now **show,** for **GA** (and hence for **PR**), scheme

$$\mathrm{V}_4 \quad \frac{\begin{array}{l} f,\ g,\ h : A \times \mathbb{N} \to \mathbb{N} \\ f(a,0) = g(a,0) : A \to \mathbb{N} \\ f(a,sn) = f(a,n) + h(a,n) : A \times \mathbb{N} \to \mathbb{N} \\ g(a,sn) = g(a,n) + h(a,n) : A \times \mathbb{N} \to \mathbb{N} \end{array}}{f(a,n) = g(a,n).}$$

Rule V_4 can be **derived,** by applying rule U_1 to the distance map

$$d(a,n) = |f(a,n), g(a,n)| = |f(a,n) - g(a,n)|$$
$$=_{\text{by def}} \ (f(a,n) \dot{-} g(a,n)) + (g(a,n) \dot{-} f(a,n)) :$$
$$A \times \mathbb{N} \to \mathbb{N}^2 \xrightarrow{+} \mathbb{N} :$$

$$
\begin{aligned}
d(a,0) &= (f(a,0) \dot{-} g(a,0)) + (g(a,0) \dot{-} f(a,0)) = 0 \\
d(a,sn) &= (f(a,sn) \dot{-} g(a,sn)) + (g(a,sn) \dot{-} f(a,sn)) \\
&= (f(a,n) + h(a,n)) \dot{-} (g(a,n) + h(a,n)) \\
&\quad + (g(a,n) + h(a,n)) \dot{-} (f(a,n) + h(a,n)) \\
&= (f(a,n) \dot{-} g(a,n)) + (g(a,n) \dot{-} f(a,n)) \\
&= d(a,n) : A \times \mathbb{N} \to \mathbb{N},
\end{aligned}
$$

whence, by U_1:

$$
\begin{aligned}
&d(a,n) = d(a,0) = 0, \text{ i.e.} \\
&(f(a,n) \dot{-} g(a,n)) + (g(a,n) \dot{-} f(a,n)) = 0, \text{ whence} \\
&f(a,n) \dot{-} g(a,n) = 0 = g(a,n) \dot{-} f(a,n) : A \times \mathbb{N} \to \mathbb{N},
\end{aligned}
$$

and hence

$$
\begin{aligned}
f(a,n) &= f(a,n) + (g(a,n) \dot{-} f(a,n)) \\
&= \max(f(a,n), g(a,n)) \\
&= \max(g(a,n), f(a,n)) \\
&= g(a,n) + (f(a,n) \dot{-} g(a,n)) \\
&= g(a,n) \quad \textbf{q.e.d.}
\end{aligned}
$$

individual equality, equality *predicate*

$$[\, m \doteq n \,] : \mathbb{N}^2 \to \mathbb{2}$$

is **defined** via weak order as follows:

$$\begin{aligned}
&[\,m \leq n\,] \ =_{\text{def}} \ \neg\,[m \dot{-} n] : \mathbb{N}^2 \to \mathbb{N} \to \mathbb{N}, \ \text{where} \\
&\neg\, n \ =_{\text{def}} \ 1 \dot{-} n, \ \text{directly p.r. defined by} \\
&\neg\, 0 \ =_{\text{def}} \ 1 \equiv \text{true} : \mathbb{1} \to \mathbb{N}, \\
&\neg\, s\, n \ =_{\text{def}} \ 0 \equiv \text{false} : \mathbb{1} \to \mathbb{N}.
\end{aligned}$$

This order on $\mathbb{N}$ is **reflexive** and **transitive.**

Individual equality—first on $\mathbb{N}$—then is easily **defined** by

$$[\,m \doteq n\,] \ =_{\text{def}} \ [\,m \leq n \ \wedge \ n \leq m\,] : \mathbb{N}^2 \to \mathbb{N}.$$

Almost by **definition,** the triple $\{\leq, \ \doteq, \ \geq\} : \mathbb{N}^2 \to \mathbb{N}$ fullfills the **law of trichotomy,** and $\max(a, b) : \mathbb{N}^2 \to \mathbb{N}$ above is characterised as the *maximum* map with respect to the order $[\,a \leq b\,] : \mathbb{N}^2 \to \mathbb{N}$ just introduced, a posteriori.

We now have at our disposition all ingredients for the

Equality definability theorem:

$$\text{(EqDef)} \quad \frac{\begin{array}{l} f = f(a) : A \to B, \ g = g(a) : A \to B \text{ in } \mathbf{PR}, \\ \mathbf{PR} \vdash \ \text{true}_A = [\, f(a) \doteq_B g(a)\,] : \\ \qquad A \xrightarrow{\Delta} A \times A \xrightarrow{f \times g} B \times B \xrightarrow{\doteq_B} \mathbb{2} \end{array}}{\mathbf{PR} \vdash \ f = g : A \to B, \ \text{i.e. } f =^{\mathbf{PR}} g : A \to B.}$$

Proof:

We begin with the special case $B = \mathbb{N}$: Let $f,\ g : A \to \mathbb{N}$ **PR**-maps satisfying the **antecedent** of (EqDef). Then

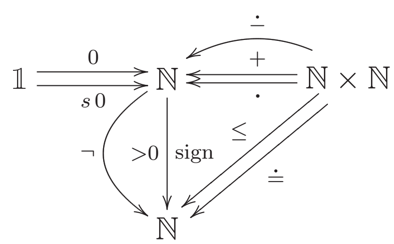
$$\begin{aligned}\mathbf{PR} \vdash\ f(a) &= f(a) + 0 = f(a) + (g(a) \dot{-} f(a)) \\ &= \max(f(a), g(a)) \\ &= \max(g(a), f(a)) \\ &= g(a) : A \to B.\end{aligned}$$

The general case for codomain object B follows, since *individual equality* on (binary) cartesian Products is canonically **defined** *componentwise,* and B is a cartesian product of $\mathbb{N}$'s **q.e.d.**

These *fundamentals* given, we can continue with properties of the algebraic structure on $\mathbb{N}$.

Algebra, Order and Logic on $\mathbb{N}$:

- $\mathbb{N}$ admits the structure



  of a **unary, commutative semiring with zero**—properties of $\dot{-}$, sign : $\mathbb{N} \to \mathbb{N}$ ("positiveness"), order, and equality $\doteq$ see below.

- $\mathbb{N}$ admits a foundational important additional algebraic structure, namely **truncated subtraction** $m \dot{-} n : \mathbb{N}^2 \to \mathbb{N}$, with its *simplification properties,* such that multiplication *distributes* over this kind of subtraction.

This distributivity will further entail that of multiplication over "full", not truncated subtraction within

$$\mathbb{Z} \;=_{\text{def}}\; (\mathbb{N}\times\mathbb{N})/\doteq_{\mathbb{Z}},$$

with **defining** equality *predicate*

$$[\,(p,q) \doteq_{\mathbb{Z}} (p',q')\,] \;=_{\text{def}}\; [\,p+q' \doteq q+p'\,]:$$

$$\mathbb{N}^2\times\mathbb{N}^2 \to \mathbb{N}\times\mathbb{N} \xrightarrow{\doteq} \mathbb{N}.$$

- $\mathbb{N}$ admits linear **order** $[\,m \leq n\,] : \mathbb{N}\times\mathbb{N} \to \mathbb{N} \subset \mathbb{N}$ as a weak reflexive and transitive *predicate*—this order is p. r. *decidable.*
- As basic logical structures, $\mathbb{N}$ admits **negation**

$$\neg = \neg\, n : \mathbb{N} \to \mathbb{N}, \text{ as well as}$$

$$\text{sign} = \text{sign}\, n = \neg\,\neg\, n : \mathbb{N} \to \mathbb{N},$$

$\text{sign}(n)$ is directly p. r. **defined** by

$$\text{sign}\, 0 \;=_{\text{def}}\; 0 \equiv \text{false},\ \text{sign}\, s\, n \;=_{\text{def}}\; 1 \equiv s\, 0:$$

$\text{sign}\, n = [\,n > 0\,] : \mathbb{N} \to \mathbb{N}$ PR decides on *positiveness.*

Furthermore, we have a fundamental *equality* ***predicate***

$$[\,m \doteq n\,] \;=_{\text{by def}}\; [\,m \leq n\,] \;\wedge\; [\,m \geq n\,] : \mathbb{N}\times\mathbb{N} \to \mathbb{N},$$

$$[\,a \wedge b \;=_{\text{def}}\; \text{sign}(a \cdot b) \text{ logical `and'}\,],$$

which is an *equivalence predicate,* and which makes up a **trichotomy** with strict order

$$[\,m < n\,] \;=_{\text{def}}\; \text{sign}(n \dot{-} m)$$
$$= [\,m \leq n\,] \;\wedge\; \neg\,[\,m \doteq n\,] : \mathbb{N}^2 \to \mathbb{N},$$

Proof of the latter equation is left as an **Exercise.**

- object $\mathbb{N}$ admits **definition** of (Boolean) "logical functions" by **truth tables,** as does set $\mathbb{2}$ classically—and below in theory **PRa** = **PR** + (abstr) of primitive recursion with predicate abstraction: draw the commuting diagrams.

- **Algebra Combined with Order:** As expected, addition is strongly monotonic in both arguments, multiplication is strongly monotonic for both arguments strictly greater than zero, and truncated subtraction is weakly monotonic in its first argument and weakly antitonic in its second.

Theorem: In free–variables arithmetics the **commutative law** for **multiplication:** $n \cdot m = m \cdot n$, holds.

Proof: We need the following

Lemma:

(i) $0 \cdot n = 0$

(ii) $sa \cdot n = a \cdot n + n$

Proof:

(i) $0 \cdot 0 = 0$ and
$0 \cdot sn = 0 \cdot (n+1) = 0 \cdot n + 0 = 0 \cdot n = 0 \cdot 0 = 0$.

(ii) We show $f(a,n) := sa \cdot n = g(a,n) := a \cdot n + n$ using V_4: $f(a,0) = g(a,0)$ because for $n = 0$ we get $(sa) \cdot 0 = 0$ as well as $a \cdot 0 + 0 = a \cdot 0 = 0$.

$$\begin{aligned}
f(a,sn) &= (sa) \cdot (sn) = (a+1) \cdot (n+1) \\
&= (a+1) \cdot n + (a+1) = (sa) \cdot n + sa \\
&= f(a,n) + h(a,n), \qquad \text{with} \qquad h(a,n) := sa \\
g(a,sn) &= a \cdot (sn) + sn = a \cdot (n+1) + (n+1) \\
&= a \cdot n + a + n + 1 = a \cdot n + n + a + 1 \\
&= a \cdot n + n + sa \\
&= g(a,n) + h(a,n).
\end{aligned}$$

So V_4 gives $f(a,n) = g(a,n)$ i.e. $sa \cdot n = a \cdot n + n$.

q.e.d.

We continue with the proof of $a \cdot n = n \cdot a$:

From $a \cdot 0 = 0 = 0 \cdot a$ and $a \cdot sn = a \cdot n + n = sn \cdot a$ by the Lemma, we conclude $a \cdot n = n \cdot a$ by V_4.[4]

q.e.d.

[4] corrected by S. Lee may 21, 2013

Theorem In free–variable arithmetics multiplication distributes over addition: $a \cdot (m+n) = a \cdot m + a \cdot n$.

Proof:

Case $n = 0$ is trivial by definition of $+$ and $\cdot$.

From the hypothesis $a \cdot (m+n) = a \cdot m + a \cdot n$ we infer the next step $a \cdot (m+sn) = a \cdot m + a \cdot sn$ by rule V_4 above—with passive parameter (a,m)—as follows:

$$\begin{aligned} \text{with} \quad & f((a,m),n) := a \cdot (m+n), \\ & g((a,m),n) := a \cdot m + a \cdot n \qquad \text{and} \\ & h((a,m),n) := a \end{aligned}$$

we have

$$\begin{aligned} f((a,m),sn) &= a \cdot (m+sn) = a \cdot (m+(n+1)) \\ &= a \cdot ((m+n)+1) = a \cdot (m+n) + a \\ &= f((a,m),n) + h((a,m),n) \\ g((a,m),sn) &= a \cdot m + a \cdot sn = a \cdot m + a \cdot (n+1) \\ &= a \cdot m + a \cdot n + a \\ &= g((a,m),n) + h((a,m),n). \end{aligned}$$

So by V_4 we get $f((a,m),n) = g((a,m),n)$, i. e. $a \cdot (m+n) = a \cdot m + a \cdot n$.

q.e.d.

Theorem: In free–variable arithmetics the associative law holds, i. e. $a \cdot (m \cdot n) = (a \cdot m) \cdot n$.

Proof: We prove the law applying rule V_4 with "active" parameter n and passive parameter (a, m) to

$$f((a,m),n) := a \cdot (m \cdot n),$$
$$g((a,m),n) := (a \cdot m) \cdot n \quad \text{and}$$
$$h((a,m),n) := a \cdot m.$$

For $n = 0$ we have: $a \cdot (m \cdot n) = a \cdot 0 = 0$, and on the other hand: $(a \cdot m) \cdot 0 = 0$.

For V_4–step we have:

$$\begin{aligned}
f((a,m),sn) &= a \cdot (m \cdot sn) = a \cdot (m \cdot (n+1)) \\
&= a \cdot (m \cdot n + m) = a \cdot (m \cdot n) + a \cdot m \\
&= f((a,m),n) + h((a,m),n) \\
g((a,m),sn) &= (a \cdot m) \cdot (n+1) = (a \cdot m) \cdot n + a \cdot m \\
&= g((a,m),n) + h((a,m),n).
\end{aligned}$$

By V_4 we get $f((a,m),n) = g((a,m),n)$, i. e. $a \cdot (m \cdot n) = (a \cdot m) \cdot n$.

q.e.d.

Distributivity theorem: In free–variable arithmetics *multiplication distributes* over *truncated subtraction:*

$$a \cdot (m \dot{-} n) = a \cdot m \;\dot{-}\; a \cdot n.$$

Proof by **equality definability,** namely

$$[\, f = g \quad \text{iff} \quad [\, f \doteq g \,] = \mathit{true} \,],$$

it is sufficient to show

$$f((a,m),n) := a \cdot (m \dot{-} n) \doteq a \cdot m \ \dot{-} \ a \cdot n =: g((a,m),n)] = true.$$

Proof of this law becomes comparitively easy with *diagonal induction* out of Pfender, Kröplin, Pape 1994:

Anchoring ($m = 0$ resp. $n = 0$):

$$a \cdot (0 \dot{-} n) = a \cdot 0 = 0 = 0 \dot{-} a \cdot n = a \cdot 0 \ \dot{-} \ a \cdot n, \qquad \text{as well as}$$
$$a \cdot (m \dot{-} 0) = a \cdot m = a \cdot m \dot{-} 0 = a \cdot m \dot{-} a \cdot 0.$$

Diagonal induction **step:**

$$f(a,m,n) := a \cdot (m \dot{-} n) \doteq a \cdot m \dot{-} a \cdot n =: g(a,m,n)$$
$$\Longrightarrow f(a,sm,sn) = a \cdot (sm \dot{-} sn) \doteq a \cdot sm \dot{-} a \cdot sn = g(a,sm,sn),$$

since

$$\begin{aligned}
f(a,sm,sn) &= a \cdot (sm \dot{-} sn) = a \cdot (m \dot{-} n) \\
&= f(a,m,n), \\
g(a,sm,sn) &= a \cdot sm \dot{-} a \cdot sn = a \cdot (m+1) \dot{-} a \cdot (n+1) \\
&= (a \cdot m + a) \dot{-} (a \cdot n + a) \\
&= a \cdot m \dot{-} a \cdot n \qquad \text{by absorption law for } \dot{-} \\
&= a \cdot (m \dot{-} n) \\
&= g(a,m,n).
\end{aligned}$$

q.e.d.

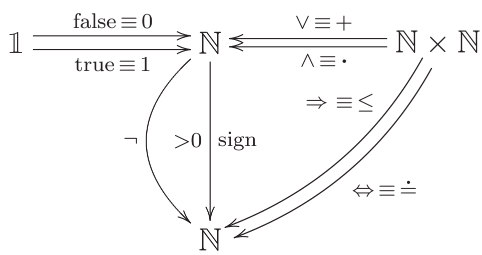
Proposition: Addition and multiplication in free-variable arithmetics are weakly monotonous, i. e.

$$
\begin{aligned}
m \leq n &\implies m \dot{-} n = 0 \\
&\implies (a+m) \dot{-} (a+n) \doteq 0 \qquad \text{by absorption law for } \dot{-} \\
&\implies a+m \ \leq \ a+n \\
m \leq n &\implies m \dot{-} n = 0 \\
&\implies (a \cdot m) \dot{-} (a \cdot n) \doteq a \cdot (m \dot{-} n) \doteq 0 \\
&\implies a \cdot m \ \leq \ a \cdot n
\end{aligned}
$$

q.e.d.

Boolean Structure on $\mathbb{N}$

In present framework **GA** of **Goodstein Arithmetic** we introduce on NNO $\mathbb{N}$ the following *proto Boolean* structure:

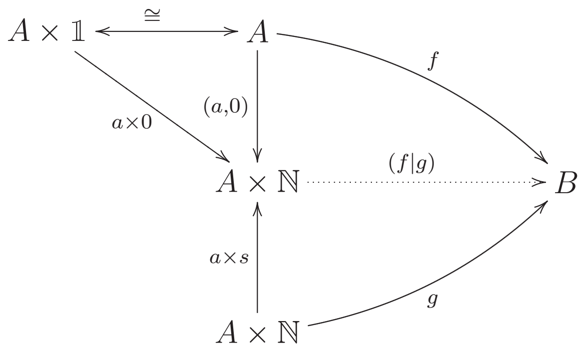


[Successors are all viewed logically to represent truth value true.]

1.7 Sum objects and definition by distinction of cases

"Hilbert's infinite hotel" $\mathbb{N} \cong \mathbb{1} + \mathbb{N}$:

Consider the **sum** diagram



where

$$(f|g) \ =_{\text{def}} \ \text{pr}[f : A \to B,\ g \circ \ell : (A \times \mathbb{N}) \times B \to A \times \mathbb{N} \to B]$$

is the *unique* commutative fill-in into this *sum diagram:* full scheme (pr) of primitive recursion. Symbolically:

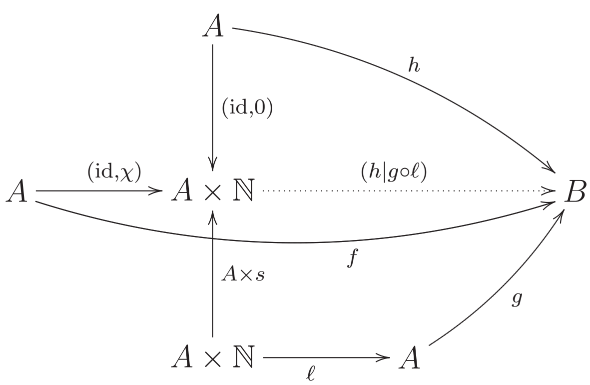
$$A \times \mathbb{N} = A + (A \times \mathbb{N}) \cong (A \times \mathbb{1}) + (A \times \mathbb{N}).$$

An important **consequence** is the following scheme of **map defini-**

tion by case distinction:

$$
\text{(IF)} \quad \frac{\begin{array}{l} \chi = \mathrm{sign} \circ \chi : A \to \mathbb{N} \text{ p.r. predicate,} \\ g, h : A \to B \text{ p.r. maps} \end{array}}{\begin{array}{l} f = \mathrm{if}[\chi, (g|h)] \text{ ``if } \chi \text{ then } g \text{ else } h\text{''} \\ =_{\mathrm{def}} \ (h|g \circ \ell) \circ (\mathrm{id}_A, \chi) : \\ A \to A \times \mathbb{N} \to B, \\ \chi(a) \implies \mathrm{if}[\chi, (g|h)] \doteq g(a), \\ \neg\chi(a) \implies \mathrm{if}[\chi, (g|h)] \doteq h(a). \end{array}}
$$

Proof: Commuting DIAGRAM:



with $(h|g\,\ell) : A \times \mathbb{N} = A + (A \times \mathbb{N}) \to B$ the induced map out of the sum ("coproduct"), coproduct *injections* $(\mathrm{id}, 0), A \times s$.

free-variable notation:

$$f = f(a) = \mathrm{if}[\chi, (g|h)](a)$$
$$= \begin{cases} g(a) \text{ if } \chi(a) \\ h(a) \text{ if } \neg\, \chi(a) \textit{ (otherwise).} \end{cases}$$

This terminates presentation (and discussion) of terms and equational **axioms** presenting *fundamental categorical free variables theory* **PR** of *primitive recursion.*

Note:

In PFENDER, KRÖPLIN, PAPE 1994 section 4, D. Pape has adapted the classical concept of primitive recursion out of YASHUHARA 1971 to the (free-variables) categorical setting, and shown equivalence with fundamental theory **PR** above.

1.8 Substitutivity and Peano induction

Leibniz substitutivity theorem for predicative equality:

$$\frac{f : A \to B \ \ \mathbf{PR}\text{-map}}{a \doteq a' \implies f(a) \doteq f(a') : A \times A \to \mathbb{N}.}$$

Proof by structural induction on f :

- $f = 0 : \mathbb{1} \to \mathbb{N}$: clear since $0 \doteq 0 : \mathbb{1} \to \mathbb{N} \times \mathbb{N} \xrightarrow{\doteq} \mathbb{N}$.

- $f = s : \mathbb{N} \to \mathbb{N}$: Use $[s\,m \dot{-} s\,n] = [m \dot{-} n]$ and $[a \doteq b] = [a \leq b] \wedge [b \leq a] = \neg[a \dot{-} b] \wedge \neg[b \dot{-} a]$.
- $f = \Pi : A \to \mathbb{1}$: trivial since $\doteq_{\mathbb{1}} = \ \mathrm{true}_{\mathbb{1}\times\mathbb{1}}$.
- $f = \ell : A \times B \to A$:
  $(a, b) \doteq (a', b') \iff [a \doteq a'] \wedge [b \doteq b']$
  $\implies [a \doteq a'] \iff [\ell(a, b) \doteq \ell(a', b')]$:
  $(A \times B) \times (A \times B) \to \mathbb{N}$.
- $f = r : A \times B \to B$: analogous.

  Further **recursively:**
- for a composition $g \circ f : A \to B \to C$:

$$\begin{aligned} a \doteq a' &\implies f\,a \doteq f\,a' \ (\mathit{hypothesis}) \\ &\implies g(f\,a) \doteq g(f\,a') \ (\mathit{hypothesis}) \\ &\iff (g \circ f)(a) \doteq (g \circ f)(a') : A \times A \to \mathbb{N}. \end{aligned}$$

- for an induced $(f, g) : C \to A \times B$:

$$\begin{aligned} c \doteq c' &\implies f(c) \doteq f(c') \wedge g(c) \doteq g(c') \ (\mathit{hypothesis}) \\ &\iff (f(c), g(c)) \doteq (f(c'), g(c')) \\ &\iff (f, g)(c) \doteq (f, g)(c') : C \times C \to \mathbb{N}. \end{aligned}$$

- for an iterated map $f^{\S} : A \times \mathbb{N} \to A$ **to show:**

$$(a, n) \doteq (a', n') \implies f^{\S}(a, n) \doteq f^{\S}(a', n) : (A \times \mathbb{N})^2 \to A.$$

Diagonal induction on $(n, n') \in \mathbb{N} \times \mathbb{N}$:

$(a, 0) \doteq (a', 0) \implies f^{\S}(a, 0) \doteq a \doteq a' \doteq f^{\S}(a', 0)$;

left axis: $(a, 0) \neq (a, s\,\mathrm{pre}(n'))$, premise fails;

right axis: $(0, a') \neq (s\,\mathrm{pre}(n), a')$, premise fails;

diagonal induction step:

$$
\begin{aligned}
&(a, s\,n) \doteq (a', s\,n') \implies a \doteq a' \wedge s\,n \doteq s\,n' \\
&\implies a \doteq a' \wedge n \doteq n' \ (\textit{injectivity of } s) \\
&\implies (a, n) \doteq (a', n') \implies f^{\S}(a, n) \doteq f^{\S}(a', n') \\
&\qquad (\textit{induction hypothesis}) \\
&\implies f^{\S}(a, s\,n) \doteq f(f^{\S}(a, n)) \doteq f(f^{\S}(a', n')) \doteq f^{\S}(a', s\,n') \\
&(\textit{structural recursion hypothesis on } f)
\end{aligned}
$$

q.e.d.

Peano's **axioms** read in categorical free-variables form:[5]

Peano theorem:

- P1: *zero is a natural number:*

  $0 : \mathbb{1} \to \mathbb{N}$ is a map constant of $\mathbb{N}$, a *natural number* as such.

  [Other natural numbers are free variables on $\mathbb{N}$]

- P2: *to any natural number (free variable) n is assigned* a *successor:*

  This *assignment* is realised categorically by *successor* ***map***

[5] see Reiter 1982 as well as Pfender, Kröplin & Pape

$s = s(n) : \mathbb{N} \to \mathbb{N}$.

Such successor $s(n)$ is *unique:*

This is given categorically by Leibniz's substitutivity for the successor map:

$$\mathbf{PR} \vdash \; m \doteq n \implies s(m) \doteq s(n) : \mathbb{N} \times \mathbb{N} \to \mathbb{N}.$$

- P3: 0 *is not a successor:*

  This follows from $sn > 0$, whence $sn \neq 0$, by definition of $m \doteq n$ via $m < n$ via $m \dot{-} n$.

- P4: *equality* $s(m) \doteq s(n)$ *implies* $m \doteq n$:

  This is **derived** *injectivity* of successor map $s : \mathbb{N} \to \mathbb{N}$ which reads in free variables:

  $$\begin{aligned} & s\,m \equiv s(m) \doteq s(n) \equiv s\,n \\ & \implies m \doteq \mathrm{pre}\,s\,m \doteq \mathrm{pre}\,s\,n \doteq n : \\ & \mathbb{N} \times \mathbb{N} \to \mathbb{N}. \end{aligned}$$

- P5: Peano-***induction***, **derived** from *uniqueness* part (pr!) of *full* scheme (pr) of primitive recursion (Freyd):

$$\text{(P5)} \quad \frac{\begin{array}{l} \varphi = \varphi(a,n) : A \times \mathbb{N} \to \mathbb{N} \quad \text{predicate} \\ \varphi(a,0) = \mathrm{true}_A(a) \quad (\textit{anchor}) \\ [\,\varphi(a,n) \implies \varphi(a,s\,n)\,] = \mathrm{true}_{A\times\mathbb{N}} \quad (\textit{induction step}) \end{array}}{\varphi(a,n) = \mathrm{true}_{A\times\mathbb{N}} \qquad (\textit{conclusio}).}$$

Proof of Peano induction principle (P5) from *full scheme* (pr) of primitive recursion:[6]

For scheme (pr!) choose as anchor map

$$g = g(a) = \varphi(a,0) = \mathrm{true}(a) : A \to \mathbb{N}, \text{and as step map}$$
$$h = h((a,n),b) = b \vee \varphi(a, s\,n) : (A \times \mathbb{N}) \times \mathbb{N} \to \mathbb{N}$$

By (pr) we get a unique $f = f(a,n) : A \times \mathbb{N} \to \mathbb{N}$ which satifies

$$f(a,0) = \varphi(a,0) = \mathrm{true}(a) \quad \text{and}$$
$$f(a, s\,n) = h((a,n), f(a,n)) = f(a,n) \vee \varphi(a, s\,n).$$

This works for $f = \mathrm{true} : A \times \mathbb{N} \to \mathbb{N}$ as well as for $f = \varphi$, the latter since

$$\varphi(a,n) \vee \varphi(a, s\,n)$$
$$= (\varphi(a,n) \vee \varphi(a, s\,n)) \wedge (\varphi(a,n) \Rightarrow \varphi(a, s\,n))$$
$$\text{by 2nd hypothesis}$$
$$= \varphi(a, s\,n) \quad \text{by boolean tautology}$$
$$(\alpha \vee \beta) \wedge (\alpha \Rightarrow \beta) = \beta :$$

test with $\beta = 0 \equiv$ false *and* $\beta = 1 \equiv$ true.

q.e.d.

By replacing predicate φ with

$$\psi(a,n) := \bigwedge_{i \le n} \varphi(a,i) : A \times \mathbb{N} \to \mathbb{N}$$

in this **proof** we get

[6] Reiter 1982 and Pfender, Kröplin, Pape 1994

Course of values induction principle:

$$
\text{(P5)}\quad \frac{\begin{array}{l} \varphi = \varphi(a,n) : A \times \mathbb{N} \to \mathbb{N} \quad \text{predicate} \\ \varphi(a,0) = \mathrm{true}_A(a) \qquad (\textit{anchor}) \\ [\bigwedge_{i \le n} \varphi(a,i) \implies \varphi(a, s\,n)\,] = \mathrm{true}_{A \times \mathbb{N}} \quad (\textit{induction step}) \end{array}}{\varphi(a,n) = \mathrm{true}_{A \times \mathbb{N}} \quad (\textit{conclusio}).}
$$

Here predicate $\bigwedge_{i \le n} \varphi(a,i) : A \times \mathbb{N} \to \mathbb{N}$ is p. r. **defined** by

$$
\begin{aligned}
&\bigwedge_{i \le 0} \varphi(a,i) = \varphi(a,0) : A \to \mathbb{N}, \\
&\bigwedge_{i \le s\,n} \varphi(a,i) = \bigwedge_{i \le n} \varphi(a,i) \wedge \varphi(a, s\,n) : A \times \mathbb{N} \to \mathbb{N}.
\end{aligned}
$$

1.9 Integer division and related

Integer division with remainder (Euclide)

$$
(a \div b, a \text{ rem } b) : \mathbb{N} \times \mathbb{N}_> \to \mathbb{N} \times \mathbb{N}
$$

is characterised by

$$
\begin{aligned}
&a \div b = \max\{c \le a \,|\, b \cdot c \le a\} : \mathbb{N} \times \mathbb{N}_> \to \mathbb{N}, \\
&a \text{ rem } b = a \dot{-} (a \div b) \cdot b : \mathbb{N} \times \mathbb{N}_> \to \mathbb{N}.
\end{aligned}
$$

[for $\mathbb{N}_> = \{n \in \mathbb{N} | n > 0\}$ and objects defined by p. r. predicate abstraction in general see next chapter.]

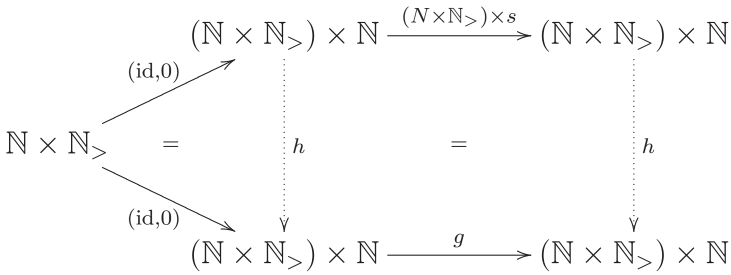
Explicitely, we **define**

$$
\div = a \div b : \mathbb{N} \times \mathbb{N}_> \to \mathbb{N}
$$

via *initialised iteration* $h = h((a,b),n)$ of

$$g = g((a,b),c) = \begin{cases} ((a,b),c) \text{ if } a < b, \\ ((a \dot{-} b, b), c+1) \text{ if } a \geq b \end{cases}$$

in



$$a \div b \ =_{\text{def}} \ r\,h((a,b),a) : \mathbb{N}\times\mathbb{N}_> \to (\mathbb{N}\times\mathbb{N}_>)\mathbb{N} \to \mathbb{N},$$
$$a \,\text{rem}\, b \ =_{\text{def}} \ \ell\ell\, h((a,b),a) = a \dot{-} b \cdot (a \div b) : \mathbb{N}\times\mathbb{N}_> \to \mathbb{N}.$$

The predicate $a|b : \mathbb{N}_> \times \mathbb{N} \to \mathbb{N}$, *a is a divisor of b*, *a divides b* is **defined** by

$$a|b = [\, b \text{ rem } a \doteq 0 \,].$$

Exercise: Construct the Gaussian algorithm for determination of the **gcd** of $a, b \in \mathbb{N}_>$ **defined** as

$$\text{gcd}(a,b) = \max\{c \leq \min(a,b) \mid c|a \wedge c|b\} : \mathbb{N}_> \times \mathbb{N}_> \to \mathbb{N}_>$$

by iteration of mutual rem.

Primes

Define the predicate *is a prime* by

$$\mathbb{P}(p) = \bigwedge_{m=1}^{p} [m|p \Rightarrow m \doteq 1 \vee m \doteq p] : \mathbb{N} \to \mathbb{2} :$$

Only 1 and p divide p.

Write $\mathbb{P}$ for $\{n \in \mathbb{N} | \mathbb{P}(n)\} \subset \mathbb{N}$ too.

The (euclidean) count $p_n : \mathbb{N} \rightarrowtail \mathbb{N}$ of all primes is given by

$$\begin{aligned} p_0 &= 2, \\ p_{n+1} &= \min\{p \in \mathbb{N} | \mathbb{P}(p), p_n < p \leq \prod_q [q \leq p_n \wedge \mathbb{P}(q)]\} + 1 \\ &= \min\{p \in \mathbb{N} | \mathbb{P}(p), p < 2p_n\} : \\ & \mathbb{P} \rightarrowtail \mathbb{P}, \end{aligned}$$

iterated binary product and iterated binary minimum.

The latter presentation is given by BERTRAND's theorem.

Notes

(a) An NNO, within a cartesian Closed category of sets, was first studied by Lawvere 1964.

(b) Eilenberg-Elgot 1970 iteration, here special case of one-successor iteration theory **PR**, is, because of Freyd's uniqueness scheme (FR!), a priori stronger than classical free-variables *primitive recursive arithmetic* **PRA** in the sense of SMORYNSKI 1977. If viewed as a subsystem of **PM**, **ZF** or **NGB**, that **PRA** is stronger than our **PR**.

(c) Within Topoi (with their cartesian closed structure), Freyd 1970 characterised Lawvere's NNO by unique initialised iteration. Such Freyd's NNO has been called later, e.g. in Maietti 2010??, *parametrised NNO*

(d) Lambek-Scott 1986 consider in parallel a *weak NNO:* uniqueness of Lawvere's sequnces $a : \mathbb{N} \to A$ not required. We need here uniqueness (of the initialised iterated) for proof of Goodstein's 1971 uniqueness rules basic for his development of p. r. arithmetic. Without the latter uniqueness requirement, the definition of parametrised (weak) NNO is equational.

(e) For uniqueness of the set of natural numbers (out of the Peano-axioms), classical set theory needs *higher order.* This corresponds in category theory to the use of free meta-variables on *maps.*

In first order classical, elementhood based Peano-arithmetic there are other models of the natural numbers, even uncountable ones. Others than the "standard" (e.g. von Neumann) model.[7]

2 Predicate Abstraction

We extend the fundamental theory **PR** of primitive recursion *definitionally* by predicate abstraction objects $\{A \,|\, \chi\} = \{a \in A \,|\, \chi(a)\}$. We get an (embedding) extension $\mathbf{PR} \sqsubset \mathbf{PRa}$ having all of the expected properties.

[7] This was brought to my attention 2013 in a seminar talk of J. Busse and A. Schlote who quote Barwise ed. 1977 as well as Ebbinghaus et al. 1996 and 2008.

2.1 Extension by predicate abstraction

We discuss a p. r. **abstraction scheme** as a definitional enrichment of **PR**, into theory **PRa** of *PR decidable objects and PR maps in between,* decidable subobjects of the objects of **PR**. The objects of **PR** are, up to isomorphism,

$$\mathbb{1},\ \mathbb{N}^1\ =_{\text{def}}\ \mathbb{N},\ \mathbb{N}^{\underline{m}+1}\ =_{\text{def}}\ (\mathbb{N}^{\underline{m}} \times \mathbb{N}).$$

[$\underline{m}$ is a free metavariable, over the NNO constants $0, 1 = s\,0,\ 2 = s\,s\,0, \ldots \in \underline{\mathbb{N}}$.]

The extension **PRa** is given by adding schemes $(\text{Ext}_{\mathbf{Obj}})$, $(\text{Ext}_{\mathbf{Map}})$, and $(\text{Ext}_{=})$ below. Together they correspond to the *scheme of abstraction* in **set** theory, and they are referred below as *schemes* of *PR abstraction.*

Our first predicate-into-object *abstraction* scheme is

$$(\text{Ext}_{\mathbf{Obj}})\quad \dfrac{\begin{array}{l} \chi : A \to \mathbb{N} \text{ a } \mathbf{PR}\text{-predicate:} \\ \text{sign} \circ \chi = \chi : A \to \mathbb{N} \to \mathbb{N}, \\ Au \xrightarrow{\ \chi\ } \mathbb{N} \xrightarrow{\ \text{sign}\ } \mathbb{N} \quad (= \chi) \end{array}}{\{A \,|\, \chi\} \text{ object (of emerging theory } \mathbf{PRa})}$$

Subobject $\{A \,|\, \chi\} \subseteq A \cong \mathbb{N}^{\underline{n}}$ may be written alternatively, with *bound* variable a, as

$$\{A \,|\, \chi\} = \{a \in A \,|\, \chi(a)\}.$$

$\{A\,|\,\chi\}$ is just another name for the (external) code $\chi \in \mathbf{PR} \subset \underline{\mathbb{N}}$, a NNO constant out of $\underline{\mathbb{N}}$, the external set of natural number constants

$$0,\ 1 \equiv s\,0,\ 2 \equiv s\,s\,0 \text{ etc. } \underline{n} \equiv s\ \ldots\ s\,0 \equiv \mathrm{num}(\underline{n}) \in \underline{\mathbb{N}} \text{ etc.}$$

The *maps* of $\mathbf{PRa} = \mathbf{PR} + (\text{abstr})$ come in by

$$(\mathrm{Ext}_{\mathbf{Map}})\quad \frac{\begin{array}{l}\{A\,|\,\chi\},\ \{B\,|\,\varphi\}\ \mathbf{PRa}\text{-objects,}\\ f : A \to B \text{ a } \mathbf{PR}\text{-map,}\\ \mathbf{PR} \vdash\ \chi(a) \implies \varphi\, f\,(a), \text{ i. e.}\\ [\,\chi \implies \varphi \circ f\,] =^{\mathbf{PR}} \mathrm{true}_A : A \xrightarrow{\Pi} \mathbb{1} \xrightarrow{1} \mathbb{N}\end{array}}{f \text{ is a } \mathbf{PRa}\text{-map } f : \{A\,|\,\chi\} \to \{B\,|\,\varphi\}}$$

In particular, if for predicates $\chi',\ \chi'' : A \to \mathbb{N}$

$\mathbf{PR} \vdash\ \chi'(a) \implies \chi''(a) : A \to \mathbb{N} \times \mathbb{N} \to \mathbb{N}$,
then $\mathrm{id}_A : \{A\,|\,\chi'\} \to \{A\,|\,\chi''\}$ in $\mathbf{PRa}$ is called an *inclusion,*
and written $\subseteq : A' = \{A\,|\,\chi'\} \to A'' = \{A\,|\,\chi''\}$ or $A' \subseteq A''$.

Nota bene: For predicate (terms!) $\chi,\varphi : A \to \mathbb{N}$ such that $\mathbf{PR} \vdash \chi = \varphi : A \to \mathbb{N}$ (logically: such that $\mathbf{PR} \vdash\ [\chi \iff \varphi]$) we have

$$\{A\,|\,\chi\} \subseteq \{A\,|\,\varphi\} \text{ and } \{A\,|\,\varphi\} \subseteq \{A\,|\,\chi\},$$

but—in general—not *equality of objects.* We only get in this case

$$\mathrm{id}_A : \{A\,|\,\chi\} \xrightarrow{\cong} \{A\,|\,\varphi\}$$

as an **PRa** *isomorphism.*

A posteriori, we introduce as Reiter does, the formal *truth Algebra* $\mathbb{2}$ as

$$\mathbb{2} \ =_{\text{def}} \ \{n \in \mathbb{N} \,|\, \chi(n)\}, \text{ where } \chi(n) = [n \leq 1] : \mathbb{N} \to \mathbb{N},$$

with proto Boolean operations on $\mathbb{N}$ restricting—in codomain and domain—to *boolean* operations on $\mathbb{2}$ resp.

$$\mathbb{2} \times \mathbb{2} \ =_{\text{def}} \ \{(m, n) \in \mathbb{N} \times \mathbb{N} \,|\, m, n \leq s\,0\},$$

by definition below of cartesian Product of objects within **PRa**.

PRa-maps with common **PRa** domain and codomain are considered equal, if their values are equal on their defining *domain predicate.* This is expressed by the scheme

$$(\text{Ext}_=) \quad \frac{\begin{array}{l} f,\, g : \{A \,|\, \chi\} \to \{B \,|\, \varphi\} \ \textbf{PRa}\text{-maps}, \\ \textbf{PR} \vdash \ \chi(a) \implies f(a) \doteq_B g(a) \end{array}}{f = g : \{A \,|\, \chi\} \to \{B \,|\, \varphi\},}$$

explicitly:

$$\begin{array}{l} f =^{\textbf{PRa}} g : \{A \,|\, \chi\} \to \{B \,|\, \varphi\}, \text{ also noted} \\ \textbf{PRa} \vdash \ f = g : \{A \,|\, \chi\} \to \{B \,|\, \varphi\}. \end{array}$$

Structure Theorem for the theory **PRa** of *primitive recursion with Predicate Abstraction:*[8]

[8] cf. Reiter 1980

PRa is a cartesian p. r. theory. The theory **PR** is cartesian p. r. embedded. The theory **PRa** has universal extensions of all of its predicates and a boolean truth object as codomain of these predicates, as well as map definition by case distinction. In detail:

(i) **PRa** inherits associative **map composition** and identities from **PR**.

(ii) **PRa** has **PR** fully **embedded** by

$$\langle f : A \to B \rangle \mapsto \langle f : \{A \,|\, \mathrm{true}_A\} \to \{B \,|\, \mathrm{true}_B\} \rangle$$

(iii) **PRa** has **cartesian product**

$$\{A \,|\, \chi\} \times \{B \,|\, \varphi\} \ =_{\mathrm{def}} \ \{A \times B \,|\, \chi \wedge \varphi : A \times B \to \mathbb{N} \times \mathbb{N} \xrightarrow{\wedge} \mathbb{N}\},$$

with *projections* and universal property inherited from **PR**.

We abbreviate $\{A|\mathrm{true}_A\}$ by A.

(iv) object $\mathbb{2}$ comes as a *sum* $\mathbb{1} \xrightarrow[0]{\mathrm{false}} \mathbb{2} \cong \mathbb{1} + \mathbb{1} \xleftarrow[1]{\mathrm{true}} \mathbb{1}$ over which cartesian product $A \times \_$ *distributes.*

This allows in fact for the usual **truth-table definitions** of all *boolean operations* on object $\mathbb{2}$ and for PR map **definition** by **case distinction.**

(v) The embedding $\sqsubset$: **PR** $\longrightarrow$ **PRa** is a **cartesian functor :** it preserves Products and their *cartesian* universal property with respect to the *projections* inherited from **PR**.

(vi) **PRa** has **extensions** of its *predicates,* namely

$$\mathrm{Ext}\,[\,\varphi : \{A \,|\, \chi\} \to \mathbb{2}\,] \;=_{\mathrm{def}}\; \{A \,|\, \chi \wedge \varphi\} \subseteq \{A \,|\, \chi\},$$

characterised as (**PRa**)-*equalisers*

$$\mathrm{Equ}\,(\chi \,\wedge\, \varphi,\ \mathrm{true}_A) : \{A \,|\, \chi\} \to \mathbb{2}$$

[mutatis mutandis: within theory **PRa**, we identify predicates $\chi = \mathrm{sign} \circ \chi : A \to \mathbb{N} \to \mathbb{N}$ with maps $\chi : A \to \mathbb{2}$.]

(vii) **PRa** has *all* **equalisers,** namely equalisers

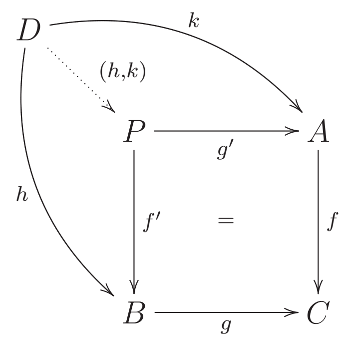
$$\mathrm{Equ}[\,f, g\,] \;=_{\mathrm{def}}\; \{a \in A \,|\, \chi(a) \,\wedge\, f(a) \doteq_B g(a)\}$$

$$= \mathrm{Ext}[\,\doteq_B \circ\, (f, g) : A' \to B' \times B' \xrightarrow{\doteq} \mathbb{2}\,],$$

of arbitrary **PRa** map pairs $f, g : A' = \{A \,|\, \chi\} \to B' = \{B \,|\, \varphi\}$,

and hence all finite projective **limits,** in particular **pullbacks,** which we will rely on later.

A *pullback,* of a map $f : A \to C$ *along* a map $g : B \to C$, also of g along f, is a square in



[I prefer this "set theoretical" way to construct extension sets out of the cartesian category structure of fundamental theory **PR**, and then I construct equalisers and the other finite limits on this basis. Another possibility—Romàn(?)—is to add equalisers as *undefined notion* and to construct directly from these and cartesian product. The relation between (vi) and (vii) is best understood set theoretically: use free variable argument chase, and recall set theoretical definition of an equaliser.]

The embedding **preserves** such limits as far as available already in **PR**. Equality *predicate* extends to cartesian Products componentwise as

$$[\,(a,b) \doteq_{A\times B} (a',b')\,] \;=_{\mathrm{def}}\; [\,a \doteq_A a'\,] \wedge [\,b \doteq_B b'\,] : (A\times B)^2 \to 2,$$

and to (predicative) subobjects $\{A\,|\,\chi\}$ by restriction.

(viii) arithmetical structure extends from **PR** to **PRa**, i. e. **PRa** admits the *iteration* scheme as well as Freyd's *uniqueness* scheme: the iterated

$$f^{\S} : \{A\,|\,\chi\} \times \{\mathbb{N}\,|\,\mathrm{true}_{\mathbb{N}}\} \to \{A\,|\,\chi\}$$

is just the *restricted* **PR**-map $f^{\S} : A \times \mathbb{N} \to A$, the uniqueness schemes follow from **definition** of $=^{\mathbf{PRa}}$ via **PRa**'s scheme $(\mathrm{Ext}_=)$ above.

(ix) In particular, our *equality predicate* $\doteq_A : A^2 \to \mathbb{N}$, restricted to subobjects $A' = \{A\,|\,\chi\} \subseteq A$, inherits all of the properties of equality on $\mathbb{N}$ and the other *fundamental objects.*

(x) **Countability:** Each **fundamental** object A i. e. A a finite power of $\mathbb{N} \equiv \{\mathbb{N} \,|\, \mathrm{true}_{\mathbb{N}}\}$, admits, by CANTOR's isomorphism

$$\mathrm{ct} = \mathrm{ct}_{\mathbb{N}\times\mathbb{N}}(n) : \mathbb{N} \xrightarrow{\cong} \mathbb{N} \times \mathbb{N},$$

a retractive count $\mathrm{ct}_A(n) : \mathbb{N} \to A$.

Problem: For which predicates $\chi : A \to \mathbb{2}$ (A fundamental) does theory **PRa** admit a retractive *count*

$$\mathrm{ct} = \mathrm{ct}_{\{A \,|\, \chi\}}(n) : \mathbb{N} \to \{A \,|\, \chi\}?$$

The difficulty is seen already in case $\emptyset_A \;=_{\text{by def}}\; \{A \,|\, \mathrm{false}_A\}$. A **sufficient condition** is $\{A \,|\, \chi\}$ to come with a *point*, $a_0 : \mathbb{1} \to \{A \,|\, \chi\}$. But there may be non-empty objects without points in suitable theories.

Remarks:

- a **PRa**-map $f : \{A \,|\, \chi\} \to \{B \,|\, \varphi\}$ can be viewed as a *defined partial* **PR** map from A to B with values in φ : the object of *defined arguments,* namely $\{a \in A \,|\, \chi(a)\}$ is p. r. *decidable.* By **definition** of **PRa**'s equality, **PR**-map $f : A \to B$ "doesn't care" about arguments a in the *complement* $\{a \in A \,|\, \neg\, \chi(a)\}$.

  So wouldn't it be easier to realise this view to *defined partial maps* just by throwing the *undefined arguments* into a *waste basket* $\{\bot\}$?

  But where to place this waste basket, this for each codomain object B? The fundamental objects have a zero-vector as a candidate. For example we could interprete truncated subtraction as a *defined partial* map

$$a \dot{-} b : \{(m, n) \in \mathbb{N} \times \mathbb{N} \,|\, m \geq n\} \to \mathbb{N},$$

and throw the complement $\{(m,n) \in \mathbb{N} \times \mathbb{N} \,|\, m < n\}$ into waste basket $\{0\} \subset \mathbb{N}$. But this is not a good interpretation of *truncated* (!) subtraction: Value 0 is *not* waste, it has an important meaning as zero.

"The" waste basket $\{\bot\}$ should be an entity with a *natural* extra representation, and we should have only one such entity in a later theory of defined partial p. r. maps to come. This theory, to be called $\mathbf{PR}\mathbb{X}\mathbf{a}$, will be constructed with the help of a *universal object* $\mathbb{X}$ which is to contain all *numerals* (codes of numbers) and all nested pairs of numerals. It then has place for LaTeXcodes of all symbols, in particular for the code $\underline{\bot}$ of *undefined value* symbol $\underline{\bot}$, in a "Hilbert's hotel".

- a **PR**-map $f : A \to B$ such that f is a **PRa**-map

  $f : \{A \,|\, \chi \vee \chi' : A \to 2\} \to \{B \,|\, \varphi\}$
  also works as a **PRa**-map
  $f : \{A \,|\, \chi\} \to \{B \,|\, \varphi\}$, and a **PRa**-map
  $g : \{A \,|\, \chi\} \to \{B \,|\, \varphi \wedge \varphi'\}$
  also works as a **PRa**-map
  $g : \{A \,|\, \chi\} \to \{B \,|\, \varphi\}$.

Since map-properties of *injectivity, epi-property* of **PR**-maps viewed as **PRa**-maps **depend** on choice of hosting **PRa** objects—**examples** above—*specification* of a **PRa** map $f : \{A \,|\, \chi\} \to \{B \,|\, \varphi\}$ must contain, besides **PR**-map $f : A \to B$, domain and codomain *objects* $\chi : A \to 2$ and $\varphi : B \to 2$ as well.

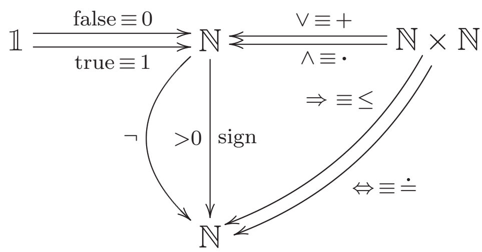
This way the members of map set family $\mathbf{PRa}(A, B)$: A, B $\mathbf{PRa}$-objects, become mutually disjoint. Inclusions $i : A' \xrightarrow{\subseteq} A''$ are realised in $\mathbf{PRa}$ as restricted $\mathbf{PR}$-identities

$\mathrm{id}_A : \{A \,|\, \chi'\} \xrightarrow{\subseteq} \{A \,|\, \chi''\},\ \chi' \implies \chi''.$

2.2 Predicate calculus

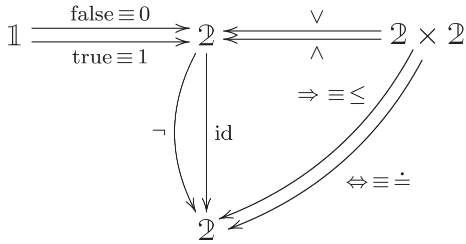
Free Variables Predicate Calculus

In the framework $\mathbf{GA} \subseteq \mathbf{PR} \sqsubset \mathbf{PRa}$ of **Goodstein Arithmetic** we have introduced on NNO $\mathbb{N}$ the following proto Boolean structure:



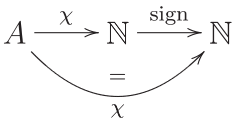
This structure is turned, within $\mathbf{PRa}$, into a two-valued Boolean algebra on object

$$\begin{aligned} \mathbb{2} \ &=_{\text{by def}} \ \{0, 1\} \\ &=_{\text{def}} \ \{n \in \mathbb{N} \,|\, n \doteq 0 \vee n \doteq 1\} \\ &=_{\text{by def}} \ \{n \in \mathbb{N} \,|\, n \leq 1\} : \end{aligned}$$



A **PR** *predicate* on an object A of **PR** has been a **PR** map $\chi : A \to \mathbb{N}$ with

sign $\circ\, \chi = \chi$,



A **PRa** predicate on an object $\{A|\chi\}$ is a **PRa** map $\varphi = \varphi(a) : \{A|\chi\} \to \mathbb{2} = \{0, 1\}$.

Using the Boolean operations on $\mathbb{2}$ above, a *free-Variables boolean predicate calculus* is easily **defined,** making the set of **PR** predicates on (any) object A of **PRa** into a boolean algebra:

- overall negation:

  $\neg\, \varphi(a) = \neg \circ \varphi : A \to \mathbb{2} \to \mathbb{2},$

- conjunction:

  $\chi(a) \wedge \varphi(a) = \wedge \circ (\chi, \varphi) : A \to \mathbb{2}^2 \to \mathbb{2},$

- disjunction:

  $\chi(a) \vee \varphi(a) = \vee \circ (\chi, \varphi) : A \to \mathbb{2}^2 \to \mathbb{2},$

- implication:

  $[\chi(a) \;\Rightarrow\; \varphi(a)] = \;\Rightarrow\; \circ\, (\chi, \varphi) : A \to \mathbb{2}^2 \to \mathbb{2},$

- equivalence:

  $[\chi(a) \Leftrightarrow \varphi(a)] = \dot{=}_{2} \circ (\chi, \varphi) : A \to 2^2 \to 2,$

Verification of the logical properties of such free-variables predicates and their interrelationships by the *truth table method* inherited from the Boolean algebra 2.

Axiomatic Images and Quantification

As a step aside, we discuss here classical quantification, introduced **axiomatically** via image predicates. These correspond to topos theoretic characteristic functions of non-necessarily monic (injective) maps. quantification + cartesian PR allows for the original version of Gödel's theorems. It seems to be necessary for that original theorems and proof, since existential quantification plays a prominent rôle in statement and proof. Nevertheless, Incompleteness can be shown in a different way for weaker theories, cf. Goodstein 1957. We do not exclude that **PR**, **PRa** turn out to be incomplete in Goodstein's sense.

Definition: A (total) predicate $\chi : B \to 2$ is a (the) *image predicate* of a map $f = f(a) : A \to B$, if

- $\chi \circ f = \mathrm{true}_A : A \to B \to 2$ and
- $\chi : B \to 2$ *minimal* in this regard i. e.

$$\frac{\varphi \circ f = \mathrm{true}_A : A \to B \to 2}{[\chi(b) \Rightarrow \varphi(b)] = \mathrm{true}_B}$$

If available, such χ, noted $\mathrm{im}[f] = \mathrm{im}[f](b) : B \to \mathbb{2}$, is unique, this by minimality and Equality Definability.

In case of $f : A \to B$ monic, such χ is just the characteristic map of f in the sense of Elementary Topos theory **ETT,** with respect to $\mathbb{2} = \{0, 1\} \subset \mathbb{N}$ taken as its *subobject classifier, truth object.*

If available, image

$$\mathrm{im}[\{A \times B|\varphi\} \xrightarrow{\subseteq} A \times B \xrightarrow{\ell} A] : A \to \mathbb{2}$$

works as *right existential quantification*

$$(\exists b \in B)\varphi(a, b) = (\exists_r \varphi)(a) : A \to \mathbb{2},$$

with the categorical properties of this quantification known from (**ETT** and categorical) **set** theory.

If available, **define** *right universal quantification*

$$(\forall b \in B)\varphi(a, b) \ =_{\mathrm{def}} \ \neg\,(\exists b \in B)\,\neg\varphi(a, b) : A \to \mathbb{2}.$$

Our (weak, categorical) **set** theories **T** will here always be **Extensions** of quantified p. r. theory **PRa**$\exists$ = **PRa** + ($\exists$), **defined** to be theory **PRa** closed under formation of images and hence closed under (two-valued) quantification $\exists, \forall$.

Comment: These *semi-classical* theories will be taken as *background* for Consistency questions: we will show differences in internal consistency between these classical **set** theories **T**, in particular between Osius' categorical pendants of the different stages of Zermelo-Fraenkel **set** theory **ZF** on one hand, and the categorical theories here: $\mathbf{PR}, \mathbf{PRa}$ above, and $\mathbf{PR}\mathbb{X}, \mathbf{PR}\mathbb{X}\mathbf{a}, \pi\mathbf{R}$ to come. For fixing ideas, you may always read **set** theory $\mathbf{T}$ as $\mathbf{T} := \mathbf{PRa}\exists$: Gödel's Incompleteness theorems apply to $\mathbf{PRa}\exists$, not to *descent* p. r. theory $\pi\mathbf{R}$ to come.

Notes

(a) we have equalisers, products distributing over sums, sums certainly stable under pullbacks, quotients by equivalence predicates (not yet quotients by equivalence relations).

(b) in comparison with doctrines: Kock-Reyes 1977, and in comparison with pretopoi: Maietti 2010??, (axiomatic) quantification is lacking for "our" strengthenings $\mathbf{S}$ of $\mathbf{PRa}$.

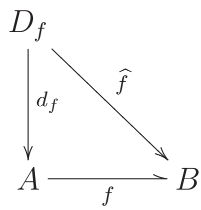
3 Partial Maps

We introduce general recursive maps as *partial p. r. maps,* coming as a p. r. enumeration of *defined arguments* together with a p. r. *rule* mapping the enumeration index of a defined argument into the *value* of that argument. This covers μ-recursive maps and content driven loops as in particular while-loops. Code evaluation will be definable as such a while-loop.

3.1 Theory of partial maps

Definition: A partial map $f : A \rightharpoonup B$ is a pair

$$f = \langle d_f : D_f \to A,\ \widehat{f} : D_f \to B \rangle : A \rightharpoonup B,$$



The pair $f = \langle d_f\,,\, \widehat{f} \rangle$ is to fullfill the **right-uniqueness condition**

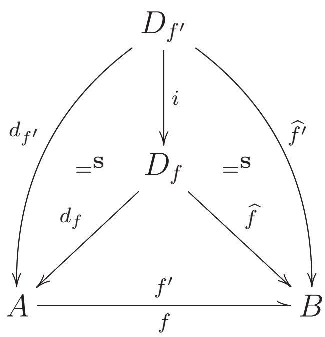
$$d_f(\hat{a}) \doteq_A d_f(\hat{a}') \implies \widehat{f}(\hat{a}) \doteq_B \widehat{f}(\hat{a}')\,:$$

We now **define** the **theory** $\widehat{\mathbf{S}}$ of *partial* **S**-maps $f : A \rightharpoonup B$.

Objects of $\widehat{\mathbf{S}}$ are those of **S**, i. e. of **PRa**. The *morphisms* of $\widehat{\mathbf{S}}$ are the *partial* **S**-maps $f : A \rightharpoonup B$.

Definition: Given $f', f : A \rightharpoonup B$ in $\widehat{\mathbf{S}}$, we say that f *extends* f' or that f' is a *restriction* of f, written $f' \,\widehat{\subseteq}\, f$, if there is given a map $i : D_{f'} \to D_f$ in **S** such that

$(f' \widehat{\subseteq} f)$



The partial maps f and f' are *equal* in $\widehat{\mathbf{S}}$, if f extends f' and f'

extends f :

$$(\mathrel{\widehat{=}}^{\mathbf{S}}) \quad \frac{f' \mathrel{\widehat{\subseteq}} f, \ f \mathrel{\widehat{\subseteq}} f' : A \rightharpoonup B}{f' \mathrel{\widehat{=}} f : A \rightharpoonup B.}$$

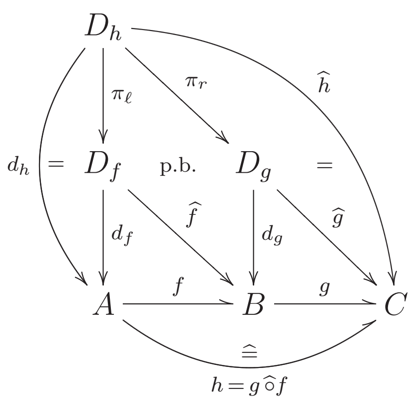
Notation: From now on, $f = g : A \to B$ will always denote equality between maps within theory $\mathbf{S}$ choosen as *basic,* cartesian p. r. theory. Equality between *partial* $\mathbf{S}$-maps, $\widehat{\mathbf{S}}$-morphisms $f, \ g : A \rightharpoonup B$ is denoted $f \mathrel{\widehat{=}} g : A \rightharpoonup B$, see the above. Pointed equality $\doteq : \mathbb{N}^2 \to \mathbb{2}$ resp. $\doteq_A : A^2 \to \mathbb{2}$ is reserved for equality *predicates* (special *maps*), *on* $\mathbb{N}$ resp. *on* objects A of $\mathbf{S}$.

Definition: *Composition* $h = g \mathbin{\widehat{\circ}} f : A \rightharpoonup B \rightharpoonup C$ of $\widehat{\mathbf{S}}$ maps

$f = \langle (d_f, \widehat{f}) : D_f \to A \times B \rangle : A \rightharpoonup B$ and

$g = \langle (d_g, \widehat{g}) : D_g \to B \times C \rangle : B \rightharpoonup C$

is **defined** by the diagram



Composition DIAGRAM for $\widehat{\mathbf{S}}$

[The idea is from BRINKMANN-PUPPE 1969: They construct composition of *relations* this way via pullback]

Remark: The *standard form* of the pullback D_h is

$$D_h = \{(\hat{a}, \hat{b}) \in D_f \times D_g \,|\, \widehat{f}(\hat{a}) \doteq_B d_g(\hat{b})\},$$

with pullback-*projections*

$$\ell = \pi_\ell = \ell \circ \subseteq : D_h \to D_f \times D_g \to D_f \text{ and}$$
$$r = \pi_r = r \circ \subseteq : D_h \to D_f \times D_g \to D_g.$$

[We may abbreviate such *restricted* projections—*pullback "projections"*— π_ℓ and π_r respectively, by ℓ, r—as suggested above]

In a sense, the pullback D_h represents the inverse image $D_h = \overset{-1}{f}[\,D_g\,]$, more precisely: $[\,D_h \xrightarrow{\ell} D_f\,] = \overset{-1}{\widehat{f}}[\,D_g \xrightarrow{d_g} B\,]$. But the definability domains d_f, d_g, d_h need not be monic (injective).

Composition $h = g \,\widehat{\circ}\, f : A \rightharpoonup B \rightharpoonup C$ gives a *well-defined* partial map h, since for $(\hat{a}, \hat{b}), (\hat{a}', \hat{b}') \in D_h$ free:

$$\begin{aligned}
& d_h(\hat{a}, \hat{b}) \doteq_A d_h(\hat{a}', \hat{b}') \iff d_f(\hat{a}) \doteq_A d_f(\hat{a}') \\
& \implies \widehat{f}(\hat{a}) \doteq_B \widehat{f}(\hat{a}') \ (f \text{ well-defined}), \\
& \iff \widehat{f}\,\ell(\hat{a}, \hat{b}) \doteq \widehat{f}\,\ell(\hat{a}', \hat{b}') \\
& \implies d_g(r(\hat{a}, \hat{b})) \doteq_B d_g(r(\hat{a}', \hat{b}')) \\
& \qquad (\ (\hat{a}, \hat{b}), (\hat{a}', \hat{b}') \in D_h, \ \text{p.b. commutes}) \\
& \iff d_g(\hat{b}) \doteq_B d_g(\hat{b}') \implies \widehat{g}(\hat{b}) \doteq_C \widehat{g}(\hat{b}') \\
& \implies \widehat{h}(\hat{a}, \hat{b}) = \widehat{g}(\hat{b}) \doteq_C \widehat{g}(\hat{b}') = \widehat{h}(\hat{a}', \hat{b}') : D_h \times D_h \to \mathbb{2}.
\end{aligned}$$

Obviously, $\widehat{\mathbf{S}}$-map $\mathrm{id}_A^{\widehat{\mathbf{S}}} =_{\mathrm{def}} \langle (\mathrm{id}_A, \mathrm{id}_A) : A \to A^2 \rangle : A \rightharpoonup A$ works as *identity* for object A with respect to composition $\widehat{\circ}$ for $\widehat{\mathbf{S}}$.

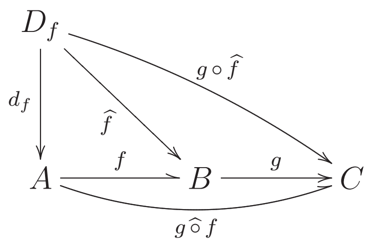
If one of two $\widehat{\mathbf{S}}$ maps to be composed, is an $\mathbf{S}$ map, $\widehat{\mathbf{S}}$ *composition* becomes simpler:

Mixed Composition Lemma:

(i) For $f : A \rightharpoonup B$ in $\widehat{\mathbf{S}}$, and $g : B \to C$ in $\mathbf{S}$:

$$g \,\widehat{\circ}\, f = \langle (d_f, g \circ \widehat{f}) : D_f \to A \times C \rangle : A \rightharpoonup C,$$

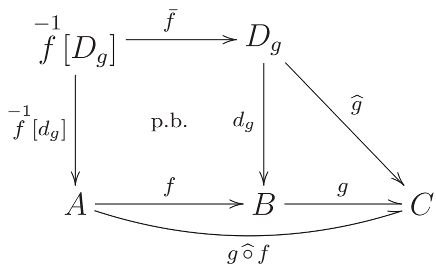
in DIAGRAM form:



(ii) For $f : A \to B$ in $\mathbf{S}$, $g : B \rightharpoonup C$ in $\widehat{\mathbf{S}}$:

$$g \,\widehat{\circ}\, f = \langle (\overset{-1}{f}[d_g], \widehat{g} \circ \bar{f}) : \overset{-1}{f}[D_g] \to A \times C \rangle : A \rightharpoonup C,$$

as DIAGRAM:

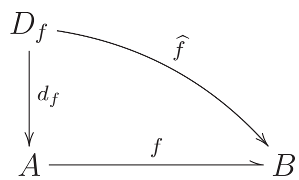


Proof: Left as an **exercise.**

3.2 Structure theorem for $\mathbf{P\widehat{R}a}$:

(i) $\widehat{\mathbf{S}}$ carries a canonical structure of a *diagonal symmetric monoidal category*, with composition $\widehat{\circ}$ and identities introduced above, monoidal product $\times$ extending $\times$ of $\mathbf{S}$, *association* ass : $(A \times B) \times C \xrightarrow{\cong} A \times (B \times C)$, *symmetry* $\Theta : A \times B \xrightarrow{\cong} B \times A$, and *diagonal* $\Delta : A \to A \times A$ inherited from $\mathbf{S}$.

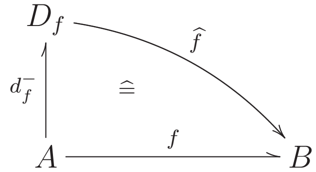
(ii) The **defining** diagram for a $\widehat{\mathbf{S}}$-map—namely



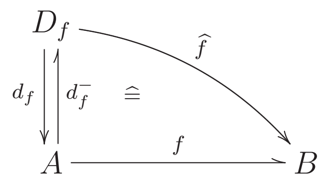
Partial Map DIAGRAM

is a **commuting** $\widehat{\mathbf{S}}$ diagram.

Conversely the *minimised opposite* $\widehat{\mathbf{S}}$ map $d_f^- : A \rightharpoonup D_f$ to $\mathbf{S}$ map $d_f : D_f \to A$ fullfills



Put together:



basic partial map DIAGRAM

(iii) **"section lemma:"** The first factor $f : A \rightharpoonup B$ in an $\widehat{\mathbf{S}}$ composition

$$h = g \,\widehat{\circ}\, f : A \rightharpoonup B \rightharpoonup C,$$

when giving an (embedded) $\mathbf{S}$ map $h : A \to C$, is itself an (embedded) $\mathbf{S}$ map:

a first composition factor of a total map is total.

So each **section** ("coretraction") of theory $\widehat{\mathbf{S}}$ is an $\mathbf{S}$ map, in particular an $\widehat{\mathbf{S}}$ section of an $\mathbf{S}$ map belongs to $\mathbf{S}$.

[We will **rely** on this **lemma** below.]

3.3 Equality definability for partials

Not needed for the Gödel discussion.

3.4 Partial-map extension as closure

Not neeeded for the discussion of the Gödel theorems.

3.5 μ-recursion without quantifiers

We **define** μ-recursion within the free-variables framework of **partial p. r. maps** as follows:

Given a **PR** predicate $\varphi = \varphi(a, n) : A \times \mathbb{N} \to \mathbb{2}$, the $\widehat{\mathbf{S}}$ morphism

$$\mu\varphi = \langle (d_{\mu\varphi}, \widehat{\mu}\varphi) : D_{\mu\varphi} \to A \times \mathbb{N} \rangle : A \rightharpoonup \mathbb{N}$$

is to have ($\mathbf{S}$) **components**

$$\begin{aligned}
&D_{\mu\varphi} \ =_{\text{def}} \ \{A \times \mathbb{N} \,|\, \varphi\} \subseteq A \times \mathbb{N}, \\
&d_{\mu\varphi} = d_{\mu\varphi}(a, n) \ =_{\text{def}} \ a = \ell \circ \subseteq : \\
&\qquad \{A \times \mathbb{N} \,|\, \varphi\} \xrightarrow{\subseteq} A \times \mathbb{N} \xrightarrow{\ell} A, \text{ and} \\
&\widehat{\mu}\varphi = \widehat{\mu}\varphi(a, n) \ =_{\text{def}} \ \min\{m \leq n \,|\, \varphi(a, m)\} : \\
&\qquad \{A \times \mathbb{N} \,|\, \varphi\} \subseteq A \times \mathbb{N} \to \mathbb{N}.
\end{aligned}$$

Comment: This **definition** of $\mu\varphi : A \rightharpoonup \mathbb{N}$ is a *static* one, by enumeration $(\ell, \widehat{\mu}\varphi) : \{A \times \mathbb{N} \,|\, \varphi\} \to A \times \mathbb{N}$ of its *graph,* as is the case in general here for *partial* p. r. maps: We start with *given* pairs in enumeration domain $\{A \times \mathbb{N} \,|\, \varphi\}$, and get *defined arguments a* "only" as $d_{\mu\varphi}$-*enumerated* "elements" (*dependent variable*) $a = d_{\mu\varphi}(\widehat{(a, n)}) = d_{\mu\varphi}(a, n)$, $\widehat{(a, n)} = (a, n)$ "already known" to lie in $D_{\mu\varphi} = \{A \times \mathbb{N} \,|\, \varphi\}$: No need—and in general no "direct" possibility—to *decide,* for a given $a \in A$, **if** a is of form $a = d_{\mu\varphi}(a, n)$ with $(a, n) \in D_{\mu\varphi}$, i. e. if *Exists* $n \in \mathbb{N}$ such that $\varphi(a, n)$. In particular, if $D_{\mu\varphi} = \{A \times \mathbb{N} \,|\, \varphi\} = \emptyset_{A \times \mathbb{N}}$, then $d_{\mu\varphi}$ as well as $\widehat{\mu}\varphi$ are empty maps.

μ-Lemma: $\widehat{\mathbf{S}}$ admits the following (free-variables) scheme (μ) combined with ($\mu!$)—*uniqueness*—as a **characterisation** of the μ-operator $\langle \varphi : A \times \mathbb{N} \to \mathbb{2} \rangle \mapsto \langle \mu\varphi : A \rightharpoonup \mathbb{N} \rangle$ above:

$$(\mu)\quad \frac{\varphi = \varphi(a,n) : A \times \mathbb{N} \to 2\!\!2 \ \mathbf{S}-\text{map (“predicate”)},}{\mu\varphi = \langle (d_{\mu\varphi}\,,\ \widehat{\mu}\varphi) : D_{\mu\varphi} \to A \times \mathbb{N}\rangle : A \rightharpoonup \mathbb{N}}$$

is an $\widehat{\mathbf{S}}$-map such that

$$\mathbf{S} \vdash\ \varphi(d_{\mu\varphi}(\hat{a}), \widehat{\mu}\varphi(\hat{a})) = \mathrm{true}_{D_{\mu\varphi}} : D_{\mu\varphi} \to 2\!\!2,$$

$+$ “argumentwise” **minimality:**

$$\mathbf{S} \vdash\ [\,\varphi(d_{\mu\varphi}(\hat{a}), n) \implies \widehat{\mu}\varphi(\hat{a}) \leq n\,] : D_{\mu\varphi} \times \mathbb{N} \to 2\!\!2$$

as well as **uniqueness**—by *maximal extension*:

$$(\mu!)\quad \frac{\begin{array}{l} f = f(a) : A \rightharpoonup \mathbb{N} \text{ in } \widehat{\mathbf{S}} \text{ such that} \\ \mathbf{S} \vdash\ \varphi(d_f(\hat{a}), \widehat{f}(\hat{a})) = \mathrm{true}_{D_f} : D_f \to 2\!\!2, \\ \mathbf{S} \vdash\ \varphi(d_f(\hat{a}), n) \implies \widehat{f}(\hat{a}) \leq n : D_f \times \mathbb{N} \to 2\!\!2 \end{array}}{\mathbf{S} \vdash\ f \mathrel{\widehat{\subseteq}} \mu\varphi : A \rightharpoonup \mathbb{N} \ \ \text{(inclusion of graphs)}}$$

[Requiring this maximality of $\mu\varphi$ is *necessary*, since—for example—(μ) alone is fulfilled already by the *empty* partial function $\emptyset : A \rightharpoonup \mathbb{N}$]

3.6 Content driven loops

By a *content driven* loop we mean an *iteration* of a given *step endo map,* whose number of performed steps is not known at *entry time* into the *loop*—as is the case for a PR iteration $f^{\S}(a,n) : A \times \mathbb{N} \to A$

with *iteration number* $n \in \mathbb{N}$—, but whose (re) entry into a "new" endo step $f : A \to A$ depends on *content* $a \in A$ reached so far:

This *(re) entry* or *exit* from the loop is now *controlled* by a *(control) predicate* $\chi = \chi(a) : A \to 2$.

First example: a <u>while</u> loop $\mathrm{wh}\,[\,\chi\,|\,f\,] : A \rightharpoonup A$, for given p. r. *control* predicate $\chi = \chi(a) : A \to 2$, and *(looping) step* endo $f : A \to A$, both in **S**, both **S**-maps for the time being, **S** as always in our present context an extension of **PRa**, admitting the scheme of (predicate) *abstraction.* Examples for the moment: $\mathbf{PRa} = \mathbf{PR} +$ (abstr) itself, Universe theory $\mathbf{PR}\mathbb{X}\mathbf{a}$ as well as $\mathbf{PA} \restriction \mathrm{PR}$, restriction of **PA** to its p. r. terms, with inheritance of all **PA**-equations for this term-restriction.

Classically, *with* variables, such $\mathrm{wh} = \mathrm{wh}\,[\,\chi\,|\,f\,]$ would be "defined"—in *pseudocode*—by

$$\begin{aligned} \mathrm{wh}(a) := [\, & a' := a; \\ & \underline{\text{while}}\ \chi(a')\ \underline{\text{do}}\ a' := f(a')\ \underline{\text{od}}; \\ & \mathrm{wh}(a) := a' \,]. \end{aligned}$$

The formal version of this—within a *classical,* element based setting—, is the following partial-(Peano)-map characterisation:

$$\mathrm{wh}(a) = \mathrm{wh}\,[\,\chi\,|\,f\,]\,(a) = \begin{cases} a \text{ if } \neg\,\chi(a) \\ \mathrm{wh}(f(a)) \text{ if } \chi(a) \end{cases} : A \rightharpoonup A.$$

But can this *dynamical, bottom up* "definition" be converted into a p. r. *enumeration* of a suitable *graph* "of all *argument-value pairs*" in terms of an $\widehat{\mathbf{S}}$-morphism

$$\mathrm{wh} = \mathrm{wh}\,[\,\chi\,|\,f\,] = \langle (d_{\mathrm{wh}}, \widehat{\mathrm{wh}}) : D_{\mathrm{wh}} \to A \times A \rangle : A \rightharpoonup A?$$

In fact, we can give such *suitable,* static **Definition** of $\mathrm{wh} = \mathrm{wh}\,[\,\chi\,|\,f\,]$: $A \rightharpoonup A$—within $\widehat{\mathbf{S}} \sqsupset \mathbf{S}$—as follows:

$$\begin{aligned}
&\mathrm{wh} \ =_{\mathrm{def}} \ f^{\S}\,\widehat{\circ}\,(\mathrm{id}_A, \mu\,\varphi_{[\,\chi\,|f\,]}) \\
&\quad =_{\text{by def}} \ f^{\S}\,\widehat{\circ}\,(A \times \mu\,\varphi_{[\,\chi\,|\,f\,]})\,\widehat{\circ}\,\Delta_A : \\
&A \to A \times A \rightharpoonup A \times \mathbb{N} \to A, \ \text{where} \\
&\varphi = \varphi_{[\,\chi\,|\,f\,]}(a,n) \ =_{\mathrm{def}} \ \neg\,\chi\,f^{\S}(a,n) : A \times \mathbb{N} \to A \to \mathbb{2} \to \mathbb{2}.
\end{aligned}$$

Within a quantified arithmetical theory like **PA**, this $\widehat{\mathbf{S}}$**-Definition** of $\mathrm{wh}\,[\,\chi\,|\,f\,] : A \rightharpoonup A$ fullfills the classical **characterisation** quoted above, as is readily shown by Peano-Induction "on" $n := \mu\,\varphi_{[\,\chi\,|\,f\,]}\,(a) : A \rightharpoonup \mathbb{N}$, at least within **PA** and its extensions.

[Classically, *partial definedness* of this—*dependent*—induction parameter n causes no problem: use a *case distinction* on definedness of $\mu\,\varphi_{\chi,f}(a)$"$\in$"$\mathbb{N}$. Even in our quantifier-free context such *dependent induction* on a *partial* dependent induction parameter will be available, see below]

In this generalised sense, we have—within theories $\widehat{\mathbf{S}} \sqsupset \mathbf{S}$—all <u>while</u> loops, for the time being at least those with *control* $\chi : A \to \mathbb{2}$ and *step* endo $f : A \to A$ within **S**.

It is obvious that such $\mathrm{wh}\,[\,\chi\,|\,f\,] : A \times A$ is in general "only" *partial*—as is trivially exemplified by integer division by *divisor* 0, which would be endlessly subtracted from the dividend, although in this case *control* and *step* are both PR.

4 Universal Sets and Universe Theories

4.1 Strings as polynomials

Strings $a_0\, a_1\, \ldots\, a_n$ of natural numbers (in set $\mathbb{N}^+ = \mathbb{N}^* \setminus \{\Box\}$ of non-empty strings) are coded as *prime power products*

$$2^{a_0} \cdot 3^{a_1} \cdot \ldots \cdot p_n^{a_n} \in \mathbb{N}_{>0} \subset \mathbb{N},\ p_j \text{ the } j\,\text{th prime number.}$$

Formally: euclidean prime power factorisation gives rise to a p. r. *projection* family

$$\pi = \pi_j(a) : \mathbb{N} \times \mathbb{N}_> \to \mathbb{N},\ a = p_0^{\pi_0(a)} \cdot p_1^{\pi_1(a)} \cdot \ldots \cdot p_a^{\pi_a(a)},$$

unique $\pi_j(a)$, $\pi_j(a) = 0$ for $j > n,\ n = n(a) : \mathbb{N}_> \to \mathbb{N}$ suitable p. r.

Strings $a_0\, a_1\, \ldots\, a_n \equiv p_0^{a_0} \cdot \ldots \cdot p_n^{a_n}$ are identified with (the coefficient lists of) "their" *polynomials*

$$p(X) = a_0 + a_1 X^1 + \ldots + a_n X^n \text{ as well as}$$
$$p(\omega) = a_0 + a_1 \omega^1 + \ldots + a_n \omega^n,$$

in *indeterminate* X resp. ω.

Componentwise addition (and truncated subtraction), as well as

$$p(\omega) \cdot \omega = \sum_{j=0}^{n} a_j \omega^{j+1} \equiv \prod_{j=0}^{n} p_{j+1}^{a_j},$$

special case of Cauchy product of polynomials.

Lexicographical **Order** of NNO strings and polynomials has—intuitively, and formally within **sets**—only *finite descending chains.*

This applies in particular to descending complexities of CCI's: *Complexity Controlled Iterations* below, with complexity values in $\mathbb{N}[\omega]$; p. r. map code *evaluation* will be resolved into such a CCI.

4.2 Universal object $\mathbb{X}$ of numerals and nested pairs

We begin the construction of Universal object by internal *numeralisation* of all objective natural numbers, of objective numerals

$$\begin{aligned}
&\mathrm{num}(0) \equiv 0 : \mathbb{1} \to \mathbb{N}, \\
&\mathrm{num}(1) \equiv 1 \ =_{\mathrm{def}} \ (s(0)) : \mathbb{1} \to \mathbb{N} \to \mathbb{N}, \\
&\mathrm{num}(2) \equiv 2 \ =_{\mathrm{def}} \ (s(s(0)) : \mathbb{1} \to \mathbb{N} \\
&\mathrm{num}(\underline{n}+1) \equiv \underline{n}+1 \ =_{\mathrm{def}} \ (s(\underline{n})) : \mathbb{1} \to \mathbb{N}, \\
&\underline{n} \in \underline{\mathbb{N}} \text{ meta-variable.}
\end{aligned}$$

Internal numerals, *numeralisation*

$$\nu = \nu(n) : \mathbb{N} \to \mathbb{N}^+ \equiv \mathbb{N}^* \setminus \{0\} \equiv \mathbb{N}_> \subset \mathbb{N} :$$

$$\begin{aligned}
&\nu(0) \ =_{\mathrm{def}} \ \ulcorner 0 \urcorner \ : \mathbb{1} \to \mathbb{N} \text{ code (\textit{goedel number}) of } 0, \\
&\nu(1) \ =_{\mathrm{def}} \ \langle \ulcorner s \urcorner \odot \nu(0) \rangle \ =_{\text{by def}} \ \langle \ulcorner s \urcorner \ \ulcorner \circ \urcorner \ \ulcorner 0 \urcorner \rangle : \mathbb{1} \to \mathbb{N},
\end{aligned}$$

abbreviation for (string) goedelisation, here in particular for LaTeXsource code

$$\begin{aligned}
&\ulcorner (\urcorner \ \ulcorner s \urcorner \ \ulcorner \circ \urcorner \nu(0) \ \ulcorner) \urcorner \ = \ \ulcorner (\urcorner \ \ulcorner s \urcorner \ \ulcorner \circ \urcorner \ \ulcorner 0 \urcorner \ \ulcorner) \urcorner \\
&\equiv p_0^{\mathbf{ASCII}[(]} \ p_1^{\mathbf{ASCII}[s]} \ p_2^{\mathbf{ASCII}[\backslash\mathrm{circ}]} \ p_3^{\mathbf{ASCII}[0]} \ p_4^{\mathbf{ASCII}[)]} \\
&\equiv 2^{40} \ 3^{115} \ 5^{\mathbf{ASCII}[\backslash\mathrm{circ}]} \ 7^{48} \ 11^{41} : \mathbb{1} \to \mathbb{N},
\end{aligned}$$

an element of $\underline{\mathbb{N}}$, a *constant* of $\mathbb{N}$,

$$\begin{aligned}
&\nu(2) \ =_{\mathrm{def}} \ \langle \ulcorner s \urcorner \odot \nu(1) \rangle = \langle \ulcorner s \urcorner \odot \langle \ulcorner s \urcorner \odot \nu(0) \rangle \rangle \quad \text{etc. PR:} \\
&\nu(n+1) \ =_{\mathrm{def}} \ \langle \ulcorner s \urcorner \odot \nu(n) \rangle \in \mathbb{N}.
\end{aligned}$$

$\nu(n)$ has n closing brackets (at end).

This internal numeralisation distributes the "elements", numbers of the NNO $\mathbb{N}$, with suitable gaps over $\mathbb{N}$: the gaps then will receive in particular codes of any other symbols of object Languages **PR** and **PRa** as well as of Universe Languages **PR**$\mathbb{X}$ and **PR**$\mathbb{X}$**a** to come.

ν-**Predicate lemma:** Enumeration $\nu : \mathbb{N} \to \mathbb{N}$ defines a characteristic predicate $\mathrm{im}[\nu] = \chi_\nu : \mathbb{N} \to \mathbb{2}$, and by this **object**

$$\nu\mathbb{N} = \{\mathbb{N}|\chi_\nu\} \subset \mathbb{N}^+$$

of internal numerals $\nu\mathbb{N} \cong \mathbb{N}$.

Proof: Use finite $\exists$—iterative '$\vee$'—for definition of $\mathrm{im}[\nu]$, as follows:

$$\begin{aligned} \chi_\nu(c) \ &=_{\mathrm{def}} \ \vee_{n \le c}[c \doteq \nu(n)] \\ &= [c \doteq \nu(0) \ \vee \ c \doteq \nu(1) \ \vee \ \ldots \ \vee \ c \doteq \nu(c)] : \mathbb{N} \to \mathbb{2} \quad \textbf{q.e.d.} \end{aligned}$$

$\nu : \mathbb{N} \to \mathbb{N}^+ \subset \mathbb{N}$ has codomain restriction

$$\nu : \mathbb{N} \to \nu\mathbb{N} \ =_{\mathrm{def}} \ \{\mathbb{N}|\chi_\nu\}$$

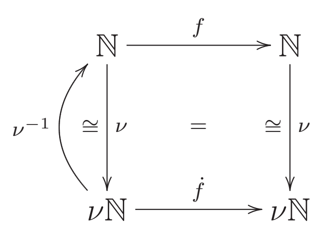
and is then an iso with p. r. inverse

$$\nu^{-1} = \nu^{-1}(c) \ =_{\mathrm{def}} \ \min_{n \le c}[\nu(n) \doteq c] : \nu\mathbb{N} \xrightarrow{\cong} \mathbb{N}.$$

For a **PR**-map $f : \mathbb{N} \to \mathbb{N}$ **define** its *numeral twin*

$$\dot{f} \ =_{\mathrm{def}} \ \nu \circ f \circ \nu^{-1} : \nu\mathbb{N} \xrightarrow{\nu^{-1}} \mathbb{N} \xrightarrow{f} \mathbb{N} \xrightarrow{\nu} \nu\mathbb{N},$$

giving trivially (local) *naturality*



Extension of numeral sets and numeralisation to all **objects** of **PR** (and of **PRa** :)

- $\nu\mathbb{1} = \{\nu 0\} = \{\ulcorner 0 \urcorner\} \subset \nu\mathbb{N} \subset \mathbb{N}$,
  $\nu_{\mathbb{1}}(0) = \nu(0) : \mathbb{1} \xrightarrow{\cong} \nu\mathbb{1} \xrightarrow{\subset} \nu\mathbb{N}$.
- recursive extension to products:

$$\frac{A, B \text{ in } \mathbf{PR}}{\begin{array}{l} \nu(A \times B) = \langle \nu A \dot{\times} \nu B \rangle \\ \ =_{\text{def}} \ \{\langle \nu A(a); \nu B(b) \rangle \mid a \in A, b \in B\} \\ \text{predicatively} \\ = \{\langle c; d \rangle \in \mathbb{N} \mid \chi_{\nu A}(c) \wedge \chi_{\nu B}(d)\}. \end{array}}$$

- Extension to (predicative) subsets:

$$\frac{\chi = \chi(a) : A \to \mathbb{N} \text{ predicate}}{\nu\{A|\chi\} \ =_{\text{def}} \ \{\nu(a) \mid a \in \{A|\chi\}\} \subseteq \nu A}$$

- **remark:** $\mathbb{X}$, $\nu\mathbb{X} \subset \mathbb{N}$, $\nu\mathbb{X} \cong \mathbb{X}$, but $\nu X \subsetneqq \mathbb{X}$, parallel to $\nu\mathbb{N} \subsetneqq \mathbb{N}$.
- ν isomorphy (and *naturality*) extend to A, B in **PR** and in **PRa**.

Universal objects $\mathbb{X}$, $\mathbb{X}_{\underline{\perp}}$ of numerals and (nested) pairs of numerals:

As code for *waste symbol* we take
$\underline{\perp} =_{\text{def}} \ulcorner \perp \urcorner \equiv \ulcorner \backslash\text{bot} \urcorner : \mathbb{1} \to \mathbb{N}$.

Define sets

$$\mathbb{X}, \mathbb{X}_{\underline{\perp}} = \{\mathbb{N} \,|\, \mathbb{X}, \mathbb{X}_{\underline{\perp}} : \mathbb{N} \to \mathbb{2}\} \subset \mathbb{N}$$

of all (codes of)

- *undefined value* $\underline{\perp}$,
- *numerals* $\nu(n) \in \nu\mathbb{N}$, and
- (possibly nested) *pairs*
  $\langle x; y \rangle =_{\text{by def}} \ulcorner (\urcorner \, x \, \ulcorner , \urcorner \, y \, \ulcorner) \urcorner$ of numerals

as follows:

- $\nu\mathbb{N} \subset \mathbb{X} \subset \mathbb{N}$, *numerals proper;* further recursively enumerated:
- $\langle \mathbb{X} \dot{\times} \mathbb{X} \rangle =_{\text{def}} \{\langle x; y \rangle \,|\, x, y \in \mathbb{X}\} \subset \mathbb{X}$,
  set of *(nested) pairs of numerals, general numerals,* in particular

$$\langle \mathbb{X} \dot{\times} \nu\mathbb{N} \rangle = \{\langle x; \nu n \rangle \,|\, x \in \mathbb{X}, n \in \mathbb{N}\} \subset \mathbb{X};$$

- $\mathbb{X}_{\underline{\perp}} =_{\text{def}} \mathbb{X} \cup \{\underline{\perp}\} \subset \mathbb{N}^+$.

$\mathbb{X}$-Predicative Lemma: $\mathbb{X}$ has predicative form

$$\mathbb{X} = \{\mathbb{N} | \chi_{\mathbb{X}}\}, \text{ and } \mathbb{X}_{\underline{\perp}} = \{\mathbb{N} \,|\, \chi_{\mathbb{X}} \vee \{\ulcorner \perp \urcorner\}\}.$$

Proof as (technically advanced) **Exercise.**

This terminates recursive **definition** of ("minimal") predicative *Universal objects* $\mathbb{X}$ and $\mathbb{X}_{\perp}$, of *nested pairs of numerals,* both

$$\mathbb{X},\ \mathbb{X}_{\perp} \subset \mathbb{N}^{+} \equiv \mathbb{N}_{>} \ =_{\text{by def}}\ \mathbb{N}_{>0} \subset \mathbb{N} \equiv \mathbb{N}^{*}.$$

Remark: A *superUniversal object* $\mathbb{U} \supset \mathbb{X}$, $\mathbb{U} \subset \mathbb{N}$ of *lists* (bracketed strings) of numerals can be **defined** p. r. by

- $\nu\mathbb{N} \subseteq \mathbb{U}$,
- $x \in \mathbb{U}, y \in \mathbb{U} \implies x; y \in \mathbb{U}$,
- $x \in \mathbb{U} \implies \langle x \rangle \in \mathbb{U}$.

(Predicative) set $\mathbb{U} \subset \mathbb{N}$ can be interpreted as set of (numeralised) coefficient lists $\mathbb{N}[X_1, X_2, \ldots, X_m, \ldots]$ of polynomials in *several indeterminates* $X_1, X_2, \ldots$ with (numeralised) coefficients out of $\nu\mathbb{N}$, written in form $\cup_m \mathbb{N}[X_1][X_2]\ldots[X_m]$.

4.3 Universe monoid PR$\mathbb{X}$

The endomorphism set $\mathbf{PR}(\mathbb{N}, \mathbb{N}) \subset \mathbf{PR}$ is itself a **monoid,** a categorical theory with just one object.

Embedded "cartesian p. r. Monoid" $\mathbf{PR}\mathbb{X}$:

- the basic, "super" object of $\mathbf{PR}\mathbb{X}$ is

  $\mathbb{X}_{\perp} = \mathbb{X} \dot{\cup} \{\perp\} = \mathbb{X} \dot{\cup} \{\ulcorner \perp \urcorner\} \subset \mathbb{N}$,

  $\mathbb{X} : \mathbb{N} \to \mathbb{N}$ in $\mathbf{PR}(\mathbb{N}, \mathbb{N})$ predicate/set of (internal) numerals and nested pairs of numerals.

- the rôle of the NNO will be taken by the above predicative subset

$$\nu\mathbb{N} = \{c \in \mathbb{N} \,|\, \chi_\nu(c)\} \subset \mathbb{X} \subset \mathbb{X}_{\underline{\perp}} \subset \mathbb{N}$$

  of the internal *numerals.*

- the basic "universe" map constants of $\mathbf{PR}\mathbb{X}$,

  ba $\in$ bas set of those maps, are

  - "identity" $\mathring{\mathrm{id}} = \mathrm{id}_{\mathbb{X}} : \mathbb{N} \supset \mathbb{X}_{\underline{\perp}} \supset \mathbb{X} \to \mathbb{X} \subset \mathbb{X}_{\underline{\perp}}$,
    $\mathbb{X} \ni x \mapsto x \in \mathbb{X}$,
    $\mathbb{N} \setminus \mathbb{X} \ni z \mapsto \underline{\perp}$ (***trash***),
    PR map code set "from" $\mathbb{N}$ "to" $\mathbb{N}$, same for all codes below.
  - "zero" (redefined for $\mathbf{PR}\mathbb{X}$) $\mathring{0} : \mathbb{X} \to \mathbb{X}_{\underline{\perp}}$,
    $\mathbb{X} \ni \nu 0 \mapsto \nu 0 \in \nu\mathbb{N} \subset \mathbb{X}$,
    $\mathbb{N} \setminus \{\nu 0\} \ni z \mapsto \underline{\perp}$,
  - "successor" $\mathring{s} : \mathbb{X}_{\underline{\perp}} \to \mathbb{X}_{\underline{\perp}}$:
    $\nu n \mapsto \nu(s\, n) =_{\text{by def}} \langle \ulcorner s \urcorner \odot \nu(n) \rangle$,
    $\mathbb{N} \setminus \nu\mathbb{N} \ni z \mapsto \underline{\perp}$.
  - "terminal map": $\mathring{\Pi} : \mathbb{X} \to \nu\mathbb{1} \subset \mathbb{X}$,
    $\mathbb{X} \ni x \mapsto \nu 0 \in \nu\mathbb{1} = \{\nu 0\} \subset \mathbb{X}$,
    $\mathbb{N} \setminus \mathbb{X} \ni z \mapsto \underline{\perp}$.
  - "left projection":
    $\mathring{\ell} : \mathbb{N} \supset \mathbb{X} \supset \langle \mathbb{X} \dot{\times} \mathbb{X} \rangle \to \mathbb{X}_{\underline{\perp}}$,
    $\langle x; y \rangle \mapsto x \in \mathbb{X}$, $\nu\mathbb{N} \ni \nu n \mapsto \underline{\perp}$, $\underline{\perp} \mapsto \underline{\perp}$.

– "right projection" $\mathring{r} \in$ bas analogous.

- close Monoid $\mathbf{PR}\mathbb{X}$ under composition of theory $\mathbf{PR}$:

$$(\circ) \quad \frac{f, g \text{ in } \mathbf{PR}\mathbb{X} \subset \mathbf{PR}(\mathbb{N}, \mathbb{N})}{(g \circ f) \text{ in } \mathbf{PR}\mathbb{X},}$$

trash propagation clear.

- "induced map":

$$(\text{ind}) \quad \frac{f, g \text{ in } \mathbf{PR}\mathbb{X}}{\langle f \,.\, g\rangle \text{ in } \mathbf{PR}\mathbb{X}, \text{ defined by}}$$

$\mathbb{X} \ni x \mapsto \langle f\,x; g\,x\rangle \in \mathbb{X}.$

- "product map":

$$(\dot{\times}) \quad \frac{f, g \text{ in } \mathbf{PR}\mathbb{X}}{\langle f \dot{\times} g\rangle \text{ in } \mathbf{PR}\mathbb{X}, \text{ defined by}}$$

$\mathbb{X} \ni \langle x; y\rangle \mapsto \langle f\,x; g\,y\rangle \in \mathbb{X},$

$\mathbb{N} \setminus \langle \mathbb{X} \dot{\times} \mathbb{X}\rangle \ni z \mapsto \underline{\perp}\,.$

- "iterated" (formally interesting, see last lines):

$$\text{(it)} \quad \frac{f : \mathbb{X} \to \mathbb{X} \ \mathbf{PR}\mathbb{X} \text{ map, in particular } \underline{\perp} \mapsto \underline{\perp}}{\begin{array}{l} f^{\dot{\S}} : \mathbb{X} \supset \langle \mathbb{X} \dot{\times} \nu\mathbb{N} \rangle \to \mathbb{X} \text{ in } \mathbf{PR}\mathbb{X}, \\ \langle x; \dot{n} \rangle \mapsto f^n(x) \in \mathbb{X}, \\ n = \nu^{-1}(\dot{n}),\ \dot{n} \in \dot{\mathbb{N}} = \nu\mathbb{N} \ =_{\text{by def}} \ \{\mathbb{N} | \chi_\nu\} \text{ free}, \\ \mathbb{N} \ni z \mapsto \underline{\perp} \text{ for } z \text{ not of form } \langle x; \dot{n} \rangle. \end{array}}$$

[Predicates $\nu\mathbb{N}$ and $\langle \mathbb{X} \dot{\times} \nu\mathbb{N} \rangle : \mathbb{N} \to \mathbb{N}$ work as auxiliary objects, subobjects of $\mathbb{X} : \mathbb{N} \to \mathbb{N}$.]

- Notion of **map equality** for theory $\mathbf{PR}\mathbb{X}$ is **inherited(!)** from $\mathbf{PR}(\mathbb{N}, \mathbb{N})$ i. e. from theory **PR**.

PR$\mathbb{X}$ **Structure theorem:** With emerging (predicative) objects $\mathbb{X}, \nu\mathbb{1}, \nu\mathbb{N}$,

$$\frac{A, B \text{ objects}}{\langle A \dot{\times} B \rangle \text{ object},}$$

constants, maps, composition above,

- $\nu\mathbb{1} = \{\nu 0\}$ taken as "terminal object",
- $\mathring{\Pi} : \mathbb{X} \to \nu\mathbb{1}$ taken as "terminal map,"

- “Product” taken

$$\langle \mathring{\ell} : \langle A \dot{\times} B \rangle \to A : \langle x; y \rangle \to x,$$
$$\mathring{r} : \langle A \dot{\times} B \rangle \to B, \langle x; y \rangle \to y \rangle,$$

- $\langle f \,.\, g \rangle : C \to \langle A \dot{\times} B \rangle,\ x \mapsto \langle f\,x; g\,x \rangle,$
  taken as “induced map,”

- $\langle f \dot{\times} g \rangle : \langle A \dot{\times} B \rangle \to \langle A' \dot{\times} B' \rangle,\ \langle x; y \rangle \mapsto \langle f\,x; g\,y \rangle,$
  taken as “map product,”

- $\langle \nu \mathbb{1} \xrightarrow{\mathring{0}} \nu \mathbb{N} \xrightarrow{\mathring{s}} \nu \mathbb{N} \rangle$ taken as NNO,

- and $f^{\dot{\S}} : \langle \mathbb{X} \dot{\times} \nu \mathbb{N} \rangle \to \mathbb{X}$ as iterated of
  PR$\mathbb{X}$ endomap $f : \mathbb{X} \to \mathbb{X},\ \langle x; \nu n \rangle \mapsto f^n(x) = f^{\S}(x, n),$

PR$\mathbb{X}$ becomes a cartesian p. r. category with universal object.

• Fundamental theory **PR** is naturally embedded into theory **PR**$\mathbb{X}$, by faithful functor **I** say.

4.4 Typed universe theory PR$\mathbb{X}$a

Let emerge within universe **monoid**/universe cartesian p. r. theory all **PRa** objects $\{A|\chi\}$ as additional objects $\nu\{A|\chi\}$ and get this way a p. r. cartesian theory **PR**$\mathbb{X}$**a** with extensions of predicates, finite limits, finite sums, coequalisers of equivalence predicates, as well as with (formal, “including”) universal object $\mathbb{X}$, of numerals and (nested) pairs of numerals.

Universal embedding theorem:

(i) $\mathbf{I}: \mathbf{PR} \longrightarrow \mathbf{PR}\mathbb{X} \subset \mathbf{PR}(\mathbb{N}, \mathbb{N})$ above is a faithful functor .

(ii) theory $\mathbf{PR}\mathbb{X}\mathbf{a}$ "inherits" from category $\mathbf{PRa}$ all of its (categorically described) structure: cartesian p. r. category structure, equality predicates on all objects, scheme of predicate abstraction, equalisers, and—trivially—the whole algebraic, logic and order structure on NNO $\nu\mathbb{N}$ and truth object $\nu\mathbb{2}$.

(iii) $\mathbf{PR}$ map embedding $\mathbf{I}$ "canonically" extends into a cartesian p. r. functorial embedding (!)

$$\mathbf{I}: \mathbf{PRa} \longrightarrow \mathbf{PR}\mathbb{X}\mathbf{a} \subset \mathbf{PR}(\mathbb{N}, \mathbb{N})$$

of theory $\mathbf{PRa} = \mathbf{PR} + (\mathrm{abstr})$ into emerging *universe theory* $\mathbf{PR}\mathbb{X}\mathbf{a}$ *with predicate abstraction.*

(iv) Embedding **I defines** a p. r. isomorphism of categories

$$\mathbf{I}: \mathbf{PRa} \xrightarrow{\cong} \mathbf{I}[\mathbf{PRa}] \sqsubset \mathbf{PR}\mathbb{X}\mathbf{a}.$$

(v) (internal) code set is

$\ulcorner \mathbb{X}, \mathbb{X} \urcorner =_{\text{by def}} \ulcorner \mathbb{X}, \mathbb{X} \urcorner_{\mathbf{PR}\mathbb{X}\mathbf{a}} = \ulcorner \mathbb{X}, \mathbb{X} \urcorner_{\mathbf{PR}\mathbb{X}} = \mathrm{PR}\mathbb{X}.$

Internal notion $\check{=}$ of equality is in both cases inherited from internal notion of equality of theories $\mathbf{PR}$, $\mathbf{PR}(\mathbb{N}, \mathbb{N})$, given as enumeration of internally equal pairs

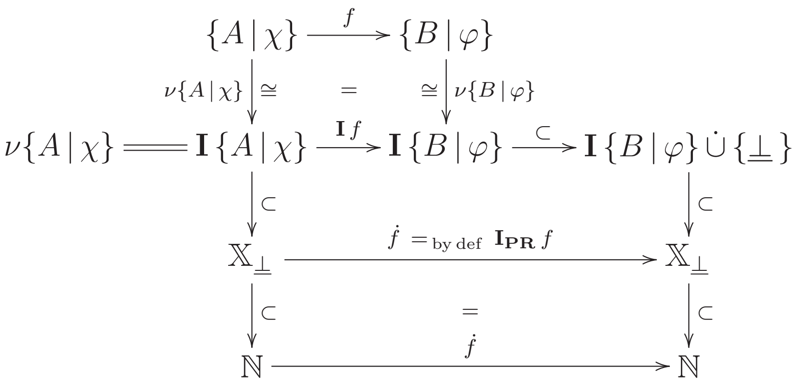
$\check{=} \ = \ \check{=}_k : \mathbb{N} \to \mathrm{PR}\mathbb{X} \times \mathrm{PR}\mathbb{X} \subset \mathbb{N} \times \mathbb{N},$

as well as predicatively as

$\check{=} \ = \ u \check{=}_k v : \mathbb{N} \times (\mathrm{PR} \times \mathrm{PR}) \to \mathbb{2} :$

kth *internal equality instance equals pair* (u, v) *of internal maps.*

(vi) put things together into the following diagram:



PRa embedding DIAGRAM for $\mathbf{I}\,f$ **q.e.d.**

5 Evaluation of p. r. map codes

5.1 Complexity controlled iteration

The data of such a **CCI** are an endomap $p = p(a) : A \to A$ (*predecessor*), and a *complexity* map $c = c(a) : A \to \mathbb{N}[\omega]$ on p's domain. Complexity *values* are taken in lexicographically ordered polynomial object $\mathbb{N}[\omega] \equiv \mathbb{N}^+ \equiv \mathbb{N}^* \smallsetminus \{\Box\} \equiv \mathbb{N}_>$.

Definition: $[c : A \to \mathbb{N}[\omega], p : A \to A]$ constitute the data of a *Complexity Controlled Iteration* $\text{CCI} = \text{CCI}[c, p]$, if

- $(a \in A)[c(a) > 0 \implies c\,p(a) < c(a)]$ *(descent)*

  as well as, for commodity,

- $(a \in A)[c(a) \doteq 0 \implies p(a) \doteq a]$ *(stationarity).*

Such data **define** a <u>while</u> loop

wh$[c > 0, p] : A \rightharpoonup A,$ more explicetly written

<u>while</u> $c(a) > 0$ <u>do</u> $a := p(a)$ <u>od</u>.

We rely on **scheme** of *non-infinite iterative descent*

$$
(\pi)\quad \frac{\begin{array}{l}
\mathrm{CCI}[c = c(a) : A \to \mathbb{N}[\omega],\ p = p(a) : A \to A] : \\
c, p \text{ make up a complexity controlled iteration,} \\
\psi = \psi(a) : A \to \mathbb{2} \ \textit{“negative” test predicate:} \\
(a \in A)(n \in \mathbb{N})[\psi(a) \implies c\,p^n(a) > 0] \\
\quad (\text{“all } n\text{”, } \textit{to be excluded})
\end{array}}{\psi(a) = \mathrm{false}_A(a) : A \to \mathbb{2}.}
$$

A predicate ψ which implies a CCI to infinitely *descend must be (overall) false.*

By contraposition this can be turned into

$$
(\pi^+)\quad \frac{\begin{array}{l}
c, p \text{ define a CCI,} \\
\varphi = \varphi(a) : A \to \mathbb{2} \ \textit{“positive” test predicate:} \\
[c\,p^n(a) \doteq 0 \implies \varphi(a)] : A \times \mathbb{N} \to \mathbb{2} \\
\quad (\text{“exists } n\text{”, } \textit{to be asserted})
\end{array}}{\varphi(a) = \mathrm{true}_A(a) : A \to \mathbb{2}.}
$$

A predicate which holds under the premise of termination of a CCI must be true by itself. This is to express that a CCI must terminate anyway. It says that the *defined arguments enumeration* of a CCI considered as a while loop is a p. r. epimorphism (not a retraction in general.) Technically, we will rely on the (negative) form (π) of the axiom.

- central **example:** *general recursive,* Ackermann type *PR-code evaluation ev* to be *resolved* into such a CCI.

- **scheme** (π) is a **theorem** for **set** theory $\mathbf{T}$ with its quantifiers $\exists$ and $\forall$, and with its having $\mathbb{N}[\omega] \equiv \omega^\omega$ as a (countable) *ordinal:* existential guarantee of finiteness of descending chains within ω^ω.

- without quantification, namely for theories like $\mathbf{PRa}, \mathbf{PR}\mathbb{X}\mathbf{a}$, we are lead to this inference-of-equations scheme guaranteeing (intuitively) termination of CCIs, in particular termination of iterative p. r. code evaluation.

Comment: The point is that (π) expresses an **axiom** which "we all" **believe** in (and which is a **theorem** in **set** theory): Nobody has pointed to—will be able (?) to point to—any *infinitly descending chain* in $\mathbb{N}[\omega] \ =_{\text{by def}} \ \mathbb{N}^+ \subset \mathbb{N}^*$ (provided with its lexicographical order), a fortiori *not* to an *iterative* such, to an infinitly descending CCI.

Definition: Call *PR descent theory* universe theory $\pi\mathbf{R} \ =_{\text{def}} \mathbf{PR}\mathbb{X}\mathbf{a} + (\pi)$ strengthened by **axiom** scheme (π) above of non-infinite descent.

5.2 PR code set

The *map code set*—set of gödel numbers—we want to ***evaluate*** is $\mathrm{PR}\mathbb{X} = \lceil \mathbb{X}, \mathbb{X} \rceil \subset \mathbb{N}$. It is p. r. **defined** as follows:

- $\ulcorner \mathrm{ba} \urcorner \in \mathrm{PR}\mathbb{X}$—formal categorically:

  $\mathrm{PR}\mathbb{X} \circ \ulcorner \mathrm{ba} \urcorner = \mathrm{true}$—this for basic map constant

  $\mathrm{ba} \in \mathrm{bas} = \{\mathring{0}, \mathring{s}, \mathring{\mathrm{id}}, \mathring{\Pi}, \mathring{\Delta}, \mathring{\ell}, \mathring{r}\}$: *zero, successor, identity, terminal map, diagonal, left and right projection.* All of these interpreted into endo map Monoid $\mathbf{PR}\mathbb{X} \subset \mathbf{PR}(\mathbb{N}, \mathbb{N})$ of fundamental cartesian p. r. theory **PR**.

- for u, v in $\mathrm{PR}\mathbb{X}$ in general add

  - internally *composed:* $\langle v \odot u \rangle = \ulcorner (\urcorner v \ulcorner \circ \urcorner u \ulcorner) \urcorner$:
    $\mathrm{PR}\mathbb{X} \times \mathrm{PR}\mathbb{X} \to \mathrm{PR}\mathbb{X}$, $u, v \in \mathrm{PR}\mathbb{X}$ both free,
    in particular $\ulcorner (g \circ f) \urcorner = \langle \ulcorner g \urcorner \odot \ulcorner f \urcorner \rangle \in \mathrm{PR}\mathbb{X}$
    for $f, g : \mathbb{X} \to \mathbb{X}$ in $\mathbf{PR}\mathbb{X}$;
  - internally *induced:* $\langle u; v \rangle = \ulcorner (\urcorner u \ulcorner , \urcorner v \ulcorner) \urcorner \in \mathrm{PR}\mathbb{X}$,
    in particular $\ulcorner (f, g) \urcorner = \langle \ulcorner f \urcorner . \ulcorner g \urcorner \rangle \in \mathrm{PR}\mathbb{X}$;
  - internal *cartesian product:* $\langle u \# v \rangle \in \mathrm{PR}\mathbb{X}$,
    $u, v \in \mathrm{PR}\mathbb{X}$ free, in particular
    $\ulcorner (f \dot{\times} g) \urcorner = \langle \ulcorner f \urcorner \# \ulcorner g \urcorner \rangle \in \mathrm{PR}\mathbb{X}$;
  - internally *iterated:* $u^{\$} = u^{\ulcorner \dot{\S} \urcorner} \in \mathrm{PR}\mathbb{X}$, $u \in \mathrm{PR}\mathbb{X}$,
    in particular $\ulcorner f^{\dot{\S}} \urcorner = \ulcorner f \urcorner^{\$} \in \mathrm{PR}\mathbb{X}$.

5.3 Iterative evaluation

For **Definition** of *evaluation ev* we first introduce *evaluation step* of form

$$e\,(u,x) = (e_{\mathrm{map}}(u,x), e_{\mathrm{arg}}(u,x)) : \mathrm{PR}\mathbb{X} \times \mathbb{X}_{\underline{\perp}} \to \mathrm{PR}\mathbb{X} \times \mathbb{X}_{\underline{\perp}}\,,$$

by primitive recursion. This within "outer" theory **PR**$\mathbb{X}$**a** which already has **PR** predicates $\mathbb{X}, \mathbb{X}_{\underline{\perp}} \ =_{\text{by def}}\ \mathbb{X} \cup \{\underline{\perp}\} = \mathbb{X} \cup \{\ulcorner \perp \urcorner\}$, and $\langle \mathbb{X} \dot{\times} \nu \mathbb{N} \rangle : \mathbb{N} \to \mathbb{N}$ as objects.

Comment: $e_{\mathrm{arg}}(u,x) \in \mathbb{X}_{\underline{\perp}}$ means here one-step u-evaluated *argument,* and $e_{\mathrm{map}}(u,x)$ denotes the remaining part of *map code* u still to be evaluated after that evaluation step.

PR **Definition** of step e, p. r. on $\mathrm{depth}(u) \in \mathbb{N}$, now runs as follows:

- $\mathrm{depth}(u) = 0$, i. e. u of form $\ulcorner \mathrm{ba} \urcorner$,

  $$\mathrm{ba} \in \mathrm{bas}\ =_{\text{by def}}\ \{\mathring{\mathrm{id}}, \mathring{0}, \mathring{s}, \mathring{\Pi}, \mathring{\Delta}, \mathring{\ell}, \mathring{r}\}$$

  one of the basic map constants of theory **PR**$\mathbb{X} \subset$ **PR** :

  $$e_{\mathrm{arg}}(\ulcorner \mathrm{ba} \urcorner, x)\ =_{\mathrm{def}}\ \mathrm{ba}(x) \in \mathbb{X}_{\underline{\perp}}\,,$$
  $$e_{\mathrm{map}}(\ulcorner \mathrm{ba} \urcorner, x)\ =_{\mathrm{def}}\ \ulcorner \mathrm{id} \urcorner \in \mathrm{PR}\mathbb{X}.$$

- cases of internal composition:

  $$e\,(\langle v \odot \ulcorner \mathrm{ba} \urcorner \rangle, x)\ =_{\mathrm{def}}\ (v, \mathrm{ba}(x)) \in \mathrm{PR}\mathbb{X} \times \mathbb{X}_{\underline{\perp}}$$

  and for $u \notin \{\ulcorner \mathrm{ba} \urcorner \mid \mathrm{ba} \in \mathrm{bas}\}$:

  $$e\,(\langle v \odot u \rangle, x)\ =_{\mathrm{def}}\ (\langle v \odot e_{\mathrm{map}}(u,x) \rangle, e_{\mathrm{arg}}(u,x)) :$$

step-evaluate first map code u, on argument x, and preserve remainder of u followed by v as map code to be step-evaluated on intermediate argument $e_{\mathrm{arg}}(u, x)$.

- cartesian cases:

$$e\,(\langle\ulcorner\mathrm{id}\urcorner\#\ulcorner\mathrm{id}\urcorner\rangle, \langle y; z\rangle) \;=_{\mathrm{def}}\; (\ulcorner\mathrm{id}\urcorner, \langle y; z\rangle) \in \mathrm{PR}\mathbb{X}\times\mathbb{X},$$

a terminating case.

For $\langle u\#v\rangle \neq \langle\ulcorner\mathrm{id}\urcorner\#\ulcorner\mathrm{id}\urcorner\rangle$:

$$e\,(\langle u\#v\rangle, \langle y; z\rangle)$$
$$=_{\mathrm{def}}\;(\langle e_{\mathrm{map}}(u, y)\#e_{\mathrm{map}}(v, z)\rangle, \langle e_{\mathrm{arg}}(u, y); e_{\mathrm{arg}}(v, z)\rangle),$$

evaluate u and v in parallel.

Here free variable x on $\mathbb{X}$ legitimatly runs only on $\langle\mathbb{X}\,\dot{\times}\,\mathbb{X}\rangle \subset \mathbb{X}$, takes there the pair form $\langle y; z\rangle$. $x \in \mathbb{X} \setminus \langle\mathbb{X}\,\dot{\times}\,\mathbb{X}\rangle$ results in present evaluation case into $\underline{\perp}$.

- Cases of an induced (redundant via $\ulcorner\Delta\urcorner$ and $\odot$):

$$e\,(\langle\ulcorner\mathrm{id}\urcorner;\ulcorner\mathrm{id}\urcorner\rangle, z) \;=_{\mathrm{def}}\; (\ulcorner\mathrm{id}\urcorner, \langle z; z\rangle),$$

a terminating case.

For $\langle u; v\rangle \neq \langle\ulcorner\mathrm{id}\urcorner;\ulcorner\mathrm{id}\urcorner\rangle$:

$$e\,(\langle u; v\rangle, z)$$
$$=_{\mathrm{def}}\;(\langle e_{\mathrm{map}}(u, z); e_{\mathrm{map}}(v, z)\rangle, \langle e_{\mathrm{arg}}(u, z); e_{\mathrm{arg}}(v, z)\rangle),$$

evaluate both components u and v.

- iteration case, with $\$:= \ulcorner\S\urcorner$ designating internal *iteration:*

  $$e\,(u^{\$}, \langle y; \nu n\rangle) = (u^{[n]}, y) :$$
  $$\mathrm{PR}\mathbb{X} \times \mathbb{X} \supset \mathrm{PR}\mathbb{X} \times \langle \mathbb{X} \dot{\times} \nu\mathbb{N}\rangle \to \mathrm{PR}\mathbb{X} \times \mathbb{X}.$$

  Here $\nu n \in \nu\mathbb{N}$ free, $n := \nu^{-1}(\nu n) \in \mathbb{N}$, and $u^{[n]}$ is given by *code expansion* as

  $$u^{[0]} \;=_{\mathrm{def}}\; \ulcorner\mathrm{id}\urcorner,\; u^{[n+1]} \;=_{\mathrm{def}}\; \langle u \odot u^{[n]}\rangle.$$

- trash case $e\,(u, x) = (\ulcorner\mathrm{id}\urcorner, \perp\!\!\!\perp) \in \mathrm{PR}\mathbb{X} \times \mathbb{X}_{\perp\!\!\!\perp}$ if (u, x) in none of the above—regular—cases.

For to convince ourselves on termination of iteration of step $e : \mathrm{PR}\mathbb{X} \times \mathbb{X}_{\perp\!\!\!\perp} \to \mathrm{PR}\mathbb{X} \times \mathbb{X}_{\perp\!\!\!\perp}$—on a pair of form $(\ulcorner\mathrm{id}\urcorner, x)$—we **introduce:**

(*Descending*) *complexity*

$$c_{ev}(u, x) = c\,(u) : \mathrm{PR}\mathbb{X} \times \mathbb{X} \xrightarrow{\ell} \mathrm{PR}\mathbb{X} \to \mathbb{N}[\omega]$$

defined p. r. as

$$c\,(\ulcorner\mathrm{id}\urcorner) \;=_{\mathrm{def}}\; 0 = 0 \cdot \omega \in \mathbb{N}[\omega],$$
$$c\,(\ulcorner\mathrm{ba}'\urcorner) \;=_{\mathrm{def}}\; 1 \in \mathbb{N}[\omega]$$

for ba' one of the other basic map constants in bas,

$$c\,\langle v \odot u\rangle \;=_{\mathrm{def}}\; c\,(u) + c\,(v) + 1 = c\,(u) + c\,(v) + 1 \cdot \omega^0 \in \mathbb{N}[\omega],$$
$$c\,\langle u\#v\rangle \;=_{\mathrm{def}}\; c\,(u) + c\,(v) + 1,$$
$$c\,\langle u; v\rangle \;=_{\mathrm{def}}\; c\,(u) + c\,(v) + 1,$$
$$c\,(u^{\$}) \;=_{\mathrm{def}}\; (c\,(u) + 1) \cdot \omega^1 \in \mathbb{N}[\omega].$$

$[\,(\_)\cdot\omega^1$ is to account for unknown *iteration count* n in argument $\langle x; n\rangle$ before code expansion.]

Example: Complexity of *addition* $+ \ =_{\text{by def}}\ s^{\S} : \mathbb{N}\times\mathbb{N}\to\mathbb{N}$:

$$\begin{aligned} &c\ulcorner + \urcorner = c\ulcorner s^{\S}\urcorner = c\,(\ulcorner s\urcorner^{\$}) \\ &= (c\ulcorner s\urcorner + 1)\cdot\omega^1 = 2\cdot\omega \in \mathbb{N}[\omega] \quad [\equiv\ 0;2\ \in\mathbb{N}^+\,] \end{aligned}$$

Motivation for the above **definition**—in particular for this latter iteration case—will become clear with the corresponding case in **proof** of **descent Lemma** below for *evaluation*

$$ev = ev\,(u,v)\ =_{\text{def}}\ r\,\widehat{\circ}\,\mathrm{wh}\,[\,c_{ev} > 0\,,\ e\,] : \mathrm{PR}\mathbb{X}\times\mathbb{X}_{\perp} \rightharpoonup \mathrm{PR}\mathbb{X}\times\mathbb{X}_{\perp} \xrightarrow{r} \mathbb{X}_{\perp}$$

defined by a <u>while</u> loop which reads

<u>while</u> $c_{ev}(u) > 0$ <u>do</u> $(u,x) := e(u,x)$ <u>od</u>.

Evaluation *step* and *complexity* above are in fact the right ones to give

Basic descent lemma: For formally *partially defined* and "nevertheless" *epi-terminating* evaluation map: the defined-arguments p. r. enumeration of partial map is epi—this by axiom scheme (π)—,

$$\begin{aligned} &ev = ev\,(u,x)\ =_{\text{by def}}\ r\,\widehat{\circ}\,\mathrm{wh}\,[\,c_{ev} > 0, e\,] : \\ &\mathrm{PR}\mathbb{X}\times\mathbb{X}_{\perp} \rightharpoonup \mathrm{PR}\mathbb{X}\times\mathbb{X}_{\perp} \xrightarrow{r} \mathbb{X}_{\perp} \\ &\text{(epi-terminating within theory } \pi\mathbf{R} = \mathbf{PRa} + (\pi)) \end{aligned}$$

i. e. for step $e = e\,(u,x) = (e_{\text{map}}, e_{\text{arg}}) : \mathrm{PR}\mathbb{X}\times\mathbb{X}_{\perp} \to \mathrm{PR}\mathbb{X}\times\mathbb{X}_{\perp}$ and complexity $c_{ev} = c_{ev}(u,x)\ =_{\text{def}}\ c\,(u) : \mathrm{PR}\mathbb{X}\to\mathbb{N}[\omega]$, we have descent *above* $0 \in \mathbb{N}[\omega]$, and Stationarity *at* complexity 0 :

$$\begin{aligned}
&\mathbf{PR}\mathbb{X} \vdash c_{ev}(u,x) > 0 \implies c_{ev}\,e\,(u,x) < c_{ev}(u,x) : \\
&\qquad \mathrm{PR}\mathbb{X} \times \mathbb{X}_{\perp\!\!\!\perp} \to \mathbb{N}[\omega] \times \mathbb{N}[\omega] \to \mathbb{2} \ \text{ i. e.} \\
&\mathbf{PR}\mathbb{X} \vdash c\,(u) > 0 \implies c\,e_{\mathrm{map}}(u,x) < c\,(u) \qquad \text{(Desc)} \\
&\qquad \text{as well as} \\
&\mathbf{PR}\mathbb{X} \vdash c\,(u) \doteq 0 \quad [\iff u \equiv \ulcorner \mathrm{id} \urcorner\ ] \\
&\qquad \implies c_{ev}\,e\,(u,x) \doteq 0 \ \wedge \ e\,(u,x) \doteq (u,x) \qquad \text{(Sta)}
\end{aligned}$$

This with respect to the canonical, *lexicographic,* and—intuitively—*finite-descent* order of polynomial semiring $\mathbb{N}[\omega]$.

Proof: The only non-trivial case $(v,b) \in \mathrm{PR}\mathbb{X} \times \mathbb{X}$ for descent $c_{ev}\,e\,(v,b) < c_{ev}(v,b)$ is iteration case $(v,b) = (u^{\$}, \langle x; n \rangle)$. In this "acute" iteration case we have

$$\begin{aligned}
&c\,(u^{[n]}) = c\,(\langle u \odot \langle u \ldots \odot u \rangle \ldots \rangle) \\
&= n \cdot c\,(u) + (n \dot{-} 1) < \omega \cdot (c(u) + 1) = c(u^{\$}),
\end{aligned}$$

proved in detail by induction on n **q.e.d.**

5.4 Evaluation characterisation

Dominated characterisation theorem for evaluation:

$ev = ev\,(u,a) : \mathrm{PR}\mathbb{X} \times \mathbb{X} \rightharpoonup \mathbb{X}$ is characterised by

- $\mathbf{PR}\mathbb{X}\mathbf{a} \vdash [\, ev\,(\ulcorner \mathrm{ba} \urcorner, x) \doteq \mathrm{ba}(x)\,]$

  as well as, again within $\mathbf{PR}\mathbb{X}\mathbf{a}, \pi\mathbf{R}$ and strengthenings, by:

- $[\, m \;\mathit{deff}\;\; ev\,(v \odot u, x)\,] \implies$
  $ev\,(\langle v \odot u\rangle, x) \doteq ev\,(v, ev\,(u, x))$;

  this reads: if m *defines* the left hand iteration ev, i. e. if iteration ev of *step e terminates* on the left hand argument after at most m steps, then ev terminates in at most m steps on right hand side as well, and the two evaluations have equal results.

- $[\, m \;\mathit{deff}\;\; ev\,(\langle u\#v\rangle, \langle x; y\rangle)\,] \implies$
  $ev\,(\langle u\#v\rangle, \langle x; y\rangle) \doteq \langle ev\,(u, x); ev\,(v, y)\rangle$,

  $[\, m \;\mathit{deff}\;\; ev\,(\langle u; v\rangle, z)\,] \implies$
  $ev\,(\langle u; v\rangle, z) \doteq \langle ev\,(u, z); ev\,(v, z)\rangle$.

- $ev\,(u^{\$}, \langle x; \ulcorner 0 \urcorner \rangle) \doteq x$,

  $[\, m \;\mathit{deff}\;\; ev\,(u^{\$}, \langle x; \nu(s\,n)\rangle\,] \implies:$

  $[\, m \;\mathit{deff}$ all ev below$\,] \wedge$
  $ev\,(u^{\$}, \langle x; \nu(s\,n)\rangle) \doteq ev\,(u, ev\,(u^{\$}, \langle x; \nu\,n\rangle))$.

- it *terminates,* with all properties above, when situated in a **set** theory **T**, since there complexity receiving ordinal $\mathbb{N}[\omega]$ has (only) finite descent, in terms of existential quantification.

Corollary: within **T**, we have the double recursive equations

- $ev\,(\ulcorner \mathrm{ba} \urcorner, x) \doteq \mathrm{ba}(x)$,

- $ev\,(\langle v \odot u\rangle, x) \doteq ev\,(v, ev\,(u, x))$,

- $ev\,(\langle u\#v\rangle, \langle x; y\rangle) \doteq \langle ev\,(u, x); ev\,(v, y)\rangle$,
  $ev\,(\langle u; v\rangle, z) \doteq \langle ev\,(u, z); ev\,(v, z)\rangle$,

- $ev\,(u^{\$}, \langle x;\ \ulcorner 0 \urcorner\ \rangle) \doteq x,$ and

  $ev\,(u^{\$}, \langle x; \nu(s\,n)\rangle) \doteq ev\,(u, ev\,(u^{\$}, \langle x; \nu\,n\rangle)).$

Within **T**—as well as within partial p. r. theories $\mathbf{P\widehat{R}}\mathbb{X}\mathbf{a}, \pi\mathbf{\widehat{R}}$—these equations can be taken as **definition** for $\mathbf{PR}\mathbb{X}$ code evaluation ev. Within **T**, they **define** evaluation as a total map.

Proof of **theorem** by primitive recursion (Peano Induction) on $m \in \mathbb{N}$ free, via case distinction on codes w, and arguments $z \in \mathbb{X}$ appearing in the different cases of the asserted conjunction (case w one of the basic map constants being trivial). All of the following—**induction step**—is situated in $\mathbf{PR}\mathbb{X}\mathbf{a}$, read: $\mathbf{PR}\mathbb{X}\mathbf{a} \vdash$ etc. If you are interested first in the negative results for **set** theories **T**, you can read it "$\mathbf{T} \vdash \ \ldots$" but **T** still deriving properties just of $\mathbf{PR}\mathbb{X}$ map codes.

- case $(w, z) = (\langle v \odot u\rangle, x)$ of an (internally) *composed,* subcase $u = \ulcorner \mathrm{id} \urcorner$: obvious.

  Non-trivial subcase $(w, z) = (\langle v \odot u\rangle, x)$, $u \neq \ulcorner \mathrm{id} \urcorner$:

$$m + 1 \ \mathit{deff}\ \ ev\,(\langle v \odot u\rangle, x) \implies :$$

$$ev\,(\langle v \odot u\rangle, x) \doteq e^{\S}((\langle v \odot e_{\mathrm{map}}(u, x)\rangle, e_{\mathrm{arg}}(u, x)), m)$$

by iterative definition of ev in this case

$$\doteq ev\,(v, ev\,(e_{\mathrm{map}}(u, x), e_{\mathrm{arg}}(u, x)))$$

by induction hypothesis on m

$$\implies :$$

$$m + 1 \ \mathit{deff}\ \ ev\,(v, ev\,(e_{\mathrm{map}}(u, x), e_{\mathrm{arg}}(u, x)))$$

$$\wedge\ ev\,(v, ev\,(e_{\mathrm{map}}(u, x), e_{\mathrm{arg}}(u, x))) \doteq ev\,(v, ev\,(u, x))\ :$$

The latter implication "holds" same way back, by the same induction hypothesis on m (map code v unchanged.)

- case $(w, z) = (\langle u\#v\rangle, \langle x; y\rangle)$ of an (internal) *cartesian product:* Obvious by definition of ev on a cartesian product map codes. Pay attention to arguments out of $\mathbb{X} \smallsetminus \langle \mathbb{X} \dot{\times} \mathbb{X}\rangle$ evaluated into $\underline{\perp}$ in this case (and in similar cases). In more detail:

$$
\begin{aligned}
& ev\,(w, z) := \\
& ev\,(\langle u\#v\rangle, \langle x; y\rangle) \\
& \quad =_{\text{by def}} \;\; ev\,(\langle e_{\text{map}}(u, x)\#e_{\text{map}}(v, y)\rangle, \langle e_{\text{arg}}(u, x), e_{\text{arg}}(v, y)\rangle) \\
& \quad\; \doteq \langle ev(e_{\text{map}}(u, x), e_{\text{arg}}(u, x)), ev(e_{\text{map}}(v, y), e_{\text{arg}}(v, y))\rangle \\
& \qquad \in \langle \mathbb{X} \dot{\times} \mathbb{X}\rangle
\end{aligned}
$$

- alternatively (or both): case $(w, z) = (\langle u; v\rangle, z)$ of an internal induced:

$$ev(w, z) \doteq \langle ev(u, z), ev(v, z)\rangle \in \langle \mathbb{X} \dot{\times} \mathbb{X}\rangle.$$

- case $(w, z) = (u^{\$}, \langle x; \ulcorner 0 \urcorner \rangle)$ of a null-fold (internally) iterated: again obvious.

- case $(w, z) = (u^\$, \langle x; \nu(s\,n)\rangle)$ of a genuine (internally) iterated:

$$m + 1 \ \textit{deff} \ \ ev\,(u^\$, \langle x; \nu(s\,n)\rangle) \implies$$

$$m + 1 \ \textit{deff} \text{ all instances of } ev \text{ below, and:}$$

$$\begin{aligned}
& ev\,(u^\$, \langle x; \nu(s\,n)\rangle) \\
& \doteq ev\,(e_{\mathrm{map}}(u^\$, \langle x; \nu(s\,n)\rangle), e_{\mathrm{arg}}(u^\$, \langle x; \nu(s\,n)\rangle)) \\
& \doteq ev\,(u^{[n+1]}, x) \doteq ev\,(\langle u \odot u^{[n]}\rangle, x) \doteq ev\,(u, ev\,(u^{[n]}, x)) \\
& \qquad \text{the latter by induction hypothesis on } m, \\
& \qquad \text{case of internal composed} \\
& \doteq ev\,(u, \langle ev\,(u^\$, x); \nu\, n\rangle) : \text{same way back.}
\end{aligned}$$

This shows the (remaining) predicative *iteration* equations "anchor" and "step" for an (internally) iterated $u^\$$, and so **proves** fullfillment of the above **double recursive** system of equations for $ev : \mathrm{PR}\mathbb{X}\mathrm{a} \times \mathbb{X} \rightharpoonup \mathbb{X}$ subordinated to *global* evaluation $ev : \mathrm{PR}\mathbb{X} \times \mathbb{X} \rightharpoonup \mathbb{X}$ **q.e.d.**

Characterisation corollary: Evaluation—$\mathbf{P\widehat{R}}\mathbb{X}\mathbf{a}$ map—

$$ev = ev\,(u, x) : \mathrm{PR}\mathbb{X} \times \mathbb{X} \rightharpoonup \mathbb{X}$$

defined as *complexity controlled iteration*—CCI—with complexity values in ordinal $\mathbb{N}[\omega]$, epi-terminates in theory $\pi\widehat{\mathbf{R}}$: has epimorphic defined arguments enumeration. This by definition of this theory strengthening $\mathbf{P\widehat{R}}\mathbb{X}\mathbf{a}$. **And** it satisfies there the characteristic double-recursive equations above for evaluation ev.

Objectivity theorem: Evaluation ev is *objective,* i. e. for each *single,* (meta free) $f : A \to B$ in theory $\mathbf{PR}\mathbb{X}\mathbf{a}$ itself, we have

$$\mathbf{PR}\mathbb{X}\mathbf{a}, \pi\mathbf{R} \vdash \ [\, m \textit{ deff } ev(\ulcorner f \urcorner, a)\,] \implies$$
$$ev(\ulcorner f \urcorner, a) = f(a), \text{ symbolically:}$$
$$\pi\mathbf{R} \vdash \ ev(\ulcorner f \urcorner, \_\,) = f : A \rightharpoonup B.$$

For frame a **set** theory $\mathbf{T}$, there is no need for explicit domination m *deff* etc.

Proof by substitution of codes of $\mathbf{PR}\mathbb{X}\mathbf{a}$ maps into code variables $u, v, w \in \mathrm{PR}\mathbb{X} \subset \mathbb{N}$ in Evaluation Characterisation above, in particular:

- $[\, m \textit{ deff } ev\,(\ulcorner g \circ f \urcorner, a)\,] \implies$
  $ev\,(\langle \ulcorner g \urcorner \odot \ulcorner f \urcorner \rangle, a) \doteq ev\,(\ulcorner g \urcorner, ev\,(\ulcorner f \urcorner, a)),$
  $\doteq g(f(a)) \doteq (g \circ f)(a)$ recursively (on m) and

- $[\, m \textit{ deff } ev\,(\ulcorner f^{\S} \urcorner, \langle a; \nu(s\, n)\rangle\,] \implies :$

  $[\, m \textit{ deff}$ all ev below$\,] \wedge$

  $ev\,(\ulcorner f \urcorner^{\$}, \langle a; \nu(s\, n)\rangle) \doteq ev\,(\ulcorner f \urcorner, ev\,(\ulcorner f \urcorner^{\$}, \langle a; \nu\, n\rangle))$

  $\doteq f(f^{\S}(a, \nu n)) = f^{\S}(a, \nu(s\, n))$ recursively on m.

- it *terminates,* with this objectivity, within **set** theory $\mathbf{T}$.

6 PR Decidability by Set Theory

We embed evaluation $\varepsilon(u, x) : \mathrm{PR}\mathbb{X} \times \mathbb{X} \rightharpoonup \mathbb{X}$ of PR map codes into **set** theory, theory $\mathbf{T}$.

Notion $f =^{\mathbf{PR}} g$ of p. r. maps is externally p. r. enumerated, by complexity of (binary) deduction trees.

Internalising—*formalising*—gives internal notion of PR equality (not: stronger **T**-equality)

$$u \check{=}_k v \in \mathrm{PR}\mathbb{X} \times \mathrm{PR}\mathbb{X}$$

coming by internal *deduction tree* dtree_k, which can be canonically provided with arguments in $\mathbb{X}$—top down from (suitable) argument x given to the *root* equation $u \check{=}_k v$ of dtree_k.

We denote internal deduction tree argumented this way by dtree_k/x, *root* of dtree_k/x then is $u/x \check{=}_k v/x$.

6.1 PR soundness framed by set theory

PR Evaluation *soundness* theorem Framed by set theory T : For p. r. theory **PR** with its internal notion of equality '$\check{=}$' we have:

(i) PR$\mathbb{X}$ to **T** evaluation **soundness:**

$$\mathbf{T} \vdash u \check{=}_k v \implies ev(u, x) = ev(v, x) \qquad (\bullet)$$

Substituting in the above "concrete" **PR$\mathbb{X}$a** codes into u resp. v, we get, by *objectivity* of evaluation ε :

(ii) **T**-Framed Objective soundness of **PR** :

For **PR$\mathbb{X}$a** maps $f, g : \mathbb{X} \supset A \to B \subset \mathbb{X}$:

$$\mathbf{T} \vdash \ulcorner f \urcorner \check{=} \ulcorner g \urcorner \implies f(a) = g(a).$$

(iii) Specialising to case $u := \ulcorner \chi \urcorner$, $\chi : \mathbb{X} \to \mathbb{2}$ a p. r. *predicate,* and to $v := \ulcorner \mathrm{true} \urcorner$, we get

T-framed *Logical soundness of* **PR** :

$$\mathbf{T} \vdash \ \exists\, k \, \mathrm{Prov}_{\mathbf{PR}}(k, \ulcorner \chi \urcorner) \implies \forall\, x\, \chi(x) :$$

If *a p. r. predicate is—within* **T**—**PR**-*internally* provable, *then it holds in* **T** *for all of its arguments.*

Proof of logically central assertion (•) by primitive recursion on k, dtree_k the k th deduction tree of the theory. These (argument-free) deduction trees are counted in lexicographical order.

Remark: A detailed **proof** is given for frame theory **PR**$\mathbb{X}$**a** and termination-conditioned evaluations in next section. This proof logically includes present case of frame theory a **set** theory **T** : within such **T** as frame, both evaluations, *ev* as well as *deduction tree evaluation* ev_d, terminate on all of their arguments.

Super Case of *equational* internal **axioms:**

- associativity of (internal) composition:

  $\langle\langle w \odot v \rangle \odot u \rangle \check{=}_k \langle w \odot \langle v \odot u \rangle\rangle \implies$

  $$\begin{aligned} & ev\,(\langle w \odot v \rangle \odot u, x) = ev\,(\langle w \odot v \rangle, ev\,(u, x)) \\ & = ev\,(w, ev\,(v, ev\,(u, x))) \\ & = ev\,(w, ev\,(\langle v \odot u \rangle, x)) = ev\,(w \odot \langle v \odot u \rangle, x). \end{aligned}$$

  This **proves** assertion (•) in present *associativity-of-composition* case.

- Analogous **proof** for the other **flat,** equational cases, namely *reflexivity of equality, left and right neutrality* of id $=_{\text{by def}}$ $\text{id}_{\mathbb{X}}$, all substitution equations for the map constants, Godement's equations for the induced map as well as surjective pairing and *distributivity equation for composition with an induced.*

- **proof** of (•) for the last equational **case,** the

  Iteration **step,** case of *genuine iteration equation*

  $\text{dtree}_k \;=\; \langle u^{\$} \odot \langle \ulcorner \text{id} \urcorner \# \ulcorner s \urcorner \rangle \check{=}_k u \odot u^{\$} \rangle$:

$$\begin{aligned} \mathbf{T} \vdash\ & ev\,(u^{\$} \odot \langle \ulcorner \text{id} \urcorner \# \ulcorner s \urcorner \rangle, \langle y; \nu(n) \rangle) && (1) \\ &= ev\,(u^{\$}, ev(\langle \ulcorner \text{id} \urcorner \# \ulcorner s \urcorner \rangle, \langle y; \nu(n) \rangle)) \\ &= ev\,(u^{\$}, \langle y; \nu(s\,n) \rangle) \\ &= ev\,(u, ev(u^{\$}, \langle y; \nu(n) \rangle)) \\ &= ev\,(u \odot u^{\$}, \langle y; \nu(n) \rangle). && (2) \end{aligned}$$

Proof of termination-conditioned inner soundness for the remaining *deep*—genuine HORN **cases**—for dtree_k , HORN type *deduction* of *root:*

Transitivity-of-equality case: with map code variables u, v, w we start here with argument-free deduction tree

$$\text{dtree}_k \quad = \quad \Uparrow \frac{u \,\check{=}_k\, w}{u \,\check{=}_i\, v \ \wedge\ v \,\check{=}_j\, w}$$

Evaluate at argument x and get in fact

$$\mathbf{T} \vdash u \check{=}_k w$$
$$\implies ev(u,x) = ev(v,x) \wedge ev(v,x) = ev(w,x)$$
(by hypothesis on $i, j < k$)
$$\implies ev(u,x) = ev(w,x):$$
transitivity export q.e.d. in this case.

Case of **symmetry** axiom scheme for equality is now obvious.

Compatibility case of composition with equality

$$\text{dedu}_k = \Uparrow \frac{\langle v \odot u \rangle \check{=}_k \langle v \odot u' \rangle}{u \check{=}_i u'}$$

By induction hypothesis on $i < k$ we have

$$\langle v \odot u \rangle \check{=}_k \langle v \odot u' \rangle \implies :$$
$$[ev(u,x) = ev(u',x) \implies$$
$$ev(v \odot u, x) = ev(v, ev(u,x)) = ev(v, ev(u',x))$$
$$= ev(v \odot u', x)]$$

by hypothesis on $u \check{=}_i u'$ and by Leibniz' substitutivity, q.e.d. in this 1st compatibility case.

Case of composition with equality in second composition factor:

$$\text{dedu}_k = \Uparrow \frac{\langle v \odot u \rangle \check{=}_k \langle v' \odot u \rangle}{v \check{=}_i v'}$$

[Here dtree_i is not (yet) provided with all of its arguments, it *is* completly argumented during top down tree evaluation.]

$$\begin{aligned}
&\langle v \odot u \rangle \check{=}_k \langle v' \odot u \rangle \implies : \\
&ev(\langle v \odot u \rangle, x) = ev(v, ev(u, x)) = ev(v', ev(u, x)) \qquad (*) \\
&= ev(\langle v' \odot u \rangle, x).
\end{aligned}$$

$(*)$ holds by $v \check{=}_i v'$, induction hypothesis on $i < k$, and Leibniz' substitutivity: same argument into equal maps.

This proves soundness assertion $(\bullet)$ in this 2nd compatibility case.

(Redundant) Case of **compatibility** of forming the induced map, with equality is analogous to compatibilities above, even easier, since the two map codes concerned are independent from each other.

(Final) Case of Freyd's (internal) **uniqueness** of the *initialised iterated,* is **case**

$$\text{dedu}_k / \langle y; \nu(n) \rangle$$

$$= \frac{w/\langle y; \nu(n)\rangle \check{=}_k \langle v^{\$} \odot \langle u \# \ulcorner \text{id} \urcorner \rangle / \langle y; \nu(n) \rangle \rangle}{\text{root}\,(t_i) \qquad\qquad\qquad\qquad \text{root}\,(t_j)}$$

where

$$\begin{aligned}
&\text{root}\,(t_i) \\
&= \langle w \odot \langle \ulcorner \text{id} \urcorner ; \ulcorner 0 \urcorner \odot \ulcorner \Pi \urcorner \rangle / y \check{=}_i u/y \rangle, \\
&\text{root}\,(t_j) \\
&= \langle w \odot \langle \ulcorner \text{id} \urcorner \# \ulcorner s \urcorner \rangle / \langle y; \nu(n) \rangle \check{=}_j \langle v \odot w \rangle / \langle y; \nu(n) \rangle \rangle.
\end{aligned}$$

Comment: w is here an internal *comparison candidate* fullfilling the same internal p. r. equations as $\langle v^{\$} \odot \langle u\# \ulcorner \mathrm{id} \urcorner \rangle\rangle$. It should be—**is**: *soundness*—evaluated equal to the latter, on $\langle \mathbb{X} \dot{\times} \nu\mathbb{N} \rangle \subset \mathbb{X}$.

Soundness **assertion** ($\bullet$) for the present Freyd's *uniqueness* **case** recurs on $\check{\doteq}_i$, $\check{\doteq}_j$ turned into predicative equations '=', these being already deduced, by hypothesis on $i, j < k$. Further ingredients are transitivity of '=' and established properties of basic evaluation ev of map terms.

So here is the remaining—inductive—**proof,** prepared by

$$\mathbf{T} \vdash ev\,(w, \langle y; \nu(0)\rangle) = ev\,(u; y) \qquad (\bar{0})$$

as well as

$$\begin{aligned} & ev(w, \langle y; \nu(s\,n)\rangle) = ev\,(w, \langle y; \ulcorner s \urcorner \odot \nu(n)\rangle) \\ & = ev\,(w \odot \langle \ulcorner \mathrm{id} \urcorner \# \ulcorner s \urcorner \rangle, \langle y; \nu(n)\rangle) \\ & = ev\,(v \odot w, \langle y; \nu(n)\rangle), \end{aligned} \qquad (\bar{s})$$

the same being true for $w' := v^{\$} \odot \langle u\# \ulcorner \mathrm{id} \urcorner \rangle$ in place of w, once more by (characteristic) double recursive equations for ev, this time with respect to the *initialised internal iterated* itself.

($\bar{0}$) and ($\bar{s}$) put together for both then show, by induction on *iteration count* $n \in \mathbb{N}$—all other free variables k, u, v, w, y together form the *passive parameter* for this induction—*truncated soundness* assertion ($\bullet$) for this *Freyd's uniqueness* case, namely

$$\mathbf{T} \vdash ev\,(w, \langle y; \nu(n)\rangle) = ev\,(v^{\$} \odot \langle u\# \ulcorner \mathrm{id} \urcorner \rangle, \langle y; \nu(n)\rangle).$$

Induction runs as follows:

Anchor $n = 0$:

$$ev\,(w, \langle y; \nu(0)\rangle) = ev\,(u, y) = ev\,(w', \langle y; \nu(0)\rangle),$$

step:

$$ev\,(w, \langle y; \nu(n)\rangle) = ev\,(w', \langle y; \nu(n)\rangle) \implies :$$
$$ev\,(w, \langle y; \nu(s\,n)\rangle) = ev\,(v, ev\,(w, \langle y; \nu(n)\rangle))$$
$$= ev\,(v, ev\,(w', \langle y; \nu(n)\rangle)) = ev\,(w', \langle y; \nu(s\,n)\rangle),$$

the latter since evaluation ev preserves predicative equality '=' (Leibniz) **q.e.d.**

Comment: Already for stating the evaluations, we needed the—categorical, free-variables theories $\mathbf{PR}, \mathbf{PRa}, \mathbf{PR}\mathbb{X}, \mathbf{PR}\mathbb{X}\mathbf{a}$ of primitive recursion. Since this type of **soundness** is a corner stone in our approach, the above complicated categorical combinatorics seem to be necessary, even for the negative results on classical foundations.

6.2 PR-predicate decision by set theory

We consider here $\mathbf{PR}\mathbb{X}\mathbf{a}$ predicates for **decidability** by **set** theorie(s) $\mathbf{T}$. Basic tool is **T-framed soundness of $\mathbf{PR}\mathbb{X}\mathbf{a}$** just above, namely

$$\frac{\chi = \chi(a) : A \to 2 \ \ \mathbf{PR}\mathbb{X}\mathbf{a} \text{ predicate}}{\mathbf{T} \vdash \ \exists k\, \mathrm{Prov}_{\mathbf{PR}\mathbb{X}\mathbf{a}}(k, \ulcorner \chi \urcorner) \implies \forall a\, \chi(a).}$$

Within $\mathbf{T}$ **define** for $\chi : A \to 2$ out of $\mathbf{PR}\mathbb{X}\mathbf{a}$ a partially defined (alleged, individual) μ-recursive **decision** $\nabla\chi = \nabla^{\mathrm{PR}}\chi : \mathbb{1} \rightharpoonup 2$ by first fixing *decision domain*

$$D = D\chi := \{k \in \mathbb{N} \,|\, \neg\, \chi(\mathrm{ct}_A(k)) \vee \mathrm{Prov}_{\mathbf{PR}\mathbb{X}\mathbf{a}}(k, \ulcorner \chi \urcorner)\},$$

$\mathrm{ct}_A : \mathbb{N} \to A$ (retractive) Cantor count of A; and then, with (partial) recursive $\mu D : \mathbb{1} \rightharpoonup D \subseteq \mathbb{N}$ within $\mathbf{T}$:

$$\nabla\chi \ =_{\mathrm{def}} \begin{cases} \text{false if } \neg\,\chi(\mathrm{ct}_A(\mu D)) \\ \quad (\textit{counterexample}), \\ \text{true if } \mathrm{Prov}_{\mathbf{PR}\mathbb{X}\mathbf{a}}(\mu D, \ulcorner\chi\urcorner) \\ \quad (\textit{internal proof}), \\ \bot \ (\textit{undefined}) \text{ otherwise, i. e.} \\ \quad \text{if } \forall a\, \chi(a) \wedge \forall k\, \neg\, \mathrm{Prov}_{\mathbf{PR}\mathbb{X}\mathbf{a}}(k, \ulcorner\chi\urcorner). \end{cases}$$

[This (alleged) decision is apparently (μ-)recursive within $\mathbf{T}$, even if apriori only partially defined.]

There is a first *consistency* problem with this **definition:** are the *defined* cases *disjoint?*

Yes, within frame theory $\mathbf{T}$ which *soundly frames* theory $\mathbf{PR}\mathbb{X}\mathbf{a}$:

$$\mathbf{T} \vdash \ (\exists\, k \in \mathbb{N})\, \mathrm{Prov}_{\mathbf{PR}\mathbb{X}\mathbf{a}}(k, \ulcorner\chi\urcorner) \implies \forall a\, \chi(a).$$

$\mathbf{T}$-framed $\mathbf{PR}\mathbb{X}\mathbf{a}$-soundness leads to

Complete T derivation alternative for $\mathbf{PR}\mathbb{X}\mathbf{a}$ predicate χ :

(a) $\mathbf{T} \vdash \ \nabla\chi = \text{false}$ iff $\mathbf{T} \vdash \ \exists a\, \neg\, \chi(a)$,

(b) $\mathbf{T} \vdash \ \nabla\chi = \text{true}$ iff $\mathbf{T} \vdash \ \exists k\, \mathrm{Prov}_{\mathbf{PR}\mathbb{X}\mathbf{a}}(k, \ulcorner\chi\urcorner)$

iff $\mathbf{T} \vdash \ \exists k\, \mathrm{Prov}_{\mathbf{PR}\mathbb{X}\mathbf{a}}(k, \ulcorner\chi\urcorner) \wedge \forall a\, \chi(a)$,

the latter iff by $\mathbf{T}$-*framed soundness* of $\mathbf{PR}\mathbb{X}\mathbf{a}$.

(c) $\mathbf{T} \vdash \ \nabla\chi = \bot$ iff $\mathbf{T} \vdash \ \forall a\, \chi(a) \wedge \forall k\, \neg\, \mathrm{Prov}_{\mathbf{PR}\mathbb{X}\mathbf{a}}(k, \ulcorner\chi\urcorner)$.

Remark:

- within quantified arithmetic $\mathbf{T}$ we have the right to replace

  $\chi(\mathrm{ct}_A(\mu D))$ by $\exists\, a\,(\chi(a))$ in the above, and

  $\mathrm{Prov}_{\mathbf{PR}\mathbb{X}\mathbf{a}}(\mu D, \ulcorner\chi\urcorner)$ by $\exists\, k\,\mathrm{Prov}_{\mathbf{PR}\mathbb{X}\mathbf{a}}(k, \ulcorner\chi\urcorner)$.

- for consistent $\mathbf{T}$, χ an arbitrary $\mathbf{T}$-formula, and *Proof* $\mathrm{Prov}_{\mathbf{T}}$ in place of $\mathrm{Prov}_{\mathbf{PR}\mathbb{X}\mathbf{a}}$, *soundness*—and therefore *disjointness* of (termination) cases(a) and (b) above—does not work anymore: take for χ Gödel's undecidable formula φ with its "characteristic" property

$$\mathbf{T} \vdash \neg\varphi \iff \exists k\,\mathrm{Prov}_{\mathbf{T}}(k, \ulcorner\varphi\urcorner).$$

Merging now the (right hand sides) of the latter two cases gives the following complete alternative,

Decidability of primitive recursive free-variable predicates *by* quantified extension $\mathbf{T}$ (via μ-recursive decision algorithm $\nabla\chi : \mathbb{1} \rightharpoonup \mathbb{2}$):

For (arbitrary) $\mathbf{PR}\mathbb{X}\mathbf{a}$ predicate $\chi = \chi(a) : A \to \mathbb{2}$ we have

$\mathbf{T} \vdash \forall a\, \chi(a)$ **or**

$\mathbf{T} \vdash \exists a \neg\, \chi(a)$.

"Theorem or derivable existence of a counterexample" **q.e.d.**

Decision Remark: this does not mean a priori that *decision algorithm* $\nabla\chi$ terminates for all such predicates χ. The theorem says only that χ is ***decidable*** "by", *within* **theory T**, that it is *not independent* from **T**.

For free-variable **PR$\mathbb{X}$a** (!) predicate $\chi := \neg\,\mathrm{Prov}_{\mathbf{T}}(k, \ulcorner \mathrm{false} \urcorner) : \mathbb{N} \to \mathbb{2}$ the above entails the alternative

$$\mathbf{T} \vdash \forall k\, \neg\, \mathrm{Prov}_{\mathbf{T}}(k, \ulcorner \mathrm{false} \urcorner) \quad \textbf{or}$$
$$\mathbf{T} \vdash \exists k\, \mathrm{Prov}_{\mathbf{T}}(k, \ulcorner \mathrm{false} \urcorner),$$

will say the alternative

$$\mathbf{T} \vdash \mathrm{Con}_{\mathbf{T}} \quad \textbf{or}$$
$$\mathbf{T} \vdash \neg\, \mathrm{Con}_{\mathbf{T}},$$

i. e. ***consistency decidability*** for **set** theory **T**.

First assertion of Gödel's **2nd incompleteness theorem** says:
$\mathbf{T} \nvdash \mathrm{Con}_{\mathbf{T}}$, if **T** consistent,
whence we get **2nd alternative** above:

$$\mathbf{T} \vdash \neg\, \mathrm{Con}_{\mathbf{T}} :$$

set *theory* **T** ***derives/proves*** *its own inconsistency (formula).*

Proof of first **assertion** of 2nd incompleteness theorem in Smorynski 1977, adapted to categorical language in next **section.**

This concerns **set** theories as **PM**, **ZF**, and **NGB** as well as "already" Peano arithmetic **PA**.

6.3 Gödel's incompleteness theorems

We visit §2. Gödel's theorems, in Smorynski 1977.

First Incompleteness Theorem. *Let* **T** *be a formal theory containing arithmetic. Then there is a sentence* φ *which asserts its own unprovability and such that:*

(i) *If* $\mathbf{T}$ *is consistent,* $\mathbf{T} \nvdash \varphi$.

(ii) *If* $\mathbf{T}$ *is* ω*-consistent,* $\mathbf{T} \nvdash \neg\,\varphi$.

In §3.2.6 Smorynski discusses possible choices of *arithmetic* (theory) **S**, namely

(a) **PRA** = (classical, free-variables) primitive recursive arithmetic,

S. Feferman: "my **PRA**", in contrast to **PRa** above.

(b) **PA** = Peano's arithmetic.

Conjecture: $\mathbf{PA} \cong \mathbf{PR}\exists \sqsubset \mathbf{PRa}\exists$.

(c) **ZF** = Zermelo-Fraenkel set theory. "This is both a good and a bad example. It is bad because the whole encoding problem is more easily solved in a set theory than in an arithmetical theory. By the same token, it is a good example."

Conjecture: PRA can categorically be viewed as cartesian theory with weak NNO in Lambek's sense.

We take $\mathbf{S} := \mathbf{PRa}$, embedding extension of categorical theory **PR**, formally stronger than **PRA** because of uniqueness of maps defined by the full schema of primitive recursion, and weaker than $\mathbf{PA} \cong \mathbf{PR}\exists$.

By construction of arithmetic **PRa**, "one can adequatly encode syntax in this $\mathbf{S} = \mathbf{PRa}$," since Smorynski's conditions (i)-(iii) for the representation of p. r. functions are fulfilled.

We take for formal extension **T** of **S** one of the categorical pendants to suitable **set** theories (subsystems of **ZF**, see Osius 1974), or the *(first order) elementary theory of two-valued Topoi with NNO,* cf. Freyd 1972, or, *minimal choice,* $\mathbf{T} := \mathbf{PRa}\exists \sqsupset \mathbf{PA}$.

Derivability theorem: Our $\mathbf{S}$ encoding, extended from $\mathbf{PRa}$ to $\mathbf{T}$, meets the following (quantifier free categorically expressed) *Derivability Conditions* in §2.1 of Smorynski:

D1 $\mathbf{T} \overset{\underline{k}}{\vdash} \varphi$ infers $\mathbf{S} \vdash \mathrm{Prov}_{\mathbf{T}}(\mathrm{num}(\underline{k}), \ulcorner\varphi\urcorner)$.

D2 $\mathbf{S} \vdash \mathrm{Prov}_{\mathbf{T}}(k, \ulcorner\varphi\urcorner) \implies \mathrm{Prov}_{\mathbf{T}}(j_2(k), \mathrm{Prov}_{\mathbf{T}}(k, \ulcorner\varphi\urcorner)),$
$j_2 = j_2(k) : \mathbb{N} \to \mathbb{N}$ suitable.

D3 $\mathbf{S} \vdash \mathrm{Prov}_{\mathbf{T}}(k, \ulcorner\varphi\urcorner) \wedge \mathrm{Prov}_{\mathbf{T}}(k', \ulcorner\varphi \Rightarrow \psi\urcorner)$
$\implies \mathrm{Prov}_{\mathbf{T}}(j_3(k, k'), \ulcorner\psi\urcorner),$
$j_3 = j_3(k, k') : \mathbb{N}^2 \to \mathbb{N}$ suitable.

Smorynski's **proof** gives the *First Gödel's incompleteness theorem,* and from that the

Second incompleteness theorem: Let $\mathbf{T}$ be one of the extensions above of $\mathbf{PR}\exists$, and $\mathbf{T}$ consistent. *Then*

$$\mathbf{T} \nvdash \mathrm{Con}_{\mathbf{T}},$$

where $\mathrm{Con}_{\mathbf{T}} = \forall k \,\neg\, \mathrm{Prov}_{\mathbf{T}}(k, \ulcorner\mathrm{false}\urcorner)$ *is the sentence asserting the consistency of* $\mathbf{T}$.

From this Gödel's theorem and our *PR Decidability theorem* for quantified arithmetic $\mathbf{PRa}\exists, \mathbf{T}$ we get

Inconsistency provability theorem for quantified arithmetical (set) theories $\mathbf{T}$:

If **T** is consistent, then

$$\mathbf{T} \vdash \neg\,\mathrm{Con}_{\mathbf{T}}.$$

[If not, then it derives everything, in particular $\neg\,\mathrm{Con}_{\mathbf{T}}$. We will see that p. r. arithmetic, under a mild termination condition for external evaluation, yields inconsistency of **T**.]

7 Consistency Decision within πR

7.1 Termination conditioned evaluation soundness

ES[9] **Theorem on termination-conditioned soundness:**

For p. r. theory $\mathbf{PR}\mathbb{X}\mathbf{a}$ [10] and internal notion of equality $\check{\doteq} \;=\; \check{\doteq}_k : \mathbb{N} \to \mathrm{PR}\mathbb{X} \times \mathrm{PR}\mathbb{X}$, dtree_k the k th deduction tree of universe theory $\mathbf{PR}\mathbb{X} \subset \mathbf{PR}(\mathbb{N}, \mathbb{N})$, we have:

(i) *Termination-Conditioned* ***Inner*** *soundness:*

With $r = r(u, x) = x : \mathrm{PR}\mathbb{X} \times \mathbb{X} \to \mathbb{X}$ right projection:

$$\begin{aligned} \mathbf{PR}\mathbb{X}\mathbf{a} \vdash \langle u \,\check{\doteq}_k\, v\rangle &\doteq \mathrm{root}\ (\mathrm{dtree}_k) \\ &\wedge\ m\ \mathit{deff}\ \ ev_d\,(\mathrm{dtree}_k/x) \\ &\implies\ ev\,(u, x) \doteq ev\,(v, x)\,. \qquad (\bullet) \end{aligned}$$

---

[9] *Evaluation soundness*

[10] presumably *not* directly for $\pi\mathbf{R}$ with respect to its own internal equality, without assumption of "π-consistency," in this regard RCF 2 contains an error

explicitly:

$$\begin{aligned}\mathbf{PR}\mathbb{X}\mathbf{a} \vdash u \check{\doteq}_k v \ \wedge \ c_d\, e_d^m\,(\mathrm{dtree}_k/x) \doteq 0 \\ \Longrightarrow ev\,(u,x) \doteq e^m(u,x) \doteq e^m(v,x) \\ \doteq ev\,(v,x), \qquad (\bullet)\end{aligned}$$

free map-code variables u, v, variable x free in universal set $\mathbb{X}$.

[*Argumentation* dtree_k/x of dtree_k and definition of *argumented tree evaluation* ev_d based on its evaluation step e_d and complexity c_d is by merged recursion on $\mathrm{depth}(\mathrm{dtree}_k)$, within **proof** below]

In words, this "m-Truncated", "m-Dominated" Inner soundness says that theory **PRa** derives:

If *for an internal* **PR**$\mathbb{X}$ equation $u \check{\doteq}_k v$ argumented deduction tree dtree_k/x *for* $u \check{\doteq}_k v$, argumented *with* $x \in \mathbb{X}$, *admits complete* argumented-tree evaluation, i. e.

if *tree-evaluation becomes* ***completed*** *after a finite number m of evaluation steps,*

then *both sides of this* internal (!) equation *are completly* ***evaluated*** *on x by (at most) m steps e of* basic *evaluation ev, into* ***equal values.***

Substituting in the above "concrete" codes into u resp. v , we get, by *objectivity* of evaluation ev, formally "mutatis mutandis":

(ii) *Termination-Conditioned Objective soundness for Map Equality:*

For $\mathbf{PR}\mathbb{X}\mathbf{a}$ maps $f, g : \mathbb{A} \to B$:

$$\mathbf{PR}\mathbb{X}\mathbf{a} \vdash [\ \ulcorner f \urcorner \check{\doteq}_k \ulcorner g \urcorner \ \wedge\ m \ \mathit{deff}\ ev_d(\mathrm{dtree}_k/a)\,]$$
$$\implies f(a) \doteq_B r\ e^m(\ulcorner g \urcorner, a) \doteq_B g(a),\ a \in A \text{ free :}$$

If *an internal PR deduction-tree for (internal) equality of* $\ulcorner f \urcorner$ *and* $\ulcorner g \urcorner$ *is available, and* ***if*** *on this tree—top down argumented with* a *in* A*—tree evaluation* ***terminates, then*** *equality* $f(a) \doteq_B g(a)$ *of* f *and* g *at this argument is the consequence.*

(iii) Specialising this to case of $f := \chi : A \to \mathbb{2}$ a p. r. *predicate* and to $g := \mathrm{true}_A : A \to \mathbb{2}$ we eventually get

Termination-Conditioned Objective Logical soundness:

$$\mathbf{PR}\mathbb{X}\mathbf{a} \vdash \ \mathrm{Prov}_{\mathbf{PR}\mathbb{X}}(k, \ulcorner \chi \urcorner) \wedge m \ \mathit{deff}\ ev_d(\mathrm{dtree}_k/a) \implies \chi(a) :$$

If *tree-evaluation of an internal* ***deduction*** *tree for a free variable p. r. predicate* $\chi : A \to \mathbb{2}$*—the tree* argumented *with* $a \in A$*—* ***terminates*** *after a finite number* m *of evaluation steps,* ***then*** $\chi(a) \doteq \mathrm{true}$ *is the consequence,* within $\mathbf{PR}\mathbb{X}\mathbf{a}$ as well as within its extensions $\pi\mathbf{R}$—and **set** theory $\mathbf{T}$.

Remark to proof below: in present case of frame theory $\mathbf{PR}\mathbb{X}\mathbf{a}$ (and stronger theory $\pi\mathbf{R}$) we have to *control* all evaluation step iterations, and we do that by control of iterative evaluation ev_d of whole argumented deduction trees, whose recursive **definition** will be—merged—part of this proof.

Proof of—basic—*termination-conditioned* ***inner*** *soundness,* i. e. of implication (•) in *ES theorem* is by induction on deduction tree counting index $k \in \mathbb{N}$ counting family $\mathrm{dtree}_k : \mathbb{N} \to \mathrm{Bintree}$, starting with (flat) $\mathrm{dtree}_0 = \langle \ulcorner \mathrm{id} \urcorner \check{\doteq}_0 \ulcorner \mathrm{id} \urcorner \rangle$. $m \in \mathbb{N}$ is to dominate argumented-deduction-tree evaluation ev_d to be recursively defined below: *condition*

m *deff* $ev_d(\mathrm{dtree}_k/x)$, step e_d, complexity c_d.

We argue by *recursive case distinction* on the form of the top up-to-two layers—top (implicational) deduction—dedu_k/x of argumented deduction tree dtree_k/x at hand.

Flat **super case** $\mathrm{depth}(dtree_k) = 0$, i. e. super case of *unconditioned,* axiomatic (internal) *equation* $u \check{\doteq}_k v$:

The first involved of these cases is *associativity* of (internal) *composition:*

$$\mathrm{dtree}_k \ = \ \langle\langle w \odot v\rangle \odot u\rangle \check{\doteq}_k \langle w \odot \langle v \odot u\rangle\rangle$$

In this case—no need of a recursion on k—

$$\begin{aligned}
\mathbf{PR}\mathbb{X}\mathbf{a} \vdash m \ deff \ ev_d(\mathrm{dtree}_k/x) \ \Longrightarrow \\
[\, m \ deff \ ev\,(\langle w \odot v\rangle \odot u, x)\,] \\
\wedge\ [\, m \ deff \ ev(\langle w \odot v\rangle, ev\,(u, x)) \\
\wedge\ [\, m \ deff \ ev(w, ev(v, ev(u, x))) \\
\wedge\ [\, m \ deff \ ev(w, ev(\langle v \odot u\rangle, x)) \\
\wedge\ [\, m \ deff \ ev\,(\langle w \odot \langle v \odot u\rangle\rangle, x)\,] \wedge
\end{aligned}$$

$$\begin{aligned}
& ev\,(\langle w \odot v\rangle \odot u, x) \doteq ev\,(\langle w \odot v\rangle, ev\,(u, x)) \\
& \doteq ev\,(w, ev\,(v, ev\,(u, x))) \\
& \doteq ev\,(w, ev\,(\langle v \odot u\rangle, x)) \doteq ev\,(w \odot \langle v \odot u\rangle, x).
\end{aligned}$$

This proves assertion (•) in present *associativity-of-composition* case. [New in comparison to previous *Inconsistency* chapter is here only the "preamble" *m deff* etc.]

Analogous **proof** for the other **flat,** equational cases, namely *reflexivity of equality, left and right neutrality* of id $=_{\text{by def}}$ $\text{id}_{\mathbb{X}}$, all substitution equations for the map constants, Godement's equations for the induced map as well as surjective pairing and distributivity of composition over forming the induced map.

Godement's equations $\ell \circ (f, g) = f,\ r \circ (f, g) = g$:

$$
\begin{aligned}
&m \ \textit{deff} \ ev \text{ etc.} \implies \\
&ev(\ulcorner \mathring{\ell} \urcorner \odot \langle u; v\rangle, z) \doteq r\, e^m(\ulcorner \mathring{\ell} \urcorner \odot \langle u; v\rangle, z) \\
&\doteq \mathring{\ell}(\langle ev(u, z); ev(v, z)\rangle) \doteq ev(u, z),
\end{aligned}
$$

analogously for composition with right projection.

Fourman's equation $(\ell \circ h, r \circ h) = h$:

$$
\begin{aligned}
&m \ \textit{deff} \ ev \text{ etc.} \implies \\
&ev(\langle \ulcorner \mathring{\ell} \urcorner \odot w; \ulcorner \mathring{r} \urcorner \odot w\rangle, z) \\
&\doteq \langle ev(\ulcorner \mathring{\ell} \urcorner, ev(w, z)); ev(\ulcorner \mathring{r} \urcorner, ev(w, z))\rangle \\
&\doteq \langle \mathring{\ell}(ev(w, z)); \mathring{r}(ev(w, z))\rangle \doteq ev(w, z)
\end{aligned}
$$

by SP equation on objective level.

Now here are the **proofs**—with preambles—of (•), for the last equational case, the

Iteration **step,** case of *genuine iteration equation*

$$\mathrm{dtree}_k \;=\; \langle u^{\$} \odot \langle \ulcorner \mathrm{id} \urcorner \# \ulcorner s \urcorner \rangle \check{\doteq}_k u \odot u^{\$} \rangle :$$

$$\begin{aligned}
\mathbf{PR}\mathbb{X}\mathbf{a} \vdash\; & m \;\mathit{deff}\;\; ev_d(\mathrm{dtree}_k/\langle y; \nu(n)\rangle) \implies \\
& m \;\mathit{deff}\;\; \textit{all instances of ev below, and:} \\
& ev\,(u^{\$} \odot \langle \ulcorner \mathrm{id} \urcorner \# \ulcorner s \urcorner \rangle, \langle y; \nu(n)\rangle) \qquad (1) \\
& \doteq ev\,(u^{\$}, ev(\langle \ulcorner \mathrm{id} \urcorner \# \ulcorner s \urcorner \rangle, \langle y; \nu(n)\rangle)) \\
& \doteq ev\,(u^{\$}, \langle y; \nu(s\,n)\rangle) \\
& \doteq ev\,(u^{[s\,n]}, y) \quad \text{(by definition of } ev \text{ step } e) \\
& \doteq ev\,(u \odot u^{[n]}, y) \\
& \doteq ev\,(u, ev(u^{\$}, \langle y; \nu(n)\rangle)) \\
& \doteq ev\,(u \odot u^{\$}, \langle y; \nu(n)\rangle). \qquad (2)
\end{aligned}$$

Proof of termination-conditioned inner soundness for the remaining *deep*—genuine Horn **cases**—for dtree_k, Horn type (at least) at *deduction* of *root:*

Transitivity-of-equality case: with map code variables u, v, w we start here with argument-free deduction tree

$$\mathrm{dtree}_k \quad = \quad \cfrac{u \,\check{\doteq}_k\, w}{\cfrac{u \,\check{\doteq}_i\, v}{\mathrm{dtree}_{ii} \quad \mathrm{dtree}_{ji}} \qquad \cfrac{v \,\check{\doteq}_j\, w}{\mathrm{dtree}_{ij} \quad \mathrm{dtree}_{jj}}}$$

It is argumented with argument x say, recursively spread down:

$$\mathrm{dtree}_k/x \quad = \quad \cfrac{\cfrac{\mathrm{dtree}_{ii}/x_{ii} \quad \mathrm{dtree}_{ji}/x_{ji}}{u/x \quad v/x} \qquad \cfrac{\mathrm{dtree}_{ij}/x_{ij} \quad \mathrm{dtree}_{jj}/x_{jj}}{v/x \quad w/x}}{u/x \quad w/x}$$

Spreading down arguments from upper level down to 2nd level must/is given explicitly, further arguments spread down is then recursive by the type of deduction (sub)trees $\mathrm{dtree}_i, \mathrm{dtree}_j,\ i, j < k$.

Now by induction hypothesis on i, j we have for tree evaluation ev_d :

$$\begin{aligned}
&u \,\check{=}_k\, w \wedge m \ \mathit{deff}\ ev_d(\mathrm{dtree}_k/x) \\
&\ \Longrightarrow m \ \mathit{deff}\ ev_d(\mathrm{dtree}_i/x), ev_d(\mathrm{dtree}_j/x) \wedge \\
&ev_d(\mathrm{dtree}_i/x) \doteq \langle \ulcorner \mathrm{id} \urcorner / ev(u, x) \doteq \ulcorner \mathrm{id} \urcorner / ev(v, x) \rangle \\
&\wedge\ ev_d(\mathrm{dtree}_j/x) \doteq \langle \ulcorner \mathrm{id} \urcorner / ev(v, x) \doteq \ulcorner \mathrm{id} \urcorner / ev(w, x) \rangle \\
&\ \Longrightarrow ev(u, x) \doteq ev(v, x) \wedge ev(v, x) \doteq ev(w, x) \\
&\ \Longrightarrow ev(u, x) \doteq ev(w, x).
\end{aligned}$$

and this is what we wanted to show in present transitivity of equality case.

[Transitivity **axiom** for equality is a main reason for necessity to consider (argumented) deduction trees: intermediate map code equalities '$\check{=}$' in a transitivity chain must be each evaluated, and pertaining deduction trees may be of arbitrary high evaluation complexity]

Case of **symmetry** axiom scheme for equality is now obvious.

Compatibility Case of composition with equality[11]

$$\mathrm{dtree}_k/x \quad = \quad \cfrac{\langle v \odot u\rangle/x \,\check{\doteq}_k\, \langle v \odot u'\rangle/x}{\cfrac{u/x \,\check{\doteq}_j\, u'/x}{\mathrm{dtree}_{ij}/x \qquad \mathrm{dtree}_{jj}/x}}$$

By induction hypothesis on $j < k$

$$\begin{aligned}
&m \ \mathit{deff} \ ev_d(\mathrm{dtree}_k/x) \implies \\
&m \ \mathit{deff} \ ev_d(\mathrm{dtree}_j/x) \implies \\
&ev(u, x) \doteq ev(u', x) \implies \\
&ev(v \odot u, x) \doteq ev(v, ev(u, x)) \doteq ev(v, ev(u', x)) \\
&\doteq ev(v \odot u', x)
\end{aligned}$$

by dominated characterisic equations for ev and Leibniz' substitutivity, q.e.d. in this 1st compatibility case.

Spread down arguments is more involved in

Case of composition with equality in second composition factor: argument spread down merged with tree evaluation ev_d and proof of result.

[11] this simplified version has been suggested by Joseph

$$\text{dtree}_k/x \quad = \quad \cfrac{\langle v \odot u\rangle/x \qquad \langle v' \odot u\rangle/x}{\cfrac{v \mathrel{\check{\doteq}}_i v'}{\text{dtree}_{ii} \qquad \text{dtree}_{ji}}}$$

[Here dtree_i is not (yet) provided with argument, it *is* argumented during top down tree evaluation below]

m *deff* $ev_d(\text{dtree}_k/x) \implies$

m *deff* all instances of ev below, and:

$$ev(\langle v \odot u\rangle, x) \doteq ev(v, ev(u, x)) \doteq ev(v', ev(u, x)) \qquad (*)$$
$$\doteq ev(\langle v' \odot u\rangle, x).$$

$(*)$ holds by Leibniz' substitutivity and

m *deff* $ev_d(\text{dtree}_k/x) \implies$

m *deff* $ev_d(\text{dtree}_i/ev(u, x))$

[*argumentation of* dtree_i *with*

$ev(u, x)$—*calculated en cours de route*,

extra **definition** of e_d]

$\implies$

m *deff* $ev(v, ev(u, x)) \doteq ev(v', ev(u, x))$,

by induction hypothesis on $i < k$: The hypothesis is independent of substituted argument, provided—and this is here the case—that dtree_i is evaluated on that argument, in $m' < m$ steps, m' suitable (minimal).

This proves assertion (•) in this 2nd compatibility case.

(Redundant) case of **compatibility** of forming the induced map with map equality is analogous to compatibilities above, even easier, because of almost independence of any two inducing map codes from each other.

(Final) case of Freyd's (internal) **uniqueness** of the *initialised iterated,* is **case**

$$\mathrm{dedu}_k/\langle y;\nu(n)\rangle \quad = \quad \frac{w/\langle y;\nu(n)\rangle \,\check{\doteq}_k\, \langle v^{\$} \odot \langle u\#\ulcorner \mathrm{id} \urcorner\rangle/\langle y;\nu(n)\rangle\rangle}{\mathrm{root}\,(t_i) \qquad\qquad\qquad\qquad \mathrm{root}\,(t_j)}$$

where

$$\mathrm{root}\,(t_i) = \langle w \odot \langle\, \ulcorner \mathrm{id} \urcorner\,;\, \ulcorner 0 \urcorner \,\odot\, \ulcorner \Pi \urcorner\,\rangle/y \,\check{\doteq}_i\, u/y\rangle,$$
$$\mathrm{root}\,(t_j) = \langle w \odot \langle\, \ulcorner \mathrm{id} \urcorner \,\#\, \ulcorner s \urcorner\,\rangle/\langle y;\nu(n)\rangle \,\check{\doteq}_j\, \langle v \odot w\rangle/\langle y;\nu(n)\rangle\rangle$$

Comment: w is here an internal *comparison candidate* fullfilling the same internal PR equations as $\langle v^{\$} \odot \langle u\#\ulcorner \mathrm{id} \urcorner\rangle\rangle$. It should be—**is**: *soundness*—evaluated equal to the latter, on $\langle \mathbb{X} \dot{\times} \nu\mathbb{N}\rangle \subset \mathbb{X}$.

soundness **assertion** (•) for the present Freyd's *uniqueness* **case** recurs on $\check{\doteq}_i$, $\check{\doteq}_j$ turned into predicative equations '$\doteq$', these being already deduced, by hypothesis on $i, j < k$. Further ingredients are transitivity of '$\doteq$' and established properties of basic evaluation *ev* of map terms.

So here is the remaining—inductive—**proof,** prepared by

$$\mathbf{T} \vdash m \text{ } \mathit{deff} \text{ dtree}_k/\langle y; \nu(n)\rangle \implies$$
$$m \text{ } \mathit{deff} \text{ all of the following } \mathit{ev}\text{-terms and}$$
$$\mathit{ev}\,(w, \langle y; \nu(0)\rangle) \doteq \mathit{ev}\,(u; y) \qquad (\bar{0})$$
$$\text{as well as}$$
$$m \text{ } \mathit{deff} \text{ both of the following } \mathit{ev}\text{-terms, and}$$
$$\mathit{ev}(w, \langle y; \nu(s\,n)\rangle) \doteq \mathit{ev}\,(w, \langle y; \ulcorner s \urcorner \odot \nu(n)\rangle)$$
$$\doteq \mathit{ev}\,(w \odot \langle \ulcorner \mathrm{id} \urcorner \# \ulcorner s \urcorner \rangle, \langle y; \nu(n)\rangle)$$
$$\doteq \mathit{ev}\,(v \odot w, \langle y; \nu(n)\rangle), \qquad (\bar{s})$$

the same being true for $w' := v^{\$} \odot \langle u\# \ulcorner \mathrm{id} \urcorner \rangle$ in place of w, once more by (characteristic) double recursive equations for ev, this time with respect to the *initialised internal iterated* itself.

$(\bar{0})$ and $(\bar{s})$ put together for both then show, by induction on *iteration count* $n \in \mathbb{N}$—all other free variables k, u, v, w, y together form the *passive parameter* for this induction—*truncated soundness* assertion $(\bullet)$ for this *Freyd's uniqueness* case, namely

$$\mathbf{T} \vdash m \text{ } \mathit{deff} \text{ dtree}_k/\langle y; \nu(n)\rangle \implies$$
$$m \text{ } \mathit{deff} \text{ all of the ev-terms concerned above, and}$$
$$\mathit{ev}\,(w, \langle y; \nu(n)\rangle) \doteq \mathit{ev}\,(v^{\$} \odot \langle u\# \ulcorner \mathrm{id} \urcorner \rangle, \langle y; \nu(n)\rangle).$$

Induction runs as follows:

Anchor $n = 0$:

$\mathit{ev}\,(w, \langle y; \nu(0)\rangle) \doteq \mathit{ev}\,(u, y) \doteq \mathit{ev}\,(w', \langle y; \nu(0)\rangle),$

Step: m *deff* etc. $\implies$

$$ev\,(w, \langle y; \nu(n)\rangle) \doteq ev\,(w', \langle y; \nu(n)\rangle) \implies :$$
$$ev\,(w, \langle y; \nu(s\,n)\rangle) \doteq ev\,(v, ev\,(w, \langle y; \nu(n)\rangle))$$
$$\doteq ev\,(v, ev\,(w', \langle y; \nu(n)\rangle)) \doteq ev\,(w', \langle y; \nu(s\,n)\rangle),$$

the latter since evaluation *ev* preserves predicative equality '$\doteq$' (Leibniz) **q.e.d.** *Termination Conditioned PR soundness theorem.*

Comment: Already for stating the evaluations, we needed the—categorical, free-variables theories $\mathbf{PR}, \mathbf{PRa}, \mathbf{PR}\mathbb{X}, \mathbf{PR}\mathbb{X}\mathbf{a}$ of primitive recursion, as well as—for termination, even in classial frame $\mathbf{T}$—PR complexities within $\mathbb{N}[\omega]$. Since this type of **soundness** is a corner stone in our approach, the above complicated categorical combinatorics seem to be necessary, even for the negative results on classical Foundations.

7.2 Framed consistency

From **termination-conditioned soundness**—resp. from $\mathbf{T}$-framed PR soundness—we get

π**R-framed internal PR consistency corollary:** For *descent* theory $\pi\mathbf{R} = \mathbf{PR}\mathbb{X}\mathbf{a} + (\pi)$, **axiom** (π) stating non-infinite iterative descent in *ordinal* $\mathbb{N}[\omega]$, we have

$$\pi\mathbf{R} \vdash \mathrm{Con}_{\mathbf{PR}\mathbb{X}},\ \text{i.e. "necessarily" in } \textit{free-variables} \text{ form:}$$
$$\pi\mathbf{R} \vdash \neg\,\mathrm{Prov}_{\mathbf{PR}\mathbb{X}}(k, \ulcorner\mathrm{false}\urcorner) : \mathbb{N} \to 2,\ k \in \mathbb{N} \text{ free},$$
$$\mathbf{T} \vdash \mathrm{Con}_{\mathbf{PR}\mathbb{X}} :$$

theory $\pi\mathbf{R}$*—as well as* ***set*** *theories* $\mathbf{T}$ *as an extension of* $\pi\mathbf{R}$—derive *that no* $k \in \mathbb{N}$ *is the internal* $\mathbf{PR}\mathbb{X}$-Proof *for* $\ulcorner\text{false}\urcorner$.

Proof for this **corollary** from *termination-conditioned soundness:* By assertion (iii) of that **theorem,** with $\chi = \chi(a) := \text{false}(a) = \text{false} : \mathbb{1} \to \mathbb{2}$, we get:

Evaluation-effective internal inconsistency of $\mathbf{PR}\mathbb{X}$—i. e. availability of an *evaluation-terminating* internal *deduction tree* of $\ulcorner\text{false}\urcorner$—*implies* false :

$$\mathbf{PR}\mathbb{X}\mathbf{a},\ \pi\mathbf{R} \vdash \text{Prov}_{\mathbf{PR}\mathbb{X}}(k, \ulcorner\text{false}\urcorner) \ \wedge\ c_d\ e_d^m(\text{dtree}_k/\langle 0\rangle) \doteq 0$$
$$\implies \text{false}.$$

Contraposition to this, still with $k, m \in \mathbb{N}$ free:

$$\pi\mathbf{R} \vdash \ \text{true} \implies \neg\,\text{Prov}_{\mathbf{PR}\mathbb{X}}(k, \ulcorner\text{false}\urcorner) \vee c_d\ e_d^m(\text{dtree}_k/\langle 0\rangle) > 0,$$

i. e. by free-variables (boolean) tautology:

$$\pi\mathbf{R} \vdash \ \text{Prov}_{\mathbf{PR}\mathbb{X}}(k, \ulcorner\text{false}\urcorner) \implies c_d\ e_d^m(\text{dtree}_k/\langle 0\rangle) > 0 : \mathbb{N}^2 \to \mathbb{2}.$$

For k "fixed", the conclusion of this implication—m free—means infinite descent in $\mathbb{N}[\omega]$ of iterative argumented deduction-tree evaluation ev_d on $\text{dtree}_k/0$, which is excluded intuitively. Formally it is excluded within our theory $\pi\mathbf{R}$ taken as frame:

We apply non-infinite-descent scheme (π) to ev_d, which is given by *step* e_d and complexity c_d—the latter descends (this is *argumented-tree evaluation descent*) with each application of e_d, as long as complexity $0 \in \mathbb{N}[\omega]$ is not ("yet") reached. We combine this with—choice of—*overall "negative"* condition

$$\psi = \psi(k) := \text{Prov}_{\mathbf{PR}\mathbb{X}}(k, \ulcorner\text{false}\urcorner) : \mathbb{N} \to \mathbb{2},\ k \in \mathbb{N} \text{ free}$$

and get—by that scheme (π)—overall negation of this (overall) *excluded* predicate ψ, namely

$$\pi\mathbf{R} \vdash \neg\,\mathrm{Prov}_{\mathbf{PR}\mathbb{X}}(k, \ulcorner \mathrm{false} \urcorner) : \mathbb{N} \to 2, \; k \in \mathbb{N} \text{ free, i.e.}$$

$$\pi\mathbf{R} \vdash \mathrm{Con}_{\mathbf{PR}\mathbb{X}} \quad \textbf{q.e.d.}$$

So "slightly" strengthened theory $\pi\mathbf{R} = \mathbf{PR}\mathbb{X}\mathbf{a} + (\pi)$ derives free variables Consistency Formula for theory $\mathbf{PR}\mathbb{X}$ of primitive recursion.

Scheme (π) holds in **set** theory, since there $O := \mathbb{N}[\omega]$ is an *ordinal,* not quite to identify with *set theoretical ordinal* ω^ω, because classical ordinal addition on that ordinal ω^ω does not commute, e.g. classically $\omega + 1 \neq 1 + \omega = \omega$. As linear *orders* (with non-infinite descent) the two are identical.

As is well known, consistency provability and *soundness* of a theory are strongly tied together. We get in fact even

Theorem on πR-framed objective soundness of theory $\mathbf{PR}\mathbb{X}\mathbf{a}$:

- for a $\mathbf{PR}\mathbb{X}\mathbf{a}$ predicate $\chi = \chi(a) : A \to 2$ we have

$$\pi\mathbf{R} \vdash \mathrm{Prov}_{\mathbf{PR}\mathbb{X}}(k, \ulcorner \chi \urcorner) \implies \chi(a) : \mathbb{N} \times A \to 2.$$

- more general, for $\mathbf{PR}\mathbb{X}\mathbf{a}$-maps $f,\, g : A \to B$ we have

$$\pi\mathbf{R} \vdash \ulcorner f \urcorner \check{\doteq}_k \ulcorner g \urcorner \implies f(a) \doteq g(a).$$

[Same for **set** theory $\mathbf{T}$ taken as frame]

Proof of first assertion is a slight generalisation of proof of *framed Internal Consistency* above as follows—take predicate χ instead of false :

Use *termination-conditioned soundness,* assertion (iii) directly:

Evaluation-effective internal provabiliity of $\ulcorner \chi \urcorner$ within **PR$\mathbb{X}$a**—i. e. availability of an *evaluation-terminating* internal *deduction tree* of $\ulcorner \chi \urcorner$—*implies* $\chi(a), a \in A$ free :

$$\mathbf{PR}\mathbb{X}\mathbf{a},\ \pi\mathbf{R} \vdash \mathrm{Prov}_{\mathbf{PR}\mathbb{X}}(k,\ \ulcorner \chi \urcorner\,) \wedge\ c_d\ e_d^m(\mathrm{dtree}_k/\langle 0 \rangle) \doteq 0 \\ \implies \chi(a) : \mathbb{N}^2 \times A \to \mathbb{2}.$$

Boolean free-variables calculus, tautology

$$[\alpha \wedge \beta \Rightarrow \gamma] = [\neg\,[\alpha \Rightarrow \gamma] \Rightarrow \neg\beta]$$

(test with $\beta = 0$ as well as with $\beta = 1$),

gives from this, still with k, m, a free:

$$\pi\mathbf{R} \vdash \neg\,[\,\mathrm{Prov}_{\mathbf{PR}\mathbb{X}}(k,\ \ulcorner \chi \urcorner\,) \Rightarrow \chi(a)\,] \\ \implies c_d\ e_d^m(\mathrm{dtree}_k/\langle 0 \rangle) > 0 : (A \times \mathbb{N}) \times \mathbb{N} \to \mathbb{2}.$$

As before, we apply non-infinit scheme (π) to ev_d, in combination with—choice of—*overall "negative"* condition

$$\psi = \psi(k,a) := \neg\,[\,\mathrm{Prov}_{\mathbf{PR}\mathbb{X}}(k,\ \ulcorner \chi \urcorner\,) \Rightarrow \chi(a)\,] : \mathbb{N} \times A \to \mathbb{2},$$

and get—scheme (π)—overall negation of this (overall) *excluded* predicate ψ, namely

$$\pi\mathbf{R} \vdash\ \mathrm{Prov}_{\mathbf{PR}\mathbb{X}}(k,\ \ulcorner \chi \urcorner\,) \implies \chi(a) : \mathbb{N} \times A \to \mathbb{2}.$$

q.e.d. for first assertion.

For **proof** of second assertion, take in the above

$$\chi = \chi(a) := [\,f(a) \doteq g(a)\,] : A \to B^2 \to \mathbb{2}$$

and get

$$
\begin{aligned}
\pi\mathbf{R} \vdash\ & \ulcorner f \urcorner \check{\doteq}_k \ulcorner g \urcorner \\
& \implies \mathrm{Prov}_{\mathbf{PR}\mathbb{X}}(j(k), \ulcorner f \doteq g \urcorner\,) \\
& \quad \text{(substitutivity into } \doteq\text{)} \\
& \implies [\, f(a) \doteq g(a)\,] : \mathbb{N} \times A \to \mathbb{2} \quad \textbf{q.e.d.}
\end{aligned}
$$

7.3 $\pi\mathbf{R}$ decision

As the kernel of decision for p. r. predicate $\chi = \chi(a) : A \to \mathbb{2}$ *by* theory $\pi\mathbf{R}$ we introduce a (partially defined) μ-recursive *decision algorithm* $\nabla\chi = \nabla^{\mathrm{PR}}\chi : \mathbb{1} \rightharpoonup \mathbb{2}$ for (individual) χ. This decision algorithm is viewed as a map of theory $\pi\widehat{\mathbf{R}}$, of *partial* $\pi\mathbf{R}$ maps.

As a *partial* p. r. map it is given—see chapter 2—by three (PR) data:

- its index domain $D = D_{\nabla\chi}$, typically (and here): $D \subseteq \mathbb{N}$,
- its enumeration $d = d_{\nabla\chi} : D \to \mathbb{1}$ of its *defined arguments,* as well as
- its *rule* $\widehat{\nabla} = \widehat{\nabla}\chi : D \to \mathbb{2}$ mapping indices k, k' in D pointing to the same argument $d(k) \doteq d(k')$ in domain $\mathbb{1}$, to the same *value* $\widehat{\nabla}(k) \doteq \widehat{\nabla}(k')$.

Now **define** alleged decision algorithm by fixing its *graph*

$$\nabla\chi = \langle (d, \widehat{\nabla}) : D \to \mathbb{1} \times \mathbb{2} \rangle : \mathbb{1} \rightharpoonup \mathbb{2}$$

as follows:

Enumeration *domain for defined arguments* is to be

$$D = D_{\nabla\chi} \ =_{\text{def}} \ \{k \,|\, \neg\,\chi\,\text{ct}_A(k) \ \vee \ \text{Prov}_{\mathbf{PR}\mathbb{X}}(k, \ulcorner\chi\urcorner)\} \subset \mathbb{N},$$

with $\text{ct}_A : \mathbb{N} \to A$ (retractive) Cantor count, A assumed pointed.

Defined arguments *enumeration* is here "simply"

$$d \ =_{\text{def}} \ \Pi : D \xrightarrow{\subseteq} \mathbb{N} \xrightarrow{\Pi} \mathbb{1}$$

—not a priori a retraction or empty—, and *rule* is taken

$$\widehat{\nabla}(k) = \widehat{\nabla}\chi(k) \ =_{\text{def}} \ \begin{cases} \text{false if } \neg\,\chi\,\text{ct}_A(k), \\ \text{true if } \text{Prov}_{\mathbf{PR}\mathbb{X}}(k, \ulcorner\chi\urcorner) \end{cases} : D \to \mathbb{2}.$$

$\widehat{\nabla} : D \to \mathbb{2}$ is in fact a well defined *rule* for *enumeration* $d : D \to \mathbb{N} \to \mathbb{1}$ of *defined argument(s)* since by (earlier) *framed logical soundness theorem*

$$\pi\mathbf{R} \vdash \ \text{Prov}_{\mathbf{PR}\mathbb{X}}(k, \ulcorner\chi\urcorner) \implies \chi(a) : \mathbb{N} \times A \to \mathbb{2},$$

whence disjointness of the alternative within $D = D_{\nabla\chi}$.

This taken together means intuitively within $\pi\mathbf{R}$—and formally within **set** theory $\mathbf{T}$:

$$\nabla(k) = \nabla\chi(k) = \begin{cases} \text{false if } \neg\,\chi\,\text{ct}_A(k), \\ \text{true if } \text{Prov}_{\mathbf{PR}\mathbb{X}}(k, \ulcorner\chi\urcorner), \\ \textit{undefined otherwise.} \end{cases}$$

We have the following complete—metamathematical—**case distinction** on $D \subset \mathbb{N}$:

- **1st case,** termination: D has at least one ("total") PR point $\mathbb{1} \to D \subseteq \mathbb{N}$, and hence

$$t = t_{\nabla\chi} \ =_{\text{by def}} \ \mu D = \min\, D : \mathbb{1} \to D$$

is a (total) p. r. point.

Subcases:

  - **1.1st,** negative (total) **subcase:**
    $\neg\,\chi\,\mathrm{ct}_A(t) = \mathrm{true}.$
    [Then $\pi\mathbf{R} \vdash \ \nabla\chi = \mathrm{false}.$]
  - **1.2nd,** positive (total) **subcase:**
    $\mathrm{Prov}_{\mathbf{PR}\mathbb{X}}(t, \ulcorner\chi\urcorner) = \mathrm{true}.$
    [Then $\pi\mathbf{R} \vdash \ \nabla\chi = \mathrm{true},$
    by $\pi\mathbf{R}$-framed objective soundness of $\mathbf{PR}\mathbb{X}$.]

    These two subcases are ***disjoint,*** disjoint here by $\pi\mathbf{R}$ framed soundness of theory $\mathbf{PR}\mathbb{X}$ which reads

$$\begin{aligned} \pi\mathbf{R} \vdash \mathrm{Prov}_{\mathbf{PR}\mathbb{X}}(k, \ulcorner\chi\urcorner) \implies \chi(a) : \\ \mathbb{N} \times A \to \mathbb{2},\ k \in \mathbb{N} \text{ free, and } a \in A \text{ free,} \end{aligned}$$

    here in particular—substitute $t : \mathbb{1} \to \mathbb{N}$ into k free:

$$\pi\mathbf{R} \vdash \mathrm{Prov}_{\mathbf{PR}\mathbb{X}}(t, \ulcorner\chi\urcorner) \implies \chi(a) : A \to \mathbb{2},\ a \text{ free.}$$

    So furthermore, by this framed soundness, in present **subcase:**

$$\pi\mathbf{R} \vdash \chi(a) \wedge \mathrm{Prov}_{\mathbf{PR}\mathbb{X}}(t, \ulcorner\chi\urcorner) : A \to \mathbb{2}.$$

- **2nd case,** derived non-termination:

  $\pi\mathbf{R} \vdash\ D = \emptyset_{\mathbb{N}} \equiv \{\mathbb{N} \,|\, \mathrm{false}_{\mathbb{N}}\} \subset \mathbb{N}$

  [then in particular $\pi\mathbf{R} \vdash\ \neg\,\chi = \mathrm{false}_A : A \to 2$,

  so $\pi\mathbf{R} \vdash \chi$ in this case],

  and

  $\pi\mathbf{R} \vdash\ \neg\,\mathrm{Prov}_{\mathbf{PR}\mathbb{X}}(k, \ulcorner\chi\urcorner) : \mathbb{N} \to 2$, k free;

- **3rd,** remaining, *ill* **case** is:

  D (metamathematically) *has no (total) points* $\mathbb{1} \to D$, *but is nevertheless not empty.*

Take in the above the **(disjoint) union** of **2nd subcase** of **1st case** and of **2nd case,** last assertion. And formalise last, remaining case frame $\pi\mathbf{R}$. **Arrive at** the following

Quasi-Decidability Theorem: p. r. predicates $\chi : A \to 2$ give rise within theory $\pi\mathbf{R}$ to the following **complete (metamathematical) case distinction:**

(a) $\pi\mathbf{R} \vdash \chi : A \to 2$ **or else**

(b) $\pi\mathbf{R} \vdash \neg\,\chi\,\mathrm{ct}_A\, t : \mathbb{1} \to D_{\nabla\chi} \to 2$

(*defined counterexample*), **or else**

(c) $D = D_{\nabla\chi}$ *non-empty, pointless,* formally: in this **case** we would have within $\pi\mathbf{R}$:

$[D \,\widehat{\circ}\, \mu D \,\widehat{=}\, \mathrm{true} : \mathbb{1} \rightharpoonup \mathbb{N} \to 2]$

and "nevertheless" for each p. r. point $p : \mathbb{1} \to \mathbb{N}$

$\neg\, D \circ p = \mathrm{true} : \mathbb{1} \to \mathbb{N} \to 2.$

We **rule out** the latter—general—possibility of a *non-empty, pointless* predicate, for quantified arithmetical frame theory $\mathbf{T}$ by gödelian ***assumption*** of ω-consistency which rules out above instance of ω-*inconsistency.*

For frame $\pi\mathbf{R}$ we rule it out by (corresponding) metamathematical ***assumption*** of "μ-consistency," as follows:

Intermission on two variants of ω-consistency:
Gödelian ***assumption*** of ω-consistency—non-ω-inconsistency—for a *quantified* arithmetical theory $\mathbf{T}$ reads:

For **no** p. r. predicate $\varphi : \mathbb{N} \to \mathbb{2}$

$$\begin{aligned} &\mathbf{T} \vdash \ (\exists\, n \in \mathbb{N})\, \varphi(n) \\ &\quad \textbf{and } \text{(nevertheless)} \\ &\mathbf{T} \vdash \neg\, \varphi(0),\ \neg\, \varphi(1),\ \neg\, \varphi(2),\ \ldots \end{aligned}$$

Adaptation to (categorical) **recursive** theory $\pi\mathbf{R}$ is the following ***assumption*** of μ-consistency, non-μ-inconsistency for $\pi\mathbf{R}$:

For **no** p. r. predicate $\varphi : \mathbb{N} \to \mathbb{2}$

$$\begin{aligned} &\pi\mathbf{R} \vdash \ \varphi(\mu\varphi) \ =_{\text{by def}} \ \varphi \,\widehat{\circ}\, \mu\varphi \mathrel{\widehat{=}} \text{true} : \mathbb{1} \rightharpoonup \mathbb{2} \\ &\qquad \text{and} \\ &\pi\mathbf{R} \vdash \neg\, \varphi(0),\ \neg\, \varphi(1),\ \ldots\ ,\ \neg\, \varphi(\text{num}(\underline{n})),\ \ldots \end{aligned}$$

For quantified $\mathbf{T}$ first line reads: $\mathbf{T} \vdash \exists n\, \varphi(n)$, and hence μ-consistency is equivalent to gödelian ω-consistency for such $\mathbf{T}$.

Alternative to μ-consistency: π-consistency.

By assertion (iii) of **Structure theorem** in chapter 2—*section lemma*—for theories $\widehat{\mathbf{S}}$ of partial p. r. maps, first factor $\mu\varphi : \mathbb{1} \rightharpoonup \mathbb{N}$ of (total) p. r. map true : $\mathbb{1} \to \mathbb{2}$ above is necessarily itself a—*totally defined*—PR map: Intuitively, a first factor of a total map cannot have undefined arguments, since these would be undefined for the composition.

Now consider—here available—(external) point evaluation into numerals[12], externalisation of objective evaluation

$$ev : \lceil \mathbb{1}, \mathbb{N} \rceil \xrightarrow{\cong} \lceil \mathbb{1}, \mathbb{N} \rceil \times \mathbb{1} \overset{ev}{\rightharpoonup} \mathbb{N} \xrightarrow{\cong} \nu\mathbb{N} \subseteq \lceil \mathbb{1}, \mathbb{N} \rceil$$

of point codes into (internal) numerals, $ev(u) \check{=} u \in \lceil \mathbb{1}, \mathbb{N} \rceil$.

This externalised evaluation $\underline{ev}$ is ***assumed***—meta-**axiom** of π-consistency—to (correctly) terminate:

$$\pi\mathbf{R}(\mathbb{1}, \mathbb{N}) \supset \mathrm{num}\,\underline{\mathbb{N}} \ni \underline{ev}(p) =^{\pi}\ p \in \pi\mathbf{R}(\mathbb{1}, \mathbb{N}).$$

Comment: π-consistency means *Semantical Completeness* of descent axiom (π), this axiom is modeled into the external world of p. r. Metamathematic. But π-consistency is somewhat stronger: it assumes termination of $\underline{ev}$ instead of non-infinite descent.

Non-μ-inconsistency (of $\pi\mathbf{R}$) is then a consequence of π-consistency of theory $\pi\mathbf{R}$ above:

$$\begin{aligned} &\pi\mathbf{R} \vdash\ \mathrm{true} = \varphi(\mu\varphi) = \varphi \,\widehat{\circ}\, \mu\varphi = \varphi \circ \mu\varphi : \mathbb{1} \to \mathbb{N} \to \mathbb{2} \\ &\text{entails } \pi\mathbf{R} \vdash\ \neg\,(\neg\,\varphi(\mathrm{num}(\underline{n}_0))), \text{with}\ \ \underline{ev}(\mu\varphi) = \mathrm{num}(\underline{n}_0). \end{aligned}$$

End of Intermission.

[12] Lassmann 1981

First **consequence:** Theory $\pi\mathbf{R}$ admits **no** non-empty predicative subset $\{n \in \mathbb{N} \,|\, \varphi(n)\} \subseteq \mathbb{N}$ such that for each numeral $\mathrm{num}(\underline{n}) : \mathbb{1} \to \mathbb{N}$

$$\pi\mathbf{R} \vdash \neg\,\varphi \circ \mathrm{num}(\underline{n}) : \mathbb{1} \to \mathbb{N} \to \mathbb{2}.$$

This rules out—in *quasi-decidability* above—possibility (c) for decision domain $D = D_{\nabla_\chi} \subseteq \mathbb{N}$ of decision operator ∇_χ for predicate $\chi : A \to \mathbb{2}$, and we get two unexpected results:

Decidability theorem: Each free-variable p. r. predicate $\chi : A \to \mathbb{2}$ gives rise to the following **complete case distinction** within, by $\pi\mathbf{R}$:

- Under ***assumption*** of μ-consistency or π-consistency for $\pi\mathbf{R}$:
  - $\pi\mathbf{R} \vdash \ \chi(a) : A \to \mathbb{2}$ (*theorem*) **or**
  - $\pi\mathbf{R} \vdash \neg\,\chi\,\mathrm{ct}_A\,\mu D : \mathbb{1} \to D_{\nabla_\chi} \to \mathbb{2}$
    (*defined counterexample.*)
- Under ***assumption*** of ω-consistency for **set** theory $\mathbf{T}$:
  - $\mathbf{T} \vdash \ \chi(a) : A \to \mathbb{2}$ (*theorem*) **or**
  - $\mathbf{T} \vdash \neg\,\chi\,\mathrm{ct}_A\,\mu D : \mathbb{1} \to D_{\nabla_\chi} \to \mathbb{2}$, i. e.
    $\mathbf{T} \vdash (\exists\, a \in A)\,\neg\,\chi(a)$.

Take here, in case of **set** theory $\mathbf{T}$, for predicate χ, $\mathbf{T}$'s own free-variable consistency formula $\mathrm{Con}_\mathbf{T} = \neg\,\mathrm{Prov}_\mathbf{T}(k, \ulcorner\mathrm{false}\urcorner\,) : \mathbb{N} \to \mathbb{2}$, and get, under ***assumption*** of ω-consistency for $\mathbf{T}$, **consistency decidability** for $\mathbf{T}$.

This contradiction to (the postcedent) of Gödel's **2nd Incompleteness theorem** shows that the ***assumption*** of ω-Consistency for **set** theories $\mathbf{T}$ must fail.

Now take in the theorem for χ $\pi\mathbf{R}$'s own free variable PR consistency formula

$$\mathrm{Con}_{\pi\mathbf{R}} = \neg\,\mathrm{Prov}_{\pi\mathbf{R}}(k,\ \ulcorner \mathrm{false} \urcorner\,) : \mathbb{N} \to \mathbb{2} \text{ and get}$$

Consistency Decidability for descent theory $\pi\mathbf{R}$:

- $\pi\mathbf{R} \vdash \mathrm{Con}_{\pi\mathbf{R}} : \mathbb{1} \to \mathbb{2}$ **or else**
- $\pi\mathbf{R} \vdash \neg\,\mathrm{Con}_{\pi\mathbf{R}}$, will say

  $\pi\mathbf{R} \vdash\ \mathrm{Prov}_{\pi\mathbf{R}}(\mu\,\mathrm{Prov}_{\pi\mathbf{R}}(k,\ \ulcorner \mathrm{false} \urcorner\,),\ \ulcorner \mathrm{false} \urcorner\,) = \mathrm{true}$ **q.e.d.**

Consistency provability theorem: $\pi\mathbf{R} \vdash \mathrm{Con}_{\pi\mathbf{R}}$, under ***assumption*** of π-consistency of theory $\pi\mathbf{R}$.

Proof: Suppose we have 2nd alternative in *consistency decidability* above,

$$\pi\mathbf{R} \vdash\ \mathrm{Prov}_{\pi\mathbf{R}}(t,\ \ulcorner \mathrm{false} \urcorner\,),$$

$t\ =_{\mathrm{def}}\ \mu\,\mathrm{Prov}_{\pi\mathbf{R}}(k,\ \ulcorner \mathrm{false} \urcorner\,) : \mathbb{1} \to \mathbb{N}$, necessarily (”total”) PR. Meta p. r. point evaluation $\underline{ev}$ would turn—π-consistency—t into a numeral $\mathrm{num}(\underline{k}_0) : \mathbb{1} \to \mathbb{N}$, $\underline{k}_0 \in \underline{\mathbb{N}}$, $\mathrm{num}(\underline{k}_0) =^{\pi} t$, hence

$$\pi\mathbf{R} \vdash \mathrm{Prov}_{\pi\mathbf{R}}(\mathrm{num}(\underline{k}_0),\ \ulcorner \mathrm{false} \urcorner\,).$$

But by derivation-into-*proof* internalisation we have

$\pi\mathbf{R} \vdash \mathrm{Prov}_{\pi\mathbf{R}}(\mathrm{num}(\underline{k}),\ \ulcorner \chi \urcorner\,)$ (only) iff $\pi\mathbf{R} \vdash_{\underline{k}} \chi$, whence we would get inconsistency $\pi\mathbf{R} \vdash_{\underline{k}_0}$ false, (and an inconsistent theory derives everything.)

This rules out in fact 2nd alternative in consistency decidability and so proves the **theorem,** here our main **goal.**

For **proof** of *soundness* of $\pi\mathbf{R}$ below we need

ν**-Lemma** for theory $\pi\mathbf{R}$:

(i) family $\nu_A : A \to [\mathbb{1}, A]_\pi = \lceil \mathbb{1}, A \rceil / \check{\doteq}^\pi$ is a natural transformation, will say

$$\begin{aligned} &(\nu_B \circ f)(a) = \nu_B(f(a)) \\ &\check{\doteq}^\pi_{k(a)} \ulcorner f \urcorner \odot \nu_A(a) && (*) \\ &= [\mathbb{1}, f]_\pi(\nu_A(a)), \\ &k(a) : A \to \mathbb{N} \text{ suitable PR.} \end{aligned}$$

As a commuting DIAGRAM:

$$\begin{array}{ccc} A \ni a & \xmapsto{\nu_A} & \nu_A(a) \in \lceil \mathbb{1}, A \rceil \\ \downarrow f & & \downarrow \lceil \mathbb{1}, f \rceil \\ & & \ulcorner f \urcorner \odot \nu_A(a) \\ & & \check{\doteq}^\pi \\ B \ni f(a) & \xmapsto{\nu_B} & \nu_B f(a) \in \lceil \mathbb{1}, B \rceil \end{array}$$

(ii) $\nu = \nu(n) : \mathbb{N} \to [\mathbb{1}, \mathbb{N}]_\pi$ is injective, i. e.

$$\nu(m) \check{\doteq}^\pi \nu(n) \implies m \doteq n.$$

(iii) same for all objects A of $\pi\mathbf{R}$: $\nu_A = \nu_A(a) : A \to [\mathbb{1}, A]_\pi$ is injective.

Proof: We show assertion (i) by structural recursion on $f : A \to B$.

anchor cases $f = \mathrm{id}_A$ as well as $f = 0 : \mathbb{1} \to \mathbb{N}$ are obvious.

anchor case $f = s : \mathbb{N} \to \mathbb{N}$:

$\nu(s(a)) \ =_{\text{by def}} \ \ulcorner s \urcorner \odot \nu(a) = [\mathbb{1}, s]\,(\nu(a)).$

Map composition $g \circ f : A \to B \to C$: combine the two commuting squares for f and for g into commuting rectangle for $g \circ f$.

cartesian Structure: use

$$\nu_{(A \times B)} \ =_{\text{by def}} \ \mathrm{ind} \circ (\nu_A \times \nu_B) :$$
$$A \times B \to \lceil \mathbb{1}, A \rceil \times \lceil \mathbb{1}, B \rceil \xrightarrow{\cong} \lceil \mathbb{1}, A \times B \rceil \to [\mathbb{1}, A \times B],$$

componentwise definition of (any) equality on cartesian product, as well as the universal properties of the cartesian product $A \times B$ and $[\mathbb{1}, A \times B] \cong [\mathbb{1}, A] \times [\mathbb{1}, B]$, projections $[\mathbb{1}, \ell], [\mathbb{1}, r]$.

Iterated $f^\S(a, n) : A \times \mathbb{N} \to A$ of (already tested) endo $f : A \to A$:

Straight forward by recursion on n, since iteration is repeated composition.

Assertion (ii) on injectivity of $\nu = \nu(n) : \mathbb{N} \to [\mathbb{1}, \mathbb{N}]_\pi$:

$$\nu(m) \check{\doteq}^\pi \nu(n) \implies \ulcorner \doteq \urcorner \odot (\nu(m) \times \nu(n)) \check{\doteq}^\pi \ulcorner \mathrm{true} \urcorner$$

by internal substitutivity into predicative equality $\doteq$

$$\iff \lceil \mathbb{1}, \doteq \rceil \circ (\nu \times \nu)(m, n) \check{\doteq}^\pi \ulcorner \mathrm{true} \urcorner$$

$$\implies \nu_{\mathbb{2}}[m \doteq n] \check{\doteq}^\pi \nu_{\mathbb{2}}(\mathrm{true})$$

by naturality of transformation ν

$$\implies m \doteq n,$$ by *self-consistency* (!) of theory $\pi\mathbf{R}$.

General ν injectivity assertion (iii) now follows from that special just above, from componentwise definition of ν—and componentwise definition of injectivity—on cartesian products (and restriction of both to predicative subobjects), via naturality of transformation $[\, \nu_A : A \to [\mathbb{1}, A]_\pi \,]_{A \in \pi\mathbf{R}}$ **q.e.d.**

This is to give self-consistency $\pi\mathbf{R} \vdash \mathrm{Con}_{\pi\mathbf{R}}$ to be **equivalent** to

Objective soundness theorem for descent theory $\pi\mathbf{R}$:

- for $\pi\mathbf{R}$-maps $f,\ g : A \to B$:

$$\pi\mathbf{R} \vdash\ [\ \ulcorner f \urcorner \check{\doteq}^\pi_k \ulcorner g \urcorner\ ] \implies f(a) \doteq_B g(a) : \mathbb{N} \times A \to \mathbb{2}.$$

- this gives in particular *logical soundness* of theory $\pi\mathbf{R}$:

  For a predicate $\chi = \chi(a) : A \to \mathbb{2}$ we have

$$\pi\mathbf{R} \vdash\ \mathrm{Prov}_{\pi\mathbf{R}}(k, \ulcorner \chi \urcorner) \implies \chi(a) : \mathbb{N} \times A \to \mathbb{2},$$

  $a \in A$ free, meaning here $\forall a$, and $k \in \mathbb{N}$ free, meaning here $\exists k$.

Proof: Granted self-consistency of theory $\pi\mathbf{R}$ means just injectivity of numeralisation

$$\nu_{\mathbb{2}} : \mathbb{2} \to [\mathbb{1}, \mathbb{2}]_\pi = \lceil \mathbb{1}, \mathbb{2} \rceil / \check{\doteq}^\pi.$$

The **Lemma** deduces that this injectivity carries over first to numeralisation $\nu_{\mathbb{N}} = \nu : \mathbb{N} \to [\mathbb{1}, \mathbb{N}]_\pi$, and then to all numeralisations

$$\nu_B : B \to [\mathbb{1}, B]_\pi, \ B \text{ a } \pi\mathbf{R} \text{ object}.$$

Now compatibility of internal composition with internal equality as well as—**Lemma** again—naturality of transformmation $\nu_A : A \to [\mathbb{1}, A]_\pi$ give

$$\begin{aligned} \pi\mathbf{R} \vdash [\ \ulcorner f \urcorner \ \check{\doteq}^\pi_k \ \ulcorner g \urcorner\ ] \\ &\implies \ulcorner f \urcorner \odot \nu_A(a) \,\check{\doteq}^\pi\, \ulcorner g \urcorner \odot \nu_A(a) \\ &\implies \nu_B(f(a)) \,\check{\doteq}^\pi\, \nu_B(g(a)) \\ &\implies f(a) \doteq g(a), \end{aligned}$$

the latter implication following from injectivity of $\nu_B : B \to [\mathbb{1}, B]_\pi$ **q.e.d.**

ω-**completeness theorem** for theory $\pi\mathbf{R}$: theory $\pi\mathbf{R}$ admits the following scheme of *test by all internal numerals:*

$$\chi = \chi(a) : A \to \mathbb{2} \text{ predicate},$$

$$k = k(a) : A \to \mathbb{N} \text{ such that}$$

$$(\omega\text{−Comp}) \quad \frac{\pi\mathbf{R} \vdash \ \mathrm{Prov}_{\pi\mathbf{R}}(k(a), \ulcorner \chi \urcorner \odot \nu_A(a)) : A \to \mathbb{2}}{\pi\mathbf{R} \vdash \ \chi : A \to \mathbb{2}.}$$

Proof: By ν naturality—within $\pi\mathbf{R}$—the antecedent gives

$$\pi\mathbf{R} \vdash \ \mathrm{Prov}_{\pi\mathbf{R}}(k'(a), \nu_{\mathbb{2}} \circ \chi(a)) : A \to \mathbb{2},$$

and from this, by $\pi\mathbf{R}$ self-consistency: injectivity of $\nu_{\mathbb{2}}$ within $\pi\mathbf{R}$,

$$\pi\mathbf{R} \vdash \chi(a) : A \to \mathbb{2} \quad \textbf{q.e.d.}$$

Interpretation: The $\nu_A(a), a \in A$ are jointly epic, νA lies *dense* in $[\mathbb{1}, A]_\pi$. theory $\pi\mathbf{R}$ is in particular internally μ-consistent, object $\mathbb{1}$ is an internal separator, all of this with respect to $\pi\mathbf{R}$ maps (on object language level). Would it work for (free variable) internal map codes either?

Question: Can we then have/assume this test to work on the external level too? can we have/assume at least object $\mathbb{1}$ to be/to become a *separator* for category $\pi\mathbf{R}$?

Attempt to an answer: logic/arithmetic externalisation of **axioms** and **theorems,** as opposite to—successfull—internalisation/arithmetisation seems me to be legitimate/consistent: both internalisation and externalisation can be seen/formalised as preserving/reflecting logical *invariants.* A theory $\mathbf{T}$ for which this is not always possible—Consistency/*consistency provability*—has a defect in this regard, it is not *sound* in the technical sense, see SMORYNSKI 1977.

Conclusion: descent theory $\pi\mathbf{R}$—in the role of metamathematic—derives its own *consistency* (formula) as well as—see below—the *inconsistency* (formulae) for **set** theories $\mathbf{T}$, the latter including Peano-arithmetic $\mathbf{PA}^+$ with order of $\mathbb{N}[\omega]$ to satisfy finite descent.

All of this under ***assumption***, meta-axiom, that theory $\pi\mathbf{R}$ is π-consistent, that it externalises its **axiom** (π) into (correct) termination of (external) evaluation $\underline{ev}$.

The $\pi\mathbf{R}$ (in part) internal version of μ-consistency, consequence of π-consistency, is ω-completeness above.

Question: Are quantified arithmetical theories $\mathbf{T}$, in particular theory $\mathbf{PA}^+$, even **inconsistent?**

By Gödel's 2nd Incompleteness theorem, first assertion, $\mathbf{T} \nvdash \mathrm{Con}_\mathbf{T}$ **if** $\mathbf{T}$ consistent, hence $\pi\mathbf{R} \nvdash \mathrm{Con}_\mathbf{T}$ if $\mathbf{T}$ consistent: this since $\mathbf{T}$ is an extension of $\pi\mathbf{R}$. But **then,** by Decidability theorem above, for $\pi\mathbf{R}$ and p. r. free-variable predicate $\mathrm{Con}_\mathbf{T} = \neg\,\mathrm{Prov}_\mathbf{T}(k, \ulcorner\mathrm{false}\urcorner) : \mathbb{N} \to \mathbb{2}$,

$$\pi\mathbf{R} \vdash \neg\,\mathrm{Con}_\mathbf{T}, \quad [\text{a fortiori } \mathbf{T} \vdash \neg\,\mathrm{Con}_\mathbf{T}.]$$

Now if we take as metamathematic the external version $\underline{\mathbf{PR}}$ of fundamental theory $\mathbf{PR}$, then the consistency questions are open.

But **if** we take as metamathematic an external version $\underline{\pi\mathbf{R}}$ of descent theory $\pi\mathbf{R}$, then we get in fact consistency of p. r. theories $\mathbf{PR}, \mathbf{PRa}, \mathbf{PR}\mathbb{X}\mathbf{a}$—and of descent theory $\pi\mathbf{R}$—as well as inconsistency of **set** theories $\mathbf{T}$.

Problems:

(1) Is axiom scheme (π) redundant, $\pi\mathbf{R} \cong \mathbf{PR}\mathbb{X}\mathbf{a}$? Certainly not, since isotonic maps from lexicographically ordered $\mathbb{N}\times\mathbb{N}, \ldots, \mathbb{N}^+ \equiv \mathbb{N}[\omega] \equiv \omega^\omega$ to $\mathbb{N}$ are not available.

(2) Can we get *internal* soundness for theory $\pi\mathbf{R}$ itself? Up to now we have only *Objective* soundness: this is the one considered by

mathematical logicians. Internal soundness (of *evaluation* versus the object language level) is a challenging open Problem with present approach.

8 Discussion (tentative)

The claim *for our* set *theories is that* $\mathbf{T}$ *proves* $\neg\,\mathrm{Con}_{\mathbf{T}}$ *which formally denies Gödel's second incompleteness theorem:*

Its second postcedent and hence the ***assumption*** of ω-consistency for **PM** and **ZF**. Gödel himself was said to be not completely convinced of this assumption.

All of our theories, in particular $\mathbf{PA} \cong \mathbf{PR}\exists$, are standard recursively axiomatized extensions of primitive recursive arithmetic **PR**. Everybody then expects for these set theories $\mathbf{T}$ ω-consistency. But this is only an ***assumption.*** Remains the possibility that this text contains a formal irreparable error. If so, where?

Axiomatisation and predicate $\mathrm{Prov}_{\mathbf{T}}$ of "being a *proof* for", are constructed in categorical parallel to Smorynski

(and to Gödels predicate 45. $x\,B\,y$, x ist ein *Beweis* für die *Formel* y, not to Rosser's $\mathrm{Prov}^{R}_{\mathbf{T}}$),

no room for *"informally motivated" formal proof predicates.*

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Address of the author:

Michael Pfender
Institut f. Mathematik MA 1-1
Technische Universitaet Berlin
Str. d. 17. Juni 136
D-10623 Berlin

michael.pfender@campus.tu-berlin.de