On some Criteria for the Riemann Hypothesis

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Abstract

We present an unifying method to prove criteria for the Riemann Hypothesis involving the arithmetical functions of Euler, von Mangoldt, Möbius and Liouville.
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1. Introduction

In this paper we present an unifying method to prove criteria for the Riemann Hypothesis involving the arithmetical functions of Euler, von Mangoldt, Möbius and Liouville, based on complex integration and Fourier analysis:

Theorem 1. *Let $(a_n)_{n\geq 1} \subset \mathbb{C}$ be an arithmetical function with the associated Dirichlet series $D(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$ absolutely convergent in the half-space $\mathrm{Re}\,(s) > l$. Let $\gamma < l$. Suppose that:*
1) The function $D(s)$ admits a meromorphic continuation in $\mathrm{Re}\,(s) \geq \gamma$ and is holomorphic in $\{\mathrm{Re}\,(s) \geq l\} \setminus \{l\}$ with residue R at $s = l$;
2) The set of poles of $D(s)$ in $\gamma \leq \mathrm{Re}\,(s) < l$ is non void. Let

$$\theta := \sup\{\mathrm{Re}\,(\rho) \mid \gamma \leq \mathrm{Re}\,(\rho) < l,\ \rho \, \textit{pole of } D(s)\}.$$

3) The function $D(s)$ is of polynomial growth in every vertical strip contained in $\mathrm{Re}\,(s) > \theta$:

$$(\forall)\theta < a < b\,(\exists) k_{a,b} > 0, C_{a,b} > 0, t_{a,b} > 0 : |D(\sigma + it)| < C_{a,b}|t|^{k_{a,b}},$$

$$\forall \sigma \in [a, b], |t| > t_{a,b}.$$

Let $f:(0,\infty)\to\mathbb{C}$ *be a continuous function such that:*
4) For every $\sigma>l$ *the function*

$$x^{\sigma}\cdot f(x)$$

is bounded on $(0,\infty)$.
5) The Mellin transform $\widehat{f}(s)=\int_0^{\infty}f(x)x^{s-1}dx$ *(which by 4) exists in* $\mathrm{Re}\,(s)>l$*) admits a meromorphic continuation in* $\mathrm{Re}\,(s)\geq\gamma$*, is holomorphic in* $\{\mathrm{Re}\,(s)>\gamma\}$ *and non vanishing in* $\gamma<\mathrm{Re}\,(s)<l$*;*
6) The function $\widehat{f}$ *is of rapid decay in every vertical strip contained in* $\mathrm{Re}\,(s)>\theta$*:*

$$(\forall)\theta<a<b,(\forall)N>0,\sup_{t\in\mathbb{R},\sigma\in[a,b]}|t|^N|\widehat{f}(\sigma+it)|<\infty.$$

Then

$$\theta=\inf\{\alpha\geq\gamma\mid\sum_{n=1}^{\infty}a_nf(nx)=\frac{R\widehat{f}(l)}{x^l}+O(\frac{1}{x^{\alpha}})\;as\,x\to 0\}.$$

Applying this to the arithmetical functions φ of Euler, Λ of von Mangoldt, μ of Möbius and λ of Liouville, using as weights the kernels $f(x)=e^{-x}$ of Abel and $f(x)=\frac{x}{e^x-1}$ of Lambert we obtain the following criteria for the Riemann hypothesis:

The following assertions are equivalent:
(i) *The Riemann Hypothesis is true;*
(ii) *For every* $\varepsilon>0$*:*

$$\sum_{n=1}^{\infty}\varphi(n)e^{-nx}=\frac{6}{\pi^2}\frac{1}{x^2}+O(\frac{1}{x^{\frac{1}{2}+\varepsilon}}),x\to 0^+;$$

(iii) *For every* $\varepsilon>0$*:*

$$\sum_{n=1}^{\infty}\varphi(n)\frac{nx}{e^{nx}-1}=\frac{12\zeta(3)}{\pi^2}\frac{1}{x^2}+O(\frac{1}{x^{\frac{1}{2}+\varepsilon}}),x\to 0^+;$$

(iv) *For every* $\varepsilon>0$*:*

$$\sum_{n=1}^{\infty}\Lambda(n)e^{-nx}=\frac{1}{x}+O(\frac{1}{x^{\frac{1}{2}+\varepsilon}}),x\to 0^+;$$

(v) *For every* $\varepsilon>0$*:*

$$\sum_{n=1}^{\infty}\Lambda(n)\frac{nx}{e^{nx}-1}=\frac{\pi^2}{6x}+O(\frac{1}{x^{\frac{1}{2}+\varepsilon}}),x\to 0^+;$$

(vi) *For every* $\varepsilon > 0$*:*

$$\sum_{n=1}^{\infty} \mu(n) e^{-nx} = O(\frac{1}{x^{\frac{1}{2}+\varepsilon}}), x \to 0^+;$$

(vii) *For every* $\varepsilon > 0$*:*

$$\sum_{n=1}^{\infty} \mu(n) \frac{nx}{e^{nx}-1} = O(\frac{1}{x^{\frac{1}{2}+\varepsilon}}), x \to 0^+;$$

(viii) *For every* $\varepsilon > 0$*:*

$$\sum_{n=1}^{\infty} |\mu(n)| e^{-nx} = \frac{6}{\pi^2} \frac{1}{x} + O(\frac{1}{x^{\frac{1}{4}+\varepsilon}}), x \to 0.$$

(ix) *For every* $\varepsilon > 0$*:*

$$\sum_{n=1}^{\infty} |\mu(n)| \frac{nx}{e^{nx}-1} = \frac{1}{x} + O(\frac{1}{x^{\frac{1}{4}+\varepsilon}}), x \to 0;$$

(x) *For every* $\varepsilon > 0$*:*

$$\sum_{n=1}^{\infty} \lambda(n) e^{-nx} = O(\frac{1}{x^{\frac{1}{2}+\varepsilon}}), x \to 0^+;$$

(xi) *For every* $\varepsilon > 0$*:*

$$\sum_{n=1}^{\infty} \lambda(n) \frac{nx}{e^{nx}-1} = O(\frac{1}{x^{\frac{1}{2}+\varepsilon}}), x \to 0^+;$$

The criterion (vi) was obtained also by Gerhold in [1], Proposition 3, P. 189.

2. Proof of Theorem 1

Proof. By the fact that $D(s)$ is absolutely convergent in $\mathrm{Re}\,(s) > l$ and by the assumption 4) the function

$$F : (0, \infty) \to \mathbb{C},$$

$$F(x) = \sum_{n=1}^{\infty} a_n f(nx)$$

is continuous and satisfies

$$F(x) = O_\sigma(x^{-\sigma}), x \in (0, \infty), \forall \sigma > l.$$

The Mellin transform $\widehat{F}(s) = \int_0^\infty F(x)x^{s-1}dx$ is defined and holomorphic in $\mathrm{Re}\,(s) > l$, and

$$\widehat{F}(s) = D(s)\widehat{f}(s).$$

The Mellin inversion formula is valid :

$$\forall \sigma > l : F(x) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \widehat{F}(s)x^{-s}ds, x > 0.$$

Let

$$\xi := \inf\{\alpha \geq \gamma \mid F(x) = \frac{R\widehat{f}(l)}{x^l} + O(\frac{1}{x^\alpha}) \,\mathrm{as}\, x \to 0\}.$$

Since $F(x)$ is $O_\sigma(x^{-\sigma})$ as $x \to 0$, for all $\sigma > l$, it follows that

$$\xi \leq l.$$

Suppose that $\theta < \xi$. Let $\alpha \in (\theta, \xi)$. Let $a > l$. From the Mellin inversion formula it follows that

$$F(x) = \frac{1}{2\pi i} \int_{a - i\infty}^{a + i\infty} D(s)\widehat{f}(s)x^{-s}ds.$$

By the assumptions 1) and 5) the function

$$s \mapsto D(s)\widehat{f}(s)x^{-s}$$

is holomorphic in $\{\mathrm{Re}\,(s) \geq \alpha\} \setminus \{l\}$ with the residue $R \cdot \widehat{f}(l) \cdot x^{-l}$ at $s = l$. Let $T > 0$. Integrating on the rectangle with the edges $a - iT$, $a + iT$, $\alpha + iT$, $\alpha - iT$, one obtains:

$$\frac{1}{2\pi i} \int_{a - iT}^{a + iT} D(s)\widehat{f}(s)x^{-s}ds + \frac{1}{2\pi i} \int_{a + iT}^{\alpha + iT} D(s)\widehat{f}(s)x^{-s}ds +$$

$$+\frac{1}{2\pi i}\int_{\alpha+iT}^{\alpha-iT} D(s)\widehat{f}(s)x^{-s}ds + \frac{1}{2\pi i}\int_{\alpha-iT}^{a-iT} D(s)\widehat{f}(s)x^{-s}ds = R\widehat{f}(l)x^{-l}.$$

From the asumptions 3) and 6) it follows that

$$\lim_{T\to\infty}\int_{a+iT}^{\alpha+iT} D(s)\widehat{f}(s)x^{-s}ds = 0$$

and that

$$\lim_{T\to\infty}\int_{\alpha-iT}^{a-iT} D(s)\widehat{f}(s)x^{-s}ds = 0,$$

hence

$$F(x) = R\widehat{f}(l)x^{-l} + \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} D(s)\widehat{f}(s)x^{-s}ds.$$

Changing variable $s = \alpha + it$ it follows that

$$\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} D(s)\widehat{f}(s)x^{-s}ds = \frac{1}{2\pi}x^{-\alpha}\int_{-\infty}^{\infty} D(\alpha+it)\widehat{f}(\alpha+it)x^{-it}dt.$$

Since by 3) and 6) the function $t \mapsto D(\alpha+it)\widehat{f}(\alpha+it)$ is in $L^1(\mathbb{R})$ one can apply the Riemann-Lebesgue theorem to obtain

$$\lim_{x\to 0}\int_{-\infty}^{\infty} D(\alpha+it)\widehat{f}(\alpha+it)x^{-it} = 0,$$

so

$$F(x) = R\widehat{f}(l)x^{-l} + o(x^{-\alpha}), x \to 0,$$

and so $\alpha \geq \xi$, a contradiction. This shows that

$$\xi \leq \theta.$$

Since, by the assumption 2), $\theta \leq l$, it follows that $\xi \leq l$. If $\xi = l$, then $\xi = \theta = l$, and the theorem is proved. If $\xi < l$, let $\alpha \in (\xi, l)$ such that

$$F(x) = R\widehat{f}(l)x^{-l} + O(x^{-\alpha}), x \to 0. \tag{1}$$

In $\mathrm{Re}\,(s) > l$ one has that:

$$\widehat{F}(s) = \int_0^\infty F(x)x^{s-1}dx.$$

Let $A > 0$. Then

$$\widehat{F}(s) = \int_0^A F(x)x^{s-1}dx + \int_A^\infty F(x)x^{s-1}dx =$$

$$= \int_0^A (R\widehat{f}(l)x^{-l} + F(x) - R\widehat{f}(l)x^{-l})x^{s-1}dx + \int_A^\infty F(x)x^{s-1}dx =$$

$$= R\widehat{f}(l)\int_0^A x^{-l}\cdot x^{s-1}dx + \int_0^A (F(x) - R\widehat{f}(l)x^{-l})x^{s-1}dx + \int_A^\infty F(x)x^{s-1}dx =$$

$$= R\widehat{f}(l)\frac{A^{s-l}}{s-l} + g(s) + h(s),$$

with

$$g(s) = \int_0^A (F(x) - R\widehat{f}(l)x^{-l})x^{s-1}dx, \mathrm{Re}\,(s) > l,$$

and

$$h(s) = \int_A^\infty F(x)x^{s-1}dx.$$

By the assumption 4) the function $h(s)$ is holomorphic in $\mathbb{C}$. By (1) there exists $B > 0$ such that

$$|F(x) - R\widehat{f}(l)x^{-l}| < Bx^{-\alpha}, \forall 0 < x \leq A,$$

hence

$$\int_0^A |(F(x) - R\widehat{f}(l)x^{-l})x^{s-1}|dx < B\int_0^A x^{-\alpha}\cdot x^{\mathrm{Re}\,s-1}dx = B\frac{A^{\mathrm{Re}\,s-\alpha}}{\mathrm{Re}\,s-\alpha},$$

so $g(s)$ is holomorphic in $\mathrm{Re}\,s > \alpha$. By the assumptions 1) and 5) the function $\widehat{F}(s)$, which equals $D(s)\widehat{f}(s)$ in $\mathrm{Re}\,(s) > l$, admits a meromorphic continuation in $\mathrm{Re}\,s > \alpha$. It follows that the equality

$$\widehat{F}(s) = R\widehat{f}(l)\frac{A^{s-l}}{s-l} + g(s) + h(s)$$

holds in $\mathrm{Re}\,s > \alpha$, so $\widehat{F}(s)$ admits a holomorphic continuation in the region $\{\mathrm{Re}\,(s) > \alpha\} \setminus \{l\}$. Since $\widehat{F}(s) = D(s)\widehat{f}(s)$ and since, by the assumption 5), $\widehat{f}(s)$ is non vanishing in $\gamma < \mathrm{Re}\,(s) < l$, the function $D(s)$ is holomorphic in $\{\alpha < \mathrm{Re}\,(s) < l\}$. From the assumption 1) it follows that $D(s)$ is holomorphic in $\{\mathrm{Re}\,(s) > \alpha\} \setminus \{l\}$, so $\theta \leq \alpha$. This shows that $\theta \leq \xi$. So $\theta = \xi$. □

3. Euler's function

Let $\varphi : \mathbb{N} \to \mathbb{C}$ be Euler's function: For $n \in \mathbb{N}$ the value $\varphi(n)$ is the number of elements in the set $\{a \in \mathbb{N} \mid 1 \leq a \leq n, (a,n) = 1\}$. Let $\zeta(s)$ be Riemann's zeta-function. The Dirichlet series of $\varphi(n)$ is

$$D(s) = \sum_{n=1}^\infty \frac{\varphi(n)}{n^s} = \frac{\zeta(s-1)}{\zeta(s)}, \mathrm{Re}\,(s) > 2.$$

This is absolutely convergent in $\mathrm{Re}\,(s) > 2$. Let $l = 2$ and $\gamma = \frac{1}{2}$. The function $D(s)$ is meromorphic in $\mathbb{C}$, holomorphic in $\{\mathrm{Re}\,(s) \geq 2\} \setminus \{2\}$, with residue $R = \frac{1}{\zeta(2)} = \frac{6}{\pi^2}$ at $s = 2$. The assumption 1) of theorem 1 is fulfilled. The set of poles of $D(s)$ in $\frac{1}{2} \leq \mathrm{Re}\,(s) < 2$ is the set of zeros of $\zeta(s)$ there, hence

$$\theta = \sup\{\mathrm{Re}\,(\rho) \mid \frac{1}{2} \leq \mathrm{Re}\,(\rho) < 2, \zeta(\rho) = 0\}.$$

The assumption 2) of theorem 1 is fulfilled. Since $\zeta(s)$ is of polynomial growth in every vertical strip contained in $\mathrm{Re}\,(s) > 0$ ([4], Chapter V) and $\frac{1}{\zeta(s)}$ is of polynomial growth in every vertical strip contained in $\mathrm{Re}\,(s) > \theta$ ([2]), the assumption 3) of theorem 1 is fulfilled.

Theorem 2. *The following assertions are equivalent:*
(i) The Riemann hypothesis is true: $\theta = \frac{1}{2}$*;*
(ii) For every $\varepsilon > 0$*:*

$$\sum_{n=1}^{\infty} \varphi(n) e^{-nx} = \frac{6}{\pi^2} \frac{1}{x^2} + O(\frac{1}{x^{\frac{1}{2}+\varepsilon}}), x \to 0;$$

(iii) For every $\varepsilon > 0$*:*

$$\sum_{n=1}^{\infty} \varphi(n) \frac{nx}{e^{nx} - 1} = \frac{12\zeta(3)}{\pi^2} \frac{1}{x^2} + O(\frac{1}{x^{\frac{1}{2}+\varepsilon}}), x \to 0.$$

Proof. (i) equivalent to (ii): Let $f : (0, \infty) \to \mathbb{C}$,

$$f(x) = e^{-x},$$

be Abel's kernel. One has that

$$f(x) = O(1), x \to 0,$$

and that

$$f(x) = O(x^{-\sigma}), x \to \infty, \forall \sigma > 0,$$

hence the assumption 4) of theorem 1 is fulfilled. The Mellin transform

$$\widehat{f}(s) = \int_0^{\infty} f(x) x^{s-1} dx = \Gamma(s)$$

is Euler's Γ-function, which is holomorphic and non-vanishing in $\mathrm{Re}\,(s) > 0$, hence the assumption 5) of theorem 1) is fulfilled. One has that $\widehat{f}(2) = 1$. By Stirling's formula the assumption 6) of theorem 1 is fulfilled too. Apply

theorem 1.
(i) equivalent to (iii): Let $f:(0,\infty)\to\mathbb{C}$,

$$f(x)=\frac{x}{e^x-1},$$

be Lambert's kernel. One has that

$$f(x)=O(1), x\to 0,$$

and that

$$f(x)=O(x^{-\sigma}), x\to\infty, \forall\sigma>0,$$

hence the assumption 4) of theorem 1 is fulfilled. The Mellin transform

$$\widehat{f}(s)=\int_0^\infty f(x)x^{s-1}dx=\int_0^\infty \frac{x^s}{e^x-1}dx=\Gamma(s+1)\zeta(s+1)$$

is holomorphic and non-vanishing in $\mathrm{Re}\,(s)>0$, hence the assumption 5) of theorem 1) is fulfilled. One has that $\widehat{f}(2)=2\zeta(3)$. By Stirling's formula and by the fact that $\zeta(s)$ is of polynomial growth in every vertical strip of the half plane $\mathrm{Re}\,(s)>\frac{1}{2}$ the assumption 6) of theorem 1 is fulfilled. Apply theorem 1. □

Remark. There exist functions $f:(0,\infty)\to\mathbb{C}$ whose error term e_f in the formula

$$F(x):=\sum_{n=1}^\infty\varphi(n)f(nx)=\frac{6}{\pi^2}\widehat{f}(2)\frac{1}{x^2}+e_f(x)$$

is $O(\frac{1}{x})$, as $x\to 0$. Consider, for instance, the function:

$$f(x)=\frac{e^{-x}}{1-e^{-x}},\qquad x>0.$$

The Mellin transform

$$\widehat{f}(s)=\int_0^\infty f(x)x^{s-1}dx=\int_0^\infty \frac{x^{s-1}}{e^x-1}dx=\Gamma(s)\zeta(s)$$

is holomorphic in $\{\mathrm{Re}\,(s)>0\}\setminus\{1\}$, with a pole at $s=1$, and vanishes exactly at the zeros of $\zeta(s)$, so the non-vanishing of $\widehat{f}(s)$ in $\frac{1}{2}<\mathrm{Re}\,(s)<1$ is equivalent to the Riemann hypothesis ! One has $\widehat{f}(2)=\Gamma(2)\zeta(2)=\frac{\pi^2}{6}$. Then

$$F(x)=\sum_{n=1}^\infty\varphi(n)f(nx)=\sum_{n=1}^\infty\varphi(n)\frac{1}{e^{nx}-1}=\sum_{n=1}^\infty\varphi(n)\frac{e^{-nx}}{1-e^{-nx}}=$$

$$= \sum_{n=1}^{\infty} \varphi(n) \sum_{k=1}^{\infty} e^{-knx} = \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \varphi(n) e^{-knx} = \sum_{n=1}^{\infty} (\sum_{d|n} \varphi(d)) e^{-nx} =$$

$$= \sum_{n=1}^{\infty} n e^{-nx} = \sum_{n=1}^{\infty} (-e^{-nx})' = (-\sum_{n=1}^{\infty} e^{-nx})' = (\frac{e^{-x}}{e^{-x}-1})' = \frac{e^x}{(e^x-1)^2} = K(e^x),$$

where $K(z) = \frac{z}{(1-z)^2}$ is Koebe's function, and

$$e_f(x) = m_x(f) - \frac{6}{\pi^2} \widehat{f}(2) \frac{1}{x^2} = \frac{e^x}{(e^x-1)^2} - \frac{1}{x^2} = O(\frac{1}{x}), \text{ as } x \to 0,$$

but one cannot apply theorem 1 to conclude that the Riemann hypothesis is true.

4. Von Mangoldt's function

Let $\Lambda : \mathbb{N} \to \mathbb{C}$ be von Mangoldt's function: If $n = p^k$ with p a prime number and $k \geq 1$ then $\Lambda(n) = \log(p)$, and $\Lambda(n) = 0$ if n is not a prime power. The Dirichlet series of $\Lambda(n)$ is

$$D(s) = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} = -\frac{\zeta'(s)}{\zeta(s)}, \operatorname{Re}(s) > 1.$$

This is absolutely convergent in $\operatorname{Re}(s) > 1$. Let $l = 1$ and $\gamma = \frac{1}{2}$. The function $D(s)$ is meromorphic in $\mathbb{C}$ and, by the Prime Number Theorem, it has in $\operatorname{Re}(s) \geq 1$ exactly one pole, at $s = 1$, with residue $R = 1$. The assumption 1) of theorem 1 is fulfilled. The set of poles of $D(s)$ in $\frac{1}{2} \leq \operatorname{Re}(s) < 1$ is the set of zeros of $\zeta(s)$ there, hence

$$\theta = \sup\{\operatorname{Re}(\rho) \mid \frac{1}{2} \leq \operatorname{Re}(\rho) < 1, \zeta(\rho) = 0\}.$$

The assumption 2) of theorem 1 is fulfilled. The assumption 3) of theorem 1 is fulfilled ([3], p. 173, estimation (19)).

Theorem 3. *The following assertions are equivalent:*
(i) The Riemann hypothesis is true: $\theta = \frac{1}{2}$*;*
(ii) For every $\varepsilon > 0$*:*

$$\sum_{n=1}^{\infty} \Lambda(n) e^{-nx} = \frac{1}{x} + O(\frac{1}{x^{\frac{1}{2}+\varepsilon}}), x \to 0;$$

(iii) For every $\varepsilon > 0$:

$$\sum_{n=1}^{\infty} \Lambda(n) \frac{nx}{e^{nx}-1} = \frac{\pi^2}{6x} + O(\frac{1}{x^{\frac{1}{2}+\varepsilon}}), x \to 0.$$

Proof. (i) equivalent to (ii): Apply theorem 1 for $f(x) = e^{-x}$ and $l = 1$. In this case $\widehat{f}(1) = \Gamma(1) = 1$.
(i) equivalent to (iii): Apply theorem 1 for $f(x) = \frac{x}{e^x-1}$ and $l = 1$. In this case $\widehat{f}(1) = \Gamma(2)\zeta(2) = \frac{\pi^2}{6}$. □

5. Möbius' function

Let $\mu : \mathbb{N} \to \mathbb{C}$ be Möbius' function: $\mu(1) = 1$, $\mu(n) = (-1)^r$ if n is a product of $r \geq 1$ distinct primes, $\mu(n) = 0$ if n is divisible by a square of a prime number. The Dirichlet series of $\mu(n)$ is

$$D(s) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)}, \ \mathrm{Re}\,(s) > 1.$$

This is absolutely convergent in $\mathrm{Re}\,(s) > 1$. Let $l = 1$ and $\gamma = \frac{1}{2}$. The function $D(s)$ is meromorphic in $\mathbb{C}$ and, by the Prime Number Theorem, it has in $\mathrm{Re}\,(s) \geq 1$ no pole. The assumption 1) of theorem 1 is fulfilled. The set of poles of $D(s)$ in $\frac{1}{2} \leq \mathrm{Re}\,(s) < 1$ is the set of zeros of $\zeta(s)$ there, hence

$$\theta = \sup\{\mathrm{Re}\,(\rho) \mid \frac{1}{2} \leq \mathrm{Re}\,(\rho) < 1, \zeta(\rho) = 0\}.$$

The assumption 2) of theorem 1 is fulfilled. The assumption 3) of theorem 1 is fulfilled.

Theorem 4. *The following assertions are equivalent:*
(i) The Riemann hypothesis is true: $\theta = \frac{1}{2}$;
(ii) For every $\varepsilon > 0$:

$$\sum_{n=1}^{\infty} \mu(n) e^{-nx} = O(\frac{1}{x^{\frac{1}{2}+\varepsilon}}), x \to 0.$$

(iii) For every $\varepsilon > 0$:

$$\sum_{n=1}^{\infty} \mu(n) \frac{nx}{e^{nx}-1} = O(\frac{1}{x^{\frac{1}{2}+\varepsilon}}), x \to 0.$$

Proof. (i) equivalent to (ii): Apply theorem 1 for $f(x) = e^{-x}$ and $l = 1$, $R = 0$.
(i) equivalent to (iii): Apply theorem 1 for $f(x) = \frac{x}{e^x-1}$ and $l = 1$, $R = 0$. □

Remark. Let $f : (0, \infty) \to \mathbb{C}$,

$$f(x) = \frac{1}{e^x - 1}.$$

The Mellin transform $\widehat{f}(s) = \Gamma(s)\zeta(s)$ is holomorphic in $\{\mathrm{Re}\,(s) > 0\} \setminus \{1\}$, with a pole at $s = 1$, and vanishes exactly at the zeros of $\zeta(s)$. One has that

$$\sum_{n=1}^{\infty} \mu(n) f(nx) = \sum_{n=1}^{\infty} \mu(n) \frac{1}{e^{nx} - 1} = \sum_{n=1}^{\infty} \mu(n) \frac{e^{-nx}}{1 - e^{-nx}} =$$

$$= \sum_{n=1}^{\infty} \mu(n) \sum_{k=1}^{\infty} e^{-knx} = \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \mu(n) e^{-knx} = \sum_{n=1}^{\infty} (\sum_{d|n} \mu(d)) e^{-nx} = e^{-x}$$

which is $O(1)$ as $x \to 0$, but one cannot apply theorem 1 to conclude that the Riemann hypothesis is true.

The Dirichlet series of $|\mu(n)|$ is

$$D(s) = \sum_{n=1}^{\infty} \frac{|\mu(n)|}{n^s} = \frac{\zeta(s)}{\zeta(2s)},\ \mathrm{Re}\,(s) > 1.$$

This is absolutely convergent in $\mathrm{Re}\,(s) > 1$. Let $l = 1$ and $\gamma = \frac{1}{4}$. The function $D(s)$ is meromorphic in $\mathbb{C}$. In $\mathrm{Re}\,(s) \geq 1$ it has exactly one pole, at $s = 1$, with residue $R = \frac{1}{\zeta(2)} = \frac{6}{\pi^2}$. The assumption 1) of theorem 1 is fulfilled. The set of poles of $D(s)$ in $\frac{1}{4} \leq \mathrm{Re}\,(s) < 1$ is the set of zeros of $\zeta(2s)$ there, hence

$$\theta = \sup\{\mathrm{Re}\,(\rho) \mid \frac{1}{4} \leq \mathrm{Re}\,(\rho) < 1, \zeta(2\rho) = 0\}.$$

The assumption 2) of theorem 1 is fulfilled. The assumption 3) of theorem 1 is fulfilled.

Theorem 5. *The following assertions are equivalent:*
(i) The Riemann hypothesis is true: $\theta = \frac{1}{4}$;
(ii) For every $\varepsilon > 0$:

$$\sum_{n=1}^{\infty} |\mu(n)| e^{-nx} = \frac{6}{\pi^2} \frac{1}{x} + O(\frac{1}{x^{\frac{1}{4}+\varepsilon}}), x \to 0.$$

(iii) For every $\varepsilon > 0$:

$$\sum_{n=1}^{\infty} |\mu(n)| \frac{nx}{e^{nx}-1} = \frac{1}{x} + O(\frac{1}{x^{\frac{1}{4}+\varepsilon}}), x \to 0.$$

Proof. (i) equivalent to (ii): Apply theorem 1 for $f(x) = e^{-x}$ and $l = 1$.
(i) equivalent to (iii): Apply theorem 1 for $f(x) = \frac{x}{e^x-1}$ and $l = 1$. □

6. Liouville's function

Let $\lambda : \mathbb{N} \to \mathbb{C}$ be Liouville's function: $\lambda(1) = 1$, and $\lambda(n) = (-1)^{k_1+\ldots+k_r}$ if the prime factorization of n is $n = p_1^{k_1} \cdot \ldots \cdot p_r^{k_r}$, $k_1 > 0, \ldots, k_r > 0$. The Dirichlet series of $\lambda(n)$ is

$$D(s) = \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s} = \frac{\zeta(2s)}{\zeta(s)}, \mathrm{Re}\,(s) > 1.$$

This is absolutely convergent in $\mathrm{Re}\,(s) > 1$. Let $l = 1$ and $\gamma = \frac{1}{2}$. The function $D(s)$ is meromorphic in $\mathbb{C}$ and, by the Prime Number Theorem, it has in $\mathrm{Re}\,(s) \geq 1$ no pole. The assumption 1) of theorem 1 is fulfilled. The set of poles of $D(s)$ in $\{\frac{1}{2} \leq \mathrm{Re}\,(s) < 1\} \setminus \{\frac{1}{2}\}$ is the set of zeros of $\zeta(s)$ there, hence

$$\theta = \sup\{\mathrm{Re}\,(\rho) \mid \frac{1}{2} \leq \mathrm{Re}\,(\rho) < 1, \zeta(\rho) = 0\}.$$

The assumption 2) of theorem 1 is fulfilled. The assumption 3) of theorem 1 is fulfilled.

Theorem 6. *The following assertions are equivalent:*
(i) The Riemann hypothesis is true: $\theta = \frac{1}{2}$;
(ii) For every $\varepsilon > 0$:

$$\sum_{n=1}^{\infty} \lambda(n) e^{-nx} = O(\frac{1}{x^{\frac{1}{2}+\varepsilon}}), x \to 0;$$

(iii) For every $\varepsilon > 0$:

$$\sum_{n=1}^{\infty} \lambda(n) \frac{nx}{e^{nx}-1} = O(\frac{1}{x^{\frac{1}{2}+\varepsilon}}), x \to 0.$$

Proof. (i) equivalent to (ii): Apply theorem 1 for $f(x) = e^{-x}$ and $l = 1$, $R = 0$.
(i) equivalent to (iii): Apply theorem 1 for $f(x) = \frac{x}{e^x-1}$ and $l = 1$, $R = 0$. □

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