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A flow-on-manifold formulation of differential-algebraic equations

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Abstract

We derive a flow formulation of differential-algebraic equations (DAEs), implicit differential equations whose dynamics are restricted by algebraic constraints. Using the framework of derivatives arrays and the strangeness-index, we identify the systems that are uniquely solvable on a particular set of initial values and thus possess a flow, the mapping that uniquely relates a given initial value with the solution through this point. The flow allows to study system properties like invariant sets, stability, monotonicity or positivity. For DAEs, the flow further provides insights into the manifold onto which the system is bound to and into the dynamics on this manifold. Using a projection approach to decouple the differential and algebraic components, we give an explicit representation of the flow that is stated in the original coordinate space. This concept allows to study DAEs whose dynamics are restricted to special subsets in the variable space, like a cone or the nonnegative orthant.

Keywords: Differential-algebraic equations, flow, flow on surface, Dynamical systems.

AMS(MOS) subject classification: 34A09, 37E35, 37C10, 37Cxx.

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1 Introduction

We consider differential-algebraic equations (DAEs)

$$F(t, x, \dot{x}) = 0, \tag{1}$$

where $F \in C^k(\mathcal{I} \times \Omega_x \times \Omega_{\dot{x}}, \mathbb{R}^n)$ is defined on open sets $\mathcal{I} \in \mathbb{R}$, $\Omega_x, \Omega_{\dot{x}} \subset \mathbb{R}^n$. DAEs model dynamical processes that are constrained by auxiliary algebraic conditions, like e.g. connected joints in multibody systems, connections or loops in networks or balance equations and conservation laws in advection-diffusion equations, see e.g. [6, 9, 10, 11, 14, 16, 21, 29, 31, 35, 41, 42] and the references therein.

We derive a *flow* formulation of the DAE (1) by defining a mapping that uniquely relates an initial value with the solution through this point. For ordinary differential equations (ODEs)

$$\dot{x} = f(t, x), \tag{2}$$

the concept of the flow is well studied [1, 23, 24, 45], and allows to study properties of (2) like invariant sets, stability, monotonicity or symmetry, see e.g. [1, 23, 24, 45, 20].

Similarly, for differential equations on manifolds there exists the concept of the flow, allowing to study system properties and their preservation in a numerical simulation, see e.g. [22, 19, 20] and the references therein. Under certain smoothness assumptions, DAEs can be considered as differential equations on a manifold, cp. e.g. [19, 29, 42], thus allowing to extend the notion of a flow implicitly to implicit systems.

For DAEs in the form (1), a flow formulation has been considered in [30] to study stability properties. As stability is a coordinate invariant property, in [30] the flow is constructed using variable transformations to separate the differential and algebraic components in (1). To study coordinate dependent property like the invariance of special sets in the state space, like cones or manifolds, however, we need a flow representation that is stated in the original coordinates.

Using the framework of derivatives arrays and the strangeness-index [29], we identify those DAEs that are uniquely solvable on a particular set of initial values. Using a projection approach to decouple the differential and algebraic components without changing the original coordinate system [3], we construct an explicit representation of the flow. Considering the time-derivative of the flow, we obtain an explicit representation for the linearization of solutions of (1). Specifying our results for linear systems, we generalize Duhamel's formula to DAEs.

2 Preliminaries

We consider *time* or *time-state dependent projections*, i.e., matrix functions $P \in C^k(\mathcal{I} \times \Omega, \mathbb{R}^{n \times n})$, $k \geq 0$, that satisfy $P^2(t, x) = P(t, x)$ for every $(t, x) \in \mathcal{I} \times \Omega$. Then, the classical properties of constant projections pointwise extend to the function P, cp. [3]. In particular, $P \in \mathbb{R}^{n \times n}$ is called *orthogonal* if P is pointwise symmetric, i.e., $P^T(t, x) = P(t, x)$ on $\mathcal{I} \times \Omega$. The *complement* $P^c := I_n - P$ of a projection P is again a projection and satisfies $\text{range}(P^c(t, x)) = \ker(P(t, x))$ and $\ker(P^c(t, x)) = \text{range}(P(t, x))$.

In particular, we consider projections that are induced by the Moore-Penrose inverse. For a matrix function $E \in C^k(\Omega, \mathbb{R}^{n \times n})$, the Moore-Penrose inverse E^+ is pointwise defined like for constant matrices, cp. e.g. [4, 12, 18], i.e., $E^+(x) := (E(x))^+$, where $(EE^+E)(x) = E(x), (E^+EE^+)(x) = E^+(x), ((E^+E)(x))^T = (E^+E)(x), ((EE^+)(x))^T = (EE^+)(x)$ for $x \in \Omega$.

For every matrix $E(x) \in \mathbb{R}^{n\times n}$, there exists a unique Moore-Penrose inverse [15] and if $E(x)$ is nonsingular, then $E^+(x) = E(x)^{-1}$ [44]. If $E \in C^\ell(\mathcal{I}\times\Omega, \mathbb{R}^{m\times n})$ and $\operatorname{rank}(E(t,x)) = d$ on $\mathcal{I}\times S$, where $S \subset \Omega$ is an open set, for every $(t_0, x_0) \in \mathcal{I} \times S$, then there exist neighborhoods $\mathcal{I}_0 \subset \mathcal{I}$, $\mathcal{U}(x_0) \subset \Omega$, such that $E^+ \in C^\ell(\mathcal{I}_0 \times \mathcal{U}(x_0), \mathbb{R}^{n\times m})$ [3, Lemma 2.3]. If $E \in C^k(\mathcal{I}, \mathbb{R}^{m\times n})$ and $\operatorname{rank}(E(t)) = d$ on $\mathcal{I}$, then $E^+ \in C^k(\mathcal{I}, \mathbb{R}^{n\times m})$ [3, Lemma 2.3]. For $E \in C^\ell(\mathcal{I} \times \Omega, \mathbb{R}^{m\times n})$, the product $(EE^+)(x) \in \mathbb{R}^{m\times m}$ is the orthogonal projection with $\operatorname{range}((EE^+)(x)) = \operatorname{range}(E(x))$, $\ker((EE^+)(x)) = \operatorname{corange}(E(x))$ and $(E^+E)(x) \in \mathbb{R}^{n\times n}$ is the orthogonal projection with $\operatorname{range}((E^+E)(x)) = \operatorname{coker}(E(x))$ and $\ker((E^+E)(x)) = \ker(E(x))$, cp. [12, p. 9].

Furthermore, we use the concept of *time-varying subsets*, in particular time-varying manifolds, as they arise in the analysis of DAEs. For an interval $\mathcal{I} \subset \mathbb{R}$ and a family $\{S(t)\}_{t\in\mathcal{I}}$ of subsets $S(t) \subset \mathbb{R}^n$, such that there exists $S(t) \in \mathbb{R}^n$ for every $t \in \mathcal{I}$, we call the set $S := \bigcup_{t\in\mathcal{I}} \{\{t\}\times S(t)\}$ a time-varying subset on $\mathcal{I}$. Extending the standard definitions of charts and coverings, cp. e.g., [13, pp. 5], [32, pp. 97], [29, pp. 198], we can give a time-varying subset the structure of a manifold, cp. [3]. Here, it suffices to introduce time-varying manifolds as time-parameterized level sets as they arise in the analysis of DAEs.

Lemma 2.1. *A time-varying subset $\mathbb{S} \subset \mathbb{R}\times\mathbb{R}^n$ is a time-varying, embedded C^k-submanifold with* $\dim(\mathbb{S}) = d$ *if and only if for every $(t_0, x_0) \in \mathbb{S}$, there exist neighborhoods $\mathcal{I}_0 \subset \mathbb{R}$, $\mathcal{U}(x_0) \subset \mathbb{R}^n$ and a function $G \in C^k(\mathcal{I}_0 \times \mathcal{U}(x_0), \mathbb{R}^{n-d})$ that satisfies* $\operatorname{rank}(DG(t,x)) = \operatorname{rank}(G_x(t,x)) = n-d$ *on $G^{-1}(0)$ and $(\mathcal{I}_0 \times \mathcal{U}(x_0)) \cap \mathbb{S} = G^{-1}(0)$.*

Dropping the time-dependancy, Lemma 2.1 corresponds to the characterization of a C^k-submanifold $\mathbb{S} \subset \mathbb{R}^n$ as level set of a submersion, cp. [13, pp. 3],[32, pp. 97],[25, p. 10].

Finally, for a locally Lipschitz function $f^{\text{Lip}}_{\text{loc}} \in C(\mathcal{I}\times\Omega, \mathbb{R}^n)$ defined on an open set $\mathcal{I}\times\Omega \subset \mathbb{R}\times\mathbb{R}^n$, the ODE

$$\dot{x} = f(t, x) \tag{3}$$

is uniquely solvable for every $(t_0, x_0) \in \mathcal{I} \times \Omega$ with solution $x \in C((t_0^-, t_0^+), \Omega)$, where $t_0^\pm \in \partial\mathcal{I}$ or $\lim_{t\to t_0^\pm} \min\{\operatorname{dist}(x(t), \partial\Omega),\ \|x(t)\|^{-1}\} = 0$, cp. e.g. [5, p. 44] and [1, p. 105]. For (t_0, x_0), t_0^-, t_0^+ are called the *negative* and *positive escape time*, respectively, and (t_0, t_0^+) the *maximal interval of existence*, cp. [1, p. 101].
The unique relation between a given initial value and its associated solution motivates the definition of the *flow*, see e.g. [1, p. 133], [5, p. 49].

Lemma 2.2. *Consider the ODE (3). If $f \in C^{Lip}_{loc}(\mathcal{I} \times \Omega, \mathbb{R}^n)$, then there exists a function $\Phi_f \colon \mathcal{I} \times \mathcal{I} \times \Omega \to \mathbb{R}^n$, $(t, t_0, x_0) \mapsto \Phi^t_f(t_0, x_0)$, that satisfies the following properties for every $(t_0, x_0) \in \mathcal{I} \times \Omega$ and $t \in [t_0, t_0^+)$.*

$$\Phi^{t_0}_f(t_0, x_0) = x_0, \tag{4a}$$

$$\Phi^t_f(t_0, \Phi^s_f(t_0, x_0)) = \Phi^t_f(t_0, x_0), \tag{4b}$$

$$\dot{\Phi}^t_f(t_0, x_0) = f(t, \Phi^t_f(t_0, x_0)). \tag{4c}$$

For every $(t_0, x_0) \in \mathcal{C}_{F,\mu+1}$, on $[t_0, \hat{t}_0^+)$, the solution x of (3) is given by $x(t) = \Phi^t_f(t_0, x_0)$ and $\Phi^{(\cdot)}_f(t_0, x_0) \in C^1([t_0, t_0^+, \mathbb{R}^n)$ on $\mathcal{I} \times \Omega$.

The characteristic properties (4) reflect the unique solvability of (3) if f is locally Lipschitz on $\mathcal{I} \times \Omega$. Property (4a) uniquely relates the flow Φ_f with the initial value (t_0, x_0), property (4b) ensures that every solution can be maximally extended on Ω and property (4c) claims that $\Phi_f^t(t_0, x_0)$ solves the differential equation (3).
For *linear ODEs*

$$\dot{x} = A(t)x + b(t) =: f_{A,b}(t, x), \tag{5}$$

with $A \in C(\mathcal{I}, \mathbb{R}^{n\times n})$ and $b \in C(\mathcal{I}, \mathbb{R}^n)$, linearity implies that $f_{A,b} \in C_{\text{loc}}^{\text{Lip}}(\mathcal{I} \times \Omega, \mathbb{R}^n)$ if $f_{A,b} \in C(\mathcal{I} \times \mathbb{R}^n, \mathbb{R}^n)$. The maximal interval of existence is given by $(t_0^-, t_0^+) = \bar{\mathcal{I}}$, cp. [5, p. 48]. The flow $\Phi_{A,b} := \Phi_{f_{A,b}}$ is an affine linear transformation of the initial values, whose system matrix is given by the homogeneous flow Φ_A induced by $f_A := f_{A,0}$, cp., e.g., [46, p. 163], and that generalizes Duhamel's formula [46] to linear systems with time-varying coefficients.

Lemma 2.3. *Consider the ODE* (5) *with $f_{A,b} \in C(\mathcal{I} \times \mathbb{R}^n, \mathbb{R}^n)$. On $\mathcal{I} \times \mathcal{I} \times \mathbb{R}^n$, the flow $\Phi_{A,b}$ is given by*

$$\Phi_{A,b}^t(t_0, x_0) = \Phi_A^t(t_0)x_0 + \int_{t_0}^t (\Phi_A^t b)(s)\, ds, \tag{6}$$

where Φ_A is the homogeneous flow induced by f_A. The flow Φ_A is pointwise invertible with $(\Phi_A^t(t_0))^{-1} = \Phi_A^{t_0}(t)$.

3 A flow formula for DAEs

To define a flow for DAEs, we need a set of initial conditions on which the implicit equation (1) is uniquely solvable and solutions can be maximally extended. There are several approaches to study DAEs like derivative arrays [6, 8, 7], projector chains [16, 33, 34, 43] or a structural analysis [39, 40] that differ in the way they separate the differential and algebraic components and in the regularity assumptions on the system. Related with these approaches are different index concepts, like the differentiation or strangeness index, the tractability index or the structural index, which measure, roughly spoken, the complexity of solving a given DAE in terms of the necessary differentiations. A comparison of the different index concepts is given, e.g., in [11, 37]. We follow the concept of derivative arrays and the strangeness index as developed in [26, 27, 28, 29], because it is applicable to a large class of DAEs and provides a suitable framework to construct a flow.

3.1 Nonlinear differential-algebraic equations

For the DAE (1) with sufficiently smooth system function F, the derivative array of size ℓ, $\ell \in \mathbb{N}$, the *derivative array* of size ℓ is the inflated DAE

$$\mathcal{F}_{F,\ell}(t, x, \dot{x}, ..., x^{(\ell+1)}) := \begin{bmatrix} F(t, x, \dot{x}) \\ \frac{d}{dt}F(t, x, \dot{x}) \\ \vdots \\ \frac{d^\ell}{dt^\ell}F(t, x, \dot{x}) \end{bmatrix} = 0 \tag{7}$$

obtained by successive differentiation. Every sufficiently smooth solution of $F(t,x,\dot{x})=0$ solves the inflated system (7). Vice versa, if $(t,x,\dot{x},...,x^{(\ell)})$ solves the derivative array (7), then $(t,x,\dot{x})$ also solves $F(t,x,\dot{x})=0$. For a derivative array of suitable size, the idea of the strangeness index is to filter out a set of differential and algebraic equations that uniquely determines the x-part of this solution $(t,x,\dot{x},...,x^{(\ell)})$. This may include algebraic equations for derivatives of x, so we consider (7) formally as an algebraic equation for the algebraic variable $z_\ell := (t,x,v_1,...,v_{\ell+1})$ with $v_k = x^{(k)}(t)$, $k=1,...,\ell+1$. The algebraic solution set is denoted by

$$\mathcal{F}_{F,\ell}^{-1}(0) = \{z_\ell \in \mathcal{I}\times\mathbb{R}^n\times\ldots\times\mathbb{R}^n \,|\, F_\ell(z_\ell)=0\}. \tag{8}$$

To solve the derivative array (7) locally for $(t,x,\dot{x})$, we make following assertions on the Jacobians

$$M_\ell(z_\ell) := \partial_{v_1,\ldots,v_{\ell+1}}\mathcal{F}_{F,\ell}(z_\ell), \quad N_\ell(z_\ell) := \partial_x\mathcal{F}_{F,\ell}(z_\ell), \tag{9}$$

containing the partial derivatives of $\mathcal{F}_{F,\ell}(z_\ell)$ with respect to the variables $v_1,...,v_{\ell+1}$ and x, respectively, cp. [29, p. 155].

Hypothesis 3.1 ([29])**.** *Consider* $F\colon \mathbb{D}\to\mathbb{R}^n$*. Let there exist* $\mu,d,a\in\mathbb{N}_0$*,* $n=d+a$*, such that* $F\in C^{\mu+1}(\mathbb{D},\mathbb{R}^n)$*,* $F_\mu^{-1}(0)\neq\emptyset$ *and for every* $z_{\mu,0}\in F_\mu^{-1}(0)$*, there exists a sufficiently small neighborhood* $\mathcal{U}(z_{\mu,0})$*, such that the following properties hold.*

1. *On* $\mathcal{U}(z_{\mu,0})\cap F_\mu^{-1}(0)$*,* $\operatorname{rank}(M_\mu(z_\mu)) = (\mu+1)n-a$ *and there exists a pointwise orthogonal matrix function* $Z_2\in C^\mu(\mathcal{U}(z_{\mu,0}),\mathbb{R}^{(\mu+1)n\times a})$ *with* $\operatorname{rank}(Z_2(z_\mu))=a$ *and* $(Z_2^T M_\mu)(z_\mu)=0$*.*
2. *On* $\mathcal{U}(z_{\mu,0})\cap F_\mu^{-1}(0)$*,* $\operatorname{rank}(Z_2^T\bar{N}_\mu(z_\mu)) = a$*, where* $\bar{N}_\mu = N_\mu[I_n,0]$*, and there exists a pointwise orthogonal matrix function* $T_1\in C^\mu(\mathcal{U}(z_{\ell,0}),\mathbb{R}^{n\times d})$ *with* $\operatorname{rank}(T_1(z_\mu))=d$ *and* $(Z_2^T\bar{N}_\mu T_1)(z_\mu)=0$*.*
3. *On* $\mathcal{U}(z_{\mu,0})\cap F_\mu^{-1}(0)$*,* $\operatorname{rank}(F_{\dot{x}}(t,x,\dot{x})T_1(z_\mu)) = d$ *and there exists an orthogonal matrix* $Z_1\in\mathbb{R}^{n\times d}$ *with* $\operatorname{rank}(Z_1)=d$ *and* $\operatorname{rank}(Z_1^T F_{\dot{x}} T_1(z_\mu))=d$*.*

The minimal μ_s for which F satisfies Hypothesis 3.1 on $\mathbb{D}$, is called the *strangeness index (s-index)* of (1) [29]. If F has s-index μ_s and satisfies Hypothesis 3.1 with μ_s+1,d,a, we say that (1) has *regular s-index* μ_s [29]. If F has (regular) s-index $\mu=0$, then F is called (regular and) *s-free* [29]. If F is s-free, then the Jacobians $F_{\dot{x}}, F_x$ satisfy the assertions of Hypothesis 3.1, implying that every algebraic equation a solution of (1) satisfies is explicitly contained in (1). Conversely, if F is of higher index, then there are algebraic equations hidden in the systems and have to be filtered out by differentiation. Numerlcally, s-free systems can be solved with the same accuracy as ODEs, cp. [29, p. 251]. To match the smoothness assumptions of Hypothesis 3.1, we can reduce the domain of definition $\mathbb{D}$.

The set of functions satisfying Hypothesis 3.1 with integers μ,d,a and $\mu+1,d,a$ is denoted by

$$C^\ell_{\mu,d,a,\mathrm{reg}}(\mathbb{D},\mathbb{R}^n) := \big\{F\in C^\ell(\mathbb{D},\mathbb{R}^n) \,|\, F \text{ satisfies Hypothesis 3.1 with } \mu,d,a \text{ and } \mu+1,d,a\ \big\}, \tag{10}$$

where $\ell\geq\mu+1$. Initial values that are part of a vector in the algebraic solution set are summarized in the *set of consistent initial values*

$$\mathcal{C}_{F,\mu} := \big\{(t_0,x_0)\in\mathcal{I}\times\Omega \mid \exists\ (v_1,...,v_{\mu+1})\in\Omega_{\dot{x}}\times\mathbb{R}^n\times\ldots\times\mathbb{R}^n : (t_0,x_0,v_1,...,v_{\mu+1})\in F_\mu^{-1}(0)\big\}. \tag{11}$$

Similarly, tuples $(t_0, x_0, \dot{x}_0)$ part of a vector in $F_\mu^{-1}(0)$ are summarized in the *set of consistent initializations*

$$\mathcal{L}_{F,\mu} := \big\{(t_0, x_0, v_1) \in F^{-1}(0) \mid \exists\ (v_2, ..., v_{\mu+1}) \in \mathbb{R}^n \times \ldots \times \mathbb{R}^n \\ : (t_0, x_0, v_1, v_2 ..., v_{\mu+1}) \in F_\mu^{-1}(0)\big\}. \tag{12}$$

For functions $F \in C^{\mu+1}_{\mu,d,a,\text{reg}}(\mathbb{D}, \mathbb{R}^n)$ and initial values $(t_0, x_0) \in \mathcal{C}_{F,\mu+1}$, the DAE (1) is uniquely solvable and the solution is maximally extendable on $\mathcal{C}_{F,\mu+1}$, cp. [28] and [29, p. 163, p, 167].

Theorem 3.1. *If $F \in C^{\mu+1}_{\mu,d,a,reg}(\mathbb{D}, \mathbb{R}^n)$, then the DAE* (1) *is uniquely solvable for every $(t_0, x_0) \in \mathcal{C}_{F,\mu+1}$. The solution is $x \in C^1([t_0, \hat{t}_0^+), \mathbb{R}^n)$, where $\hat{t}_0^+ = \sup\{t \geq t_0 \mid (t, x(t)) \in \mathcal{C}_{F,\mu+1}\}$.*

The positive escape time $\hat{t}_0^+$ denotes the time where the derivative array ceases to satisfy the rank assertions of Hypothesis 3.1, for example because the Jacobians M_μ, N_μ suffer from a rank drop in $t = t_0^+$.

As a consequence of Theorem 3.2, we consider the initial value problem (IVP)

$$F(t, x, \dot{x}) = 0, \qquad F \in C^{\mu+1}_{\mu,d,a,\text{reg}}(\mathbb{D}, \mathbb{R}^n) \tag{13a}$$

$$x(t_0) = x_0, \qquad (t_0, x_0) \in \mathcal{C}_{F,\mu+1}, \tag{13b}$$

and define a flow on the set of consistent initial values $\mathcal{C}_{F,\mu+1}$.

Corollary 3.1. *Consider the DAE* (13a). *There exists a function $\Phi_F \colon \mathcal{C}_{F,\mu+1} \times \mathcal{I} \to \mathbb{R}^n$, $(t, t_0, x_0) \mapsto \Phi_F^t(t_0, x_0)$, that satisfies the following properties for every $(t_0, x_0) \in \mathcal{C}_{F,\mu+1}$ and $t \in [t_0, \hat{t}_0^+)$.*

$$\Phi_F^{t_0}(t_0, x_0) = x_0, \tag{14a}$$

$$\Phi_F^t(t_0, \Phi_F^s(t_0, x_0)) = \Phi_F^t(t_0, x_0), \tag{14b}$$

$$F(t, \Phi_F^t(t_0, x_0), \dot{\Phi}_F^t(t_0, x_0)) = 0. \tag{14c}$$

For every $(t_0, x_0) \in \mathcal{C}_{F,\mu+1}$, on $[t_0, \hat{t}_0^+)$, the solution x of (13) *is given by $x(t) = \Phi_F^t(t_0, x_0)$ and $\Phi_F^{(\cdot)}(t_0, x_0) \in C^1([t_0, \hat{t}_0^+), \mathbb{R}^n)$ on $\mathcal{C}_{F,\mu+1}$.*

The function Φ_F in Corollary 3.1 is called the *flow* associated with the DAE (13a). Like for ODEs, the characteristic properties (14) reflect the unique solvability of the IVP (13) and the extendability of solutions on the set $\mathcal{C}_{F,\mu+1}$. In contrast to the ODE flow Φ_f that is defined on the full phase space, the DAE flow Φ_F is defined only on the set of consistent initial values.

Remark 3.1. *For a particular problem or a clever formulation of the DAE* (1), *the smoothness assumptions of Theorem 3.2 may be significantly relaxed to prove the existence and uniqueness of solutions on a particular set of initial values. Consequently, for these problems, the flow can be defined under less restrictive smoothness assertions. Treating a more general class of problems, however, we have to assume that the system function is sufficiently smooth to set up the full derivative array of size $\mu + 1$, such that we can show the uniqueness and existence of solutions.*

To represent the flow and its linearization explicitly, we use the *strangeness-free* (*s-free*) *formulation* [28, 29], which gives an equivalent formulation of (13) by specifying the same solution. In contrast to (13), however, this surrogate model is s-free and regular at the solution x and the differential and algebraic equations are explicitly given.

Theorem 3.2. *[28, 29] Consider the IVP* (13) *and let $x \in C^1([t_0, \hat{t}_0^+), \mathbb{R}^n)$ be its solution. There exist functions $\hat{F}_1 \in C^1([t_0, \hat{t}_0^+) \times \hat{\mathcal{U}}(x) \times \hat{\mathcal{U}}(\dot{x}), \mathbb{R}^d)$ and $\hat{F}_2 \in C^1([t_0, \hat{t}_0^+) \times \hat{\mathcal{U}}(x), \mathbb{R}^a)$ defined on neighborhoods of x, such that on $[t_0, \hat{t}_0^+)$, the function x is also the unique solution of*

$$\hat{F}_1(t, x, \dot{x}) = 0, \quad x(t_0) = x_0, \tag{15a}$$

$$\hat{F}_2(t, x) = 0. \tag{15b}$$

In particular, $\hat{F} = [\hat{F}_1^T, \hat{F}_2^T]^T \in C^1_{0,d,a,reg}(([t_0, \hat{t}_0^+) \times \hat{\mathcal{U}}(x) \times \hat{\mathcal{U}}(\dot{x})) \cap \mathcal{L}_{\mu+1}, \mathbb{R}^n)$.

Remark 3.2. *The functions $\hat{F}_1, \hat{F}_2$ are obtained from the derivative array by choosing a suitable parameterization of the algebraic solution set $\mathcal{F}_{F,\mu}^{-1}(0)$ along the solution x of* (1)*, cp. [28] and [29, p. 163, p. 167], and are defined until x leaves the algebraic solution set $\mathcal{F}_{F,\mu}^{-1}(0)$. For a given μ and a consistent initial value $(t_0, x_0) \in \mathcal{C}_{F,\mu+1}$, the functions $\hat{F}_1, \hat{F}_2$ are specified along the solution x up to nonsingular transformations, cp. [2, Thm. 4.2.1].*

Remark 3.3. *The assertions of Hypothesis 3.1 can be checked numerically along a numerical solution z_Δ of* (1) *by computing the derivative array, e.g., by automatic differentiation [17], and SVDs for the Jacobians $M_\mu(z_\Delta)$, $N_\mu(z_\Delta)$ [15, 29]. Similarly, the construction of the s-free formulation can be incorporated in the numerical simulation, see [29, Ch. 6]. As the Jacobians $M_\mu(z_\Delta)$, $N_\mu(z_\Delta)$ only approximate $M_\mu(z)$, $N_\mu(z)$ and the computed values $\mu_\Delta, d_\Delta, a_\Delta$ are based on numerical rank decisions, these values only indicate the true values μ, d, a. In cases of doubt a higher value of μ should be chosen to ensure that all hidden constraints are explicitly given, see [29, p. 281], [36].*
To compute a consistent initial value $z_0 \in F_{\mu+1}^{-1}$, one can either use a fixpoint iteration on the derivative array, the Gauss-Newton method [28], or decompose the variables with a time-varying transformation, cp. [30]. The latter, in particular, may be very costly, however, in some cases it may be the only way to construct the needed starting point for the remodeling procedure.

We use the s-free formulation to compute the solution of (13). On $[t_0, \hat{t}_0^+) \times \hat{\mathcal{U}}(x) \times \hat{\mathcal{U}}(\dot{x})$, the s-free formulation $\hat{F}$ induces the state-dependent space decomposition

$$\mathbb{R}^n = \operatorname{coker}(\hat{F}_{\dot{x}}(z)) \oplus \ker(\hat{F}_{\dot{x}}(z)). \tag{16}$$

To implement the decomposition (16), we pursue the projection approach considered in [3]. As we use the flow formula to study *positive* DAEs, cp. [3], i.e., systems for which every solution starting with a componentwise nonnegative initial value stays componentwise nonnegative for all its lifetime, we wish to avoid the change of coordinates occurring when using variable transformations. To realize the partitioning (16), we consider the Moore-Penrose projections

$$P_{MP}(z) := \big(\hat{F}_{\dot{x}}^+ \hat{F}_{\dot{x}}\big)(z), \quad P_{MP}^\perp(z) := I_n - P_{MP}(z) \tag{17}$$

that are pointwise defined on $[t_0, \hat{t}_0^+) \times \hat{\mathcal{U}}(x) \times \hat{\mathcal{U}}(\dot{x})$. On $[t_0, \hat{t}_0^+) \times \hat{\mathcal{U}}(x) \times \hat{\mathcal{U}}(\dot{x}) \cap \mathcal{L}_{F,\mu+1}$, the *Moore-Penrose projections $P_{MP}, P_{MP}^\perp$ associated with IVP* (13) satisfy the following properties.

Lemma 3.1. *Consider the IVP* (13) *and let $x \in C^1([t_0, \hat{t}_0^+), \mathbb{R}^n)$ be its solution. Along x, there exist neighborhoods $\mathcal{U}_{P_{MP}}(x), \mathcal{U}_{P_{MP}}(\dot{x}) \subset \mathbb{R}^n$, such that $P_{MP} \in C^1([t_0, \hat{t}_0^+) \times \mathcal{U}_{P_{MP}}(x) \times \mathcal{U}_{P_{MP}}(\dot{x}), \mathbb{R}^{n \times n})$. On $([t_0, \hat{t}_0^+) \times \mathcal{U}_{P_{MP}}(x) \times \mathcal{U}_{P_{MP}}(\dot{x})) \cap \mathcal{L}_{F,\mu+1}$,* $\operatorname{rank}(P_{MP}(z)) = d$ *and P_{MP} is independent of the chosen remodeling $\hat{F}$.*

Proof. We first prove the proposed properties if P_{MP} is evaluated on the solution x and its derivative $\dot{x}$. On $(t,x,\dot{x})$, the remodeling $\hat{F}$ is s-free, cp. Theorem 3.2, implying that $\operatorname{rank}(\hat{F}_{\dot{x}}(t,x,\dot{x})) = d$ and thus $\operatorname{rank}(P_{MP}(t,x,\dot{x})) = d$ on $[t_0,\hat{t}_0^+)$, cp. [12, p. 9]. Then, there exist neighborhoods $\mathcal{U}_{P_{MP}}(x)$,$\mathcal{U}_{P_{MP}}(\dot{x})$ of x, such that $P_{MP} \in C^1([t_0,\hat{t}_0^+) \times \mathcal{U}_{P_{MP}}(x) \times \mathcal{U}_{P_{MP}}(\dot{x}), \mathbb{R}^n)$, cp. [3, Lemma 2.2]. Furthermore, on $(t,x,\dot{x})$, the remodeling $\hat{F}$ is specified up to nonsingular transformations, cp. Remark 3.2, i.e., if $U_1 \in \mathbb{R}^{d\times d}$, $U_2 \in C^2(\mathcal{U}(z_{\mu,0}), \mathbb{R}^{a\times a})$ are pointwise orthogonal matrix functions and $U = \operatorname{diag}(U_1,U_2)$, then $\tilde{F} = U^T\hat{F}$ also satisfies the assertions of Theorem 3.2 and $\tilde{F}_{\dot{x}}^+(t,x,\dot{x}) = \hat{F}_{\dot{x}}(t,x,\dot{x})U(z_\mu)$ for $z_\mu = (t,x,v_0,...,v_{\mu+1}) \in \mathcal{F}_{F,\ell}^{-1}(0)$, cp. [3, Lemma 2.4]. On $[t_0,\hat{t}_0^+)$, then it follows that

$$(\tilde{F}_{\dot{x}}^+\tilde{F}_{\dot{x}})(t,x,\dot{x}) = \hat{F}_{\dot{x}}(t,x,\dot{x})(U^TU)(z_\mu)\hat{F}_{\dot{x}}(t,x,\dot{x}) = (\hat{F}_{\dot{x}}^+\hat{F}_{\dot{x}})(t,x,\dot{x}),$$

implying that the Moore-Penrose projections provided by $\hat{F}$ and $\tilde{F}$ agree on $(t,x,\dot{x})$. As the remodeling $\hat{F} \in C^2([t_0,\hat{t}_0^+) \times \hat{\mathcal{U}}(x) \times \hat{\mathcal{U}}(\dot{x}), \mathbb{R}^n)$ is s-free and regular on $[t_0,\hat{t}_0^+) \times \hat{\mathcal{U}}(x) \times \hat{\mathcal{U}}(\dot{x}) \cap \mathcal{L}_{F,\mu+1}$, cp. Theorem 3.2, it yields a s-free formulation for every IVP (13) with initial condition $(\tilde{t}_0,\tilde{x}_0) \in ([t_0,\hat{t}_0^+) \times \hat{\mathcal{U}}(x)) \cap \mathcal{C}_{F,\mu+1}$. Repeating the given arguments for the solution $\tilde{x}$ associated with an initial condition $(\tilde{t}_0,\tilde{x}_0) \in ([t_0,\hat{t}_0^+) \times \mathcal{U}_{P_{MP}}(x)) \cap \mathcal{C}_{F,\mu+1}$, we have proved that the given assertions are satisfied pointwise on $([t_0,\hat{t}_0^+) \times \mathcal{U}_{P_{MP}}(x) \times \mathcal{U}_{P_{MP}}(\dot{x})) \cap \mathcal{L}_{F,\mu+1}$. □

For every $z \in [t_0,\hat{t}_0^+) \times \hat{\mathcal{U}}(x) \times \hat{\mathcal{U}}(\dot{x})$, the projections $P_{MP}, P_{MP}^\perp$ induce a variable decomposition $P_{MP}(z)x \in \operatorname{coker}(\hat{F}_{\dot{x}}(z))$ and $P_{MP}^\perp(z)x \ker(\hat{F}_{\dot{x}}(z))$ for $x \in \mathbb{R}^n$. For the solution x of (13), we consider the space decomposition (16) along $(t,x,\dot{x})$ and set

$$x_d := P_{MP}(t,x,\dot{x})x, \qquad x_a := P_{MP}^\perp(t,x,\dot{x})x. \tag{18}$$

Solving the s-free formulation (15) for $\dot{x}_d, x_a$, we obtain a differential equation for x_d, while the components x_a are fixed algebraically.

Theorem 3.3. *Consider the IVP (13). Let $\hat{F} \in C^2([t_0,\hat{t}_0^+) \times \hat{\mathcal{U}}(x) \times \hat{\mathcal{U}}(\dot{x}), \mathbb{R}^n)$ be an s-free remodeling and $P_{MP} \in C^1([t_0,\hat{t}_0^+) \times \mathcal{U}_{P_{MP}}(x) \times \mathcal{U}_{P_{MP}}(\dot{x}), \mathbb{R}^{n\times n})$ the associated Moore-Penrose projection. Let $x \in C^1([t_0,\hat{t}_0^+), \mathbb{R}^n)$ be the solution of (13) and let x_d, x_a be given by (18) with $z = (t,x,\dot{x})$. On $\mathcal{I}_0$, via the components x_d, x_a, x is the unique solution of*

$$\dot{x}_d = h_{MP}(t,x_d), \quad x_d(t_0) = P_{MP}(z_0)x_0, \tag{19a}$$
$$x_a = g_{MP}(t,x_d). \tag{19b}$$

The functions $g_{MP} \in C^1([t_0,\hat{t}_0^+) \times \mathcal{U}(x_a), \mathcal{U}(x_d))$ and $h_{MP} \in C^1([t_0,\hat{t}_0^+) \times \mathcal{U}(x_d), \mathbb{R}^n)$ are uniquely defined by (13) as the implicit solution of

$$\hat{F}_1\big(t, x_d + x_a, h_{MP}(t,x_d) + \dot{g}_{MP}(t_0,x_{d,0})\big) = 0, \tag{20a}$$
$$\hat{F}_2\big(t, x_d + g_{MP}(t,x_d)\big) = 0. \tag{20b}$$

Proof. ⇒ Let $x \in C^1([t_0,\hat{t}_0^+), \mathbb{R}^n)$ solve (13). First, we show that there exists $t_1 \in (t_0, t_0^+)$, such that, on $\mathcal{I}_0 := [t_0,t_1)$, x_d, x_a solve

$$\dot{x}_d = h_{MP,0}(t,x_d), \quad x_d(t_0) = P_{MP}(z_0)x_0, \tag{21a}$$
$$x_a = g_{MP,0}(t,x_d), \tag{21b}$$

where $g_{MP,0} \in C^1(\mathcal{I}_0 \times \mathcal{U}(x_{d,0}), \mathcal{U}(x_{a,0}))$ and $h_{MP,0} \in C^1(\mathcal{I}_0 \times \mathcal{U}(x_{d,0}), \mathbb{R}^n)$ are locally defined on neighborhoods of $x_{d,0}, x_{a,0}$ as implicit solutions of (20). Exploiting the uniqueness and smoothness of the s-free formulation and its Moore-Penrose projection, we can smoothly extend $g_{MP,0}, h_{MP,0}$ to functions defined on the full interval $[t_0, \hat{t}_0^+)$.

To solve the algebraic equation (15b) for the components x_a, we show that along x, $\hat{F}_2, P_{MP}$ satisfy the assertions of the projection-based Implicit Function Theorem, cp. [3, Thm. 3.1], with $Q = I_a$, i.e.,

$$\left(\hat{F}_{2,x} P_{MP}^{\perp}\right)^+ \left(\hat{F}_{2,x} P_{MP}^{\perp}\right)(z) = P_{MP}^{\perp}(z), \tag{22a}$$

$$\left(\hat{F}_{2,x} P_{MP}^{\perp}\right) \left(\hat{F}_{2,x} P_{MP}^{\perp}\right)^+(z) = I_a \tag{22b}$$

is satisfied pointwise on $([t_0, \hat{t}_0^+) \times \mathcal{U}_{P_{MP}}(x) \times \mathcal{U}_{P_{MP}}(\dot{x})) \cap \mathcal{L}_{F,\mu+1}$. Since $\operatorname{rank}(P_{MP}) = d$ on $([t_0, \hat{t}_0^+) \times \mathcal{U}_{P_{MP}}(x) \times \mathcal{U}_{P_{MP}}(\dot{x})) \cap \mathcal{L}_{F,\mu+1}$, cp. Lemma 3.1, there exist neighborhoods $\tilde{\mathcal{U}}(x), \tilde{\mathcal{U}}(\dot{x})$ and a pointwise orthogonal function $T = [T_1, T_2] \in C^1([t_0, \hat{t}_0^+) \times \tilde{\mathcal{U}}(x) \times \tilde{\mathcal{U}}(\dot{x}), \mathbb{R}^{n \times n})$ with $\operatorname{span}(T_1(z)) = \operatorname{coker}(\hat{F}_{\dot{x}}(z))$, $\operatorname{span}(T_2(z)) = \ker(\hat{F}_{\dot{x}}(z))$, such that

$$\left(\hat{F}_{2,x} P_{MP}^{\perp}\right)(z) = \left(\begin{bmatrix} 0 & \hat{F}_{x,2} T_2 \end{bmatrix} T^T\right)(z),$$

cp. [3, Lemma 2.2]. To compute $(\hat{F}_{2,x} P_{MP}^{\perp})^+$, we show that $\hat{F}_{x,2} T_2$ is pointwise nonsingular. As $\hat{F} \in C^1_{0,d,a,\mathrm{reg}}(([t_0, \hat{t}_0^+) \times \hat{\mathcal{U}}(x) \times \hat{\mathcal{U}}(\dot{x})) \cap \mathcal{L}_{\mu+1}, \mathbb{R}^n)$, on $([t_0, \hat{t}_0^+) \times \hat{\mathcal{U}}(x) \times \hat{\mathcal{U}}(\dot{x})) \cap \mathcal{L}_{\mu+1}$, Hypothesis 3.1 implies that $\operatorname{rank}(\hat{F}_{1,\dot{x}}(z)) = d$, $\operatorname{rank}(\hat{F}_{2,x}(z)) = n - d$ and $\ker(\hat{F}_{1,\dot{x}}(z)) \cap \ker(\hat{F}_{2,x}(z)) = \{0\}$. Hence, $\mathbb{R}^n \setminus (\operatorname{coker}(\hat{F}_{1,\dot{x}}(z)) \cup \operatorname{coker}(\hat{F}_{2,x}(z))) = \{0\}$, implying that, on $([t_0, \hat{t}_0^+) \times \hat{\mathcal{U}}(x) \times \hat{\mathcal{U}}(\dot{x})) \cap \mathcal{L}_{\mu+1}$, there exists a partitioning $\mathbb{R}^n = \operatorname{coker}(\hat{F}_{1,\dot{x}}(z)) \oplus \operatorname{coker}(\hat{F}_{2,x}(z))$. With $\ker(\hat{F}_{1,\dot{x}}(z)) \subset \operatorname{coker}(\hat{F}_{2,x}(z))$, it follows that $\hat{F}_{2,x}|_{\ker(\hat{F}_{1,\dot{x}}(z))}$ is pointwise nonsingular on $([t_0, \hat{t}_0^+) \times \hat{\mathcal{U}}(x) \times \hat{\mathcal{U}}(\dot{x})) \cap \mathcal{L}_{\mu+1}$. By the choice of T_2, then the Moore-Penrose inverse is given by

$$\left(\hat{F}_{2,x} P_{MP}^{\perp}\right)^+(z) = T(z) \begin{bmatrix} 0 \\ (\hat{F}_{x,2} T_2)^{-1}(z) \end{bmatrix}, \tag{23}$$

cp. [3, Lemma 2.3]. Hence, condition (22) is satisfied pointwise on $(t, x, \dot{x})$. Repeating these arguments on $([t_0, \hat{t}_0^+) \times \mathcal{U}_{P_{MP}}(x) \times \mathcal{U}_{P_{MP}}(\dot{x})) \cap \mathcal{L}_{F,\mu+1}$, we have proved the assertion. As the remodeling $\hat{F} \in C^2([t_0, \hat{t}_0^+) \times \hat{\mathcal{U}}(x) \times \hat{\mathcal{U}}(\dot{x}), \mathbb{R}^n)$ yields a s-free formulation for every IVP (13) with initial condition $(\tilde{t}_0, \tilde{x}_0) \in ([t_0, \hat{t}_0^+) \times \hat{\mathcal{U}}(x)) \cap \mathcal{C}_{F,\mu+1}$, we can repeat the given arguments for every solution $\tilde{x}$ associated with an initial conditions $(\tilde{t}_0, \tilde{x}_0) \in ([t_0, \hat{t}_0^+) \times \mathcal{U}_{P_{MP}}(x)) \cap \mathcal{C}_{F,\mu+1}$. This proves that condition (22) is satisfied pointwise on $([t_0, \hat{t}_0^+) \times \mathcal{U}_{P_{MP}}(x) \times \mathcal{U}_{P_{MP}}(\dot{x})) \cap \mathcal{L}_{F,\mu+1}$. With this observation, we can solve the algebraic equation (15b) for the components x_a using the projection-based Implicit Function Theorem, cp. [3]. Setting $y_d := P_{MP}(z)y$, $y_a := P_{MP}^{\perp}(z)y$ for $y \in \mathbb{R}^n$ and $x_{d,0} := P_{MP}(z_0)x_0$ and $x_{a,0} := P_{MP}^{\perp}(z_0)x_0$, where $z = (t, x, \dot{x})$, there exist neighborhoods $\mathcal{I}_0 \subset [t_0, \hat{t}_0^+)$, $\mathcal{U}(x_{d,0}), \mathcal{U}(x_{a,0}) \subset \mathbb{R}^n$ and a function $g_{MP,0} \in C^1(\mathcal{I}_0 \times \mathcal{U}(x_{d,0}), \mathcal{U}(x_{a,0}))$, such that (t, y) solves (15b) if and only if $(t, y_d) \in \mathcal{I}_0 \times \mathcal{U}(x_{d,0})$ and $y_a = g_{MP,0}(t, y_d)$. Choosing $\mathcal{I}_0$ sufficiently small such that $P_{MP}(z)x \in \mathcal{U}(x_{d,0})$ on $\mathcal{I}_0$, then x_a solves (21b) on $\mathcal{I}_0$.

To solve the differential equation (15a) for the derivatives $\dot{x}_d$, we again use the projection-based Implicit Function Theorem modified for the application to implicit differential equations, cp. [2, Lem. 3.1.3]. Due to the properties of the Moore-Penrose inverse, $(\hat{F}_{\dot{x}} P_{MP})(z) = \hat{F}_{\dot{x}}(z)$ is satisfied pointwise on $z = (t, x, \dot{x})$, implying that $(\hat{F}_{1,\dot{x}} P_{MP})(z) = \hat{F}_{1,\dot{x}}(z)$ as $\hat{F}_{\dot{x}} = [\hat{F}_{1,\dot{x}}^T, 0]^T$. Then,

$$\left(\hat{F}_{1,\dot{x}} P_{MP}\right)^+ \left(\hat{F}_{1,\dot{x}} P_{MP}\right)(z) = P_{MP}(z), \tag{24}$$

$$\left(\hat{F}_{1,\dot{x}} P_{MP}\right) \left(\hat{F}_{1,\dot{x}} P_{MP}\right)^+(z) = I_d, \tag{25}$$

is satisfied pointwise on $([t_0,\hat{t}_0^+)\times\mathcal{U}_{P_{MP}}(x)\times\mathcal{U}_{P_{MP}}(\dot{x}))\cap\mathcal{L}_{F,\mu+1}$ and by [2, Lem. 3.1.3], choosing $\mathcal{I}_0$ sufficiently small, there exist neighborhoods $\mathcal{U}(\dot{x}_{d,0}),\mathcal{U}(\dot{x}_{a,0})\subset\mathbb{R}^n$ and a function $h\in C^1(\mathcal{I}_0\times\mathcal{U}(x_{d,0})\times\mathcal{U}(x_{a,0})\times\mathcal{U}(\dot{x}_{a,0}),\mathcal{U}(\dot{x}_{d,0}))$, such that $y\in C^1(\mathcal{I}_0,\mathbb{R}^n)$ solves (15a) on $\mathcal{I}_0$ if and only if the components $y_d := P_{MP}(z)y$, $y_a := P_{MP}^\perp(z)y$ satisfy $(t,y_d,y_a,\dot{y}_a)\in\mathcal{I}_0\times\mathcal{U}(x_{d,0})\times\mathcal{U}(x_{a,0})\times\mathcal{U}(\dot{x}_{a,0})$ and $\dot{x}_d = h(t,x_d,x_a,\dot{x}_a)$, where the function h solves

$$\hat{F}_1\big(t,x_d+x_a,h(t,x_d,x_a,\dot{x}_a)+\dot{x}_a\big)=0.$$

As $(\hat{F}_{\dot{x}}P_{MP}^\perp)(z)=0$ on $([t_0,\hat{t}_0^+)\times\hat{\mathcal{U}}(x)\times\hat{\mathcal{U}}(\dot{x}))\cap\mathcal{L}_{F,\mu+1}$, neither $\hat{F}_1$ nor the implicit function h depend on the particular value of $\dot{x}_a$ and we set $h(t,x_d,x_a)=h(t,x_d,x_a,\dot{x}_{a,0})$. For the solution x, this implies that $\dot{x}_d = h(t,x_d,x_a)$ for $(t,x_d,x_a)\in\mathcal{I}_0\times\mathcal{U}(x_{d,0})\times\mathcal{U}(x_{a,0})$. Replacing x_a and $\dot{x}_{a,0}$ using equation (21b), we get that

$$h(t,x_d) := h\big(t,x_d,g(t,x_0),\dot{g}_{MP}(t_0,x_0)\big).$$

Choosing $\mathcal{I}_0$ sufficiently small, such that $(x_d,x_a,\dot{x}_a)\in\mathcal{U}(x_{d,0})\times\mathcal{U}(x_{a,0})\times\mathcal{U}(\dot{x}_{a,0})$ for $t\in\mathcal{I}_0$, we find that x solves (21a).

Now, we show that the implicit functions $g_{MP,0},h_{MP,0}$ can be extended onto the full interval $[t_0,\hat{t}_0^+)$. We set $x_1 := x(t_1)$, where $\mathcal{I}_0=[t_0,t_1)$. Then, $(t_1,x_1)\in\mathcal{C}_{F,\mu+1}$ and the IVP

$$F(t,x,\dot{x})=0 \quad x(t_1)=x_1 \tag{26}$$

is uniquely solvable with $x\in C^1([t_1,t_0^+),\mathbb{R}^n)$. In particular, (26) can be remodeled along x using the $\hat{F}\in C^2([t_0,\hat{t}_0^+)\times\hat{\mathcal{U}}(x)\times\hat{\mathcal{U}}(\dot{x}),\mathbb{R}^n)$ serving as s-free remodeling of (13). Then, the Moore-Penrose projections induced by (13) and (26) as well as the differential and algebraic components x_d,x_a agree. However, we assume that the implicit functions solving $\hat{F}(t,x,\dot{x})=0$ in the neighborhood of (t_1,x_1) are given by $g_{MP,1}\in C^1(\mathcal{I}_1\times\mathcal{U}(x_{d,1}),\mathcal{U}(x_{a,1}))$ and $h_{MP,1}\in C^1(\mathcal{I}_1\times\mathcal{U}(x_{d,1}),\mathcal{U}(\dot{x}_{d,1}))$. The domains of definition of $g_{MP,i},h_{MP,i}$, $i=1,2$, are open and we can assume without loss of generality, that there exists a nonempty interval $\mathcal{I}_\cap\subset\mathcal{I}_0\cap\mathcal{I}_1$ such that $x_d(t)\in\mathcal{U}(x_{d,\cap}) := \mathcal{U}(x_{d,0})\cap\mathcal{U}(x_{d,1})$ on $\mathcal{I}_\cap$. On $\mathcal{I}_\cap$, both $g_{MP,0}$ and $g_{MP,1}$ specify the components x_a, implying that

$$x_a(t)=g_{MP,0}(t,x_d(t))=g_{MP,1}(t,x_d(t)). \tag{27}$$

Since $g_{MP\,i}\in C^1(\mathcal{I}_i\times\mathcal{U}(x_{d,i}),\mathcal{U}(x_{a,i}))$ for $i=0,1$, and $g_{MP\,i}\in C^1(\mathcal{I}_\cap\times\mathcal{U}(x_{d,\cap}),\mathbb{R}^n)$ in particular, the identity (27) implies that the composition

$$g_{MP}(t,x_d) := \begin{cases} g_{MP,0}(t,x_d(t)), & t\in[t_0,t_m),\\ g_{MP,1}(t,x_a(t)), & t\in[t_m,t_{1,r}),\end{cases} \tag{28}$$

satisfies $g_{MP}\in C^1((\mathcal{I}_0\cup\mathcal{I}_1)\times(\mathcal{U}(x_{d,0})\cup\mathcal{U}(x_{d,1})),\mathcal{U}_{ex})$, where $\mathcal{U}_{ex}\subset\mathcal{U}(x_{a,0})\cup\mathcal{U}(x_{a,1})$ and $t_{1,r}=\sup\mathcal{I}_1$. Similarly, on $\mathcal{I}_\cap$, the components $\dot{x}_d$ are equally and uniquely specified by the functions $h_{MP,0}$ and $h_{MP,1}$, implying that

$$\dot{x}_d(t)=h_{MP,0}(t,x_d(t))=h_{MP,1}(t,x_d(t)). \tag{29}$$

Since $h_{MP\,i}\in C^1(\mathcal{I}_i\times\mathcal{U}(x_{d,i}),\mathbb{R}^n)$ for $i=0,1$ and $h_{MP,0},h_{MP,1}\in C^1(\mathcal{I}_\cap\times\mathcal{U}(x_{d,\cap}),\mathbb{R}^n)$ in particular, then (29) implies that the composition

$$h_{MP}(t,x_d) := \begin{cases} h_{MP,0}(t,x_d(t)), & t\in[t_0,t_m),\\ h_{MP,1}(t,x_d(t)), & t\in[t_m,t_{1,r}),\end{cases} \tag{30}$$

satisfies $h_{MP} \in C^1((\mathcal{I}_0 \cup \mathcal{I}_1) \times \mathcal{U}(x_{d,0}) \cup \mathcal{U}(x_{d,1}), \mathbb{R}^n)$. Repeating this continuation process along x, we can successively extend g_{MP}, h_{MP} onto $[t_0, \hat{t}_0^+)$.

It remains to show that g_{MP}, h_{MP} do not depend on the choice of the s-free formulation. If

$$\hat{F}(t, x, \dot{x}) = 0,\ x(t_0) = x_0, \qquad \tilde{F}(t, x, \dot{x}) = 0,\ x(t_0) = x_0 \tag{31}$$

are two s-free formulations of (13) then there exist pointwise orthogonal matrix functions $U_1 \in \mathbb{R}^{d\times d}$, $U_2 \in C^2(\mathcal{U}(z_{\mu,0}), \mathbb{R}^{a\times a})$, such that

$$\tilde{F}(t, x, \dot{x}) = U^T(z_\mu)\hat{F}(t, x, \dot{x}), \tag{32}$$

where $z_\mu = (t, x, v_0, ..., v_{\mu+1})$, cp. Remark 3.2. Both systems (31) can be remodeled as described above, i.e., there exist implicitly defined functions $g_{MP}, \tilde{g}_{MP} \in C^1(\mathcal{I}_0 \times \mathcal{U}(x_{d,0}), \mathcal{U}(x_{a,0}))$ and $h_{MP}, h_{MP,0} \in C^1(\mathcal{I}_0 \times \mathcal{U}(x_{d,0}), \mathcal{U}(\dot{x}_{d,0}))$ satisfying

$$\hat{F}(t, x_d + g_{MP}(t, x_d), h_{MP}(t, x_d) + \dot{g}_{MP}(t, x_d)) = 0, \tag{33a}$$

$$\tilde{F}(t, x_d + \tilde{g}_{MP}(t, x_d), \tilde{h}_{MP,0}(t, x_d) + \dot{\tilde{g}}_{MP}(t, x_d)) = 0. \tag{33b}$$

Along x, the Moore-Penrose projections induced by $\hat{F}$ and $\tilde{F}$ agree, cp. Lemma 3.1 and the differential and algebraic variables x_d, x_a coincide. Regarding relation (32) and noting that U is pointwise orthogonal, the functions $h_{MP,0}, \tilde{g}_{MP}$ solving (33b) also satisfy (33a). As the functions g_{MP}, h_{MP} are the unique solutions of the implicit equation $\hat{F}(t, x_d + g_{MP}(t, x_d), h(t, x_d) = 0$, cp. [3, Thm. 3.1] and [2, Lem. 3.1.3], it follows that $\tilde{h}_{MP} = h_{MP}$ and $\tilde{g}_{MP} = g_{MP}$.

$\Leftarrow$ Let $x \in C^1([t_0, \hat{t}_0^+), \mathbb{R}^n)$ be the solution of (13) and let (19) be constructed along x. Let $y \in C^1([t_0, \hat{t}_0^+), \mathbb{R}^n)$ solve (19) on $[t_0, \hat{t}_0^+)$ via the components y_d and y_a. We prove that $x = y$ on $[t_0, \hat{t}_0^+)$. If $y_a = g_{MP}(t, y_d)$ on $[t_0, \hat{t}_0^+)$, then

$$\hat{F}_2(t, y) = 0 \tag{34}$$

on $[t_0, \hat{t}_0^+)$ by the construction of g_{MP}. If, in addition, $\dot{y}_d = h(t, y_d)$ on $[t_0, \hat{t}_0^+)$, then

$$\hat{F}_1(t, y, \dot{y}) = \hat{F}_1(t, y, h_{MP}(t, y_d) + \dot{g}_{MP}(t, y_d)).$$

By the construction of h_{MP}, noting that $P_{MP}^\perp(z)\dot{g}_{MP}(t, y_d) = \dot{P}_{MP}(z)(y_d + y_a) + \dot{g}_{MP}(t, y_d)$ due to $\dot{y}_a = P_{MP}^\perp(z)\dot{y}_a - \dot{P}_{MP}(z)y_a$, this equation reads

$$\begin{aligned}
\hat{F}_1(t, y, \dot{y}) &= \hat{F}_1\big(t, y, \hat{h}\big(t, y_d, g_{MP}(t, y_d), \dot{g}_{MP}(t_0, y_{d,0}) + \dot{P}_{MP}^\perp(z_0)(y_{d,0} + y_{a,0})\big) \\
&\quad + \dot{P}_{MP}(z)(y_d + y_a) + \dot{g}_{MP}(t, x_d)\big) \\
&= \hat{F}_1\big(t, y, \hat{h}\big(t, y_d, g_{MP}(t, y_d), \dot{g}_{MP}(t_0, y_{d,0}) + \dot{P}_{MP}^\perp(z_0)(y_{d,0} + y_{a,0})\big) \\
&\quad + P_{MP}^\perp(z)\dot{g}_{MP}(t, y_d)).
\end{aligned}$$

Using that $\dot{y}_a = P_{MP}^\perp(z)\dot{y}_a - \dot{P}_{MP}(z)y_a$ and $\mathrm{range}(P_{MP}^\perp(z)) = \ker(\hat{F}_{\dot{x}}(z)))$, we find that

$$\hat{F}_{\dot{x}_a}(t, y, \dot{y}) = \big(\hat{F}_{\dot{x}}P_{MP}^\perp\big)(t, y, \dot{y}) = 0.$$

As $\hat{F}_{\dot{x}} = [\hat{F}_{1,\dot{x}}^T, 0]^T$, it follows that $\hat{F}_1$ is independent of $\dot{x}_a$. By the definition of $\hat{h}$, then

$$\begin{aligned}
0 &= \hat{F}_1(t, y, \hat{h}\big(t, y_d, g_{MP}(t, y_d), \dot{g}_{MP}(t_0, x_{d,0}) + \dot{P}_{MP}^\perp(z_0)(x_{d,0} + x_{a,0})\big) + P_{MP}^\perp(z_0)\dot{g}_{MP}(t_0, y_{d0})) \\
&= \hat{F}_1(t, y, \dot{y}).
\end{aligned} \tag{35}$$

In combination, (34) and (35) imply that y solves (15) on $[t_0, \hat{t}_0^+)$. As $F \in C^{\mu+2}_{\mu,d,a,\mathrm{reg}}(\mathbb{D}, \mathbb{R}^{n\times n})$ and $(t_0, x_0) \in \mathcal{C}_{F,\mu+1}$, then y solves the original problem (13) and since the solution is unique, it follows that $x = y$ on $[t_0, \hat{t}_0^+)$. □

We call (19) and the functions h_{MP}, g_{MP} the *Moore-Penrose* remodeling of the IVP (13).

Remark 3.4. *In Theorem 3.3, we decouple the differential and algebraic variables using that the differential and algebraic equations* (15a), (15b) *are explicitly given in the s-free formulation. To remodel a general s-free DAE*

$$F(t, x, \dot{x}) = 0, \tag{36}$$

we can filter out the differential and algebraic equations using the Moore-Penrose projections $Q_{MP}(z) = F_{\dot{x}}(z)F_{\dot{x}}^+(z)$, $Q_{MP}^\perp = I_n - Q_{MP}$. *If* (36) *is s-free, then and only then*

$$\big(Q_{MP}^\perp \hat{F}_x P_{MP}^\perp\big)^+\big(Q_{MP}^\perp \hat{F}_x P_{MP}^\perp\big)(z) = P_{MP}^\perp(z), \tag{37a}$$

$$\big(Q_{MP}^\perp \hat{F}_x P_{MP}^\perp\big)\big(Q_{MP}^\perp \hat{F}_x P_{MP}^\perp\big)^+(z) = Q_{MP}^\perp(z) \tag{37b}$$

is satisfied pointwise on $F^{-1}(0)$ *and we can remodel the DAE* (36) *as in Theorem 3.3, solving* $(Q_{MP}^\perp F)(t, x, \dot{x}) = 0, (QF)(t, x, \dot{x}) = 0$ *for* $\dot{x}_d, x_a$*, respectively, cp. [2, Thm. 4.3.1].*
Condition (22) *is satisfied if and only if the matrix* $(\hat{F}_{x,2}T_2)(z)$ *is nonsingular, i.e., if and only if the remodeling* $\hat{F}$ *is s-free, cp. Hypothesis* (3.1)*. Thus, condition* (22) *allows to check if the computed remodeling* $\hat{F}$ *indeed is s-free. Similarly, the DAE* (36) *is s-free if and only if* $(S_2\hat{F}_{x,2}T_2)(z)$*, where* $P_{MP}^\perp = T_2T_2^T$ *and* $Q_{MP}^\perp = S_2S_2^T$*, is nonsingular.*

Solving the decoupled system (19), we find that the differential components x_d are evolved by the flow $\Phi_{h_{MP}}$ induced by the function h_{MP}, while the algebraic components x_a are coupled to this evolution by the function g_{MP}. In combination, we obtain an additively composed solution formula of (13) consisting of a dynamic part related with $\Phi_{h_{MP}}$ and a constrained part specified by g_{MP}.

Lemma 3.2. *Consider the IVP* (13) *and let* $x \in C^1([t_0, \hat{t}_0^+), \mathbb{R}^n)$ *be its solution. Set* $z_0 := (t_0, x_0, \dot{x}(t_0))$*. On* $[t_0, \hat{t}_0^+)$*, the solution* x *is given by*

$$x(t) = \Phi_{h_{MP}}^t\big(t_0, P_{MP}(z_0)x_0\big) + g_{MP}\big(t, \Phi_{h_{MP}}^t(t_0, P_{MP}(z_0)x_0)\big), \tag{38}$$

where $P \in C^1([t_0, \hat{t}_0^+) \times \mathcal{U}_{P_{MP}}(x) \times \mathcal{U}_{P_{MP}}(\dot{x}), \mathbb{R}^{n\times n})$*,* $g_{MP} \in C^1([t_0, \hat{t}_0^+) \times \mathcal{U}(x_a), \mathcal{U}(x_d))$ *and* $h_{MP} \in C^1([t_0, \hat{t}_0^+) \times \mathcal{U}(x_d), \mathbb{R}^n)$ *are the Moore-Penrose projection and remodeling induced by* (13) *and* $\Phi_{h_{MP}}$ *is the flow associated with* h_{MP}*.*

Proof. Along the solution x, we can decouple the IVP (13) as decoupled system (19) for the components x_d, x_a. With $h_{MP} \in C^1([t_0, \hat{t}_0^+) \times \mathcal{U}(x_d), \mathbb{R}^n)$, the ODE (19a) induces the flow $\Phi_{h_{MP}}$, such that, on $[t_0, \hat{t}_0^+)$,

$$x_d(t) = \Phi_{h_{MP}}^t(t_0, P_{MP}(z_0)x_0). \tag{39a}$$

Inserting (39a) into the algebraic equation (19b), we obtain that

$$x_a(t) = g_{MP}(t, \Phi_{h_{MP}}^t(t_0, P_{MP}(z_0)x_0)). \tag{39b}$$

With $x = x_d + x_a$, we have proven the representation (38). Noting that P_{MP}, g_{MP}, h_{MP} are C^1-functions, we have verified that the representation (38) is continuously differentiable on $[t_0, \hat{t}_0^+)$. □

The solution formula (38) is defined for every consistent initial value and on the full interval of existence of the associated solution. Thus, it gives rise to an explicit representation of the flow Φ_F.

Theorem 3.4. *Consider the DAE* (13a)*. For every* $(t_0, x_0) \in \mathcal{C}_{F,\mu+1}$ *and* $z_0 = (t_0, x_0, v_0) \in \mathcal{L}_{\mu+1}$*, on* $[t_0, \hat{t}_0^+)$ *the flow* Φ_F *is given by*

$$\Phi_F^t(t_0, x_0) = \Phi_{h_{MP}}^t\big(t_0, P_{MP}(z_0)x_0\big) + g_{MP}\big(t, \Phi_{h_{MP}}^t(t_0, P_{MP}(z_0)x_0)\big), \tag{40}$$

where $P \in C^1([t_0, \hat{t}_0^+) \times \mathcal{U}_{P_{MP}}(x) \times \mathcal{U}_{P_{MP}}(\dot{x}), \mathbb{R}^{n\times n})$*,* $g_{MP} \in C^1([t_0, \hat{t}_0^+) \times \mathcal{U}(x_a), \mathcal{U}(x_d))$ *and* $h_{MP} \in C^1([t_0, \hat{t}_0^+) \times \mathcal{U}(x_d), \mathbb{R}^n)$ *are the Moore-Penrose projection and remodeling induced by* (13) *and* $\Phi_{h_{MP}}$ *is the flow associated with* h_{MP}*. Furthermore, the flow* Φ_F *satisfies*

$$P_{MP}\big(t, \Phi_F^t(t_0, x_0), \dot{\Phi}_F^t(t_0, x_0)\big)\, \Phi_F^t(t_0, x_0) = \Phi_{h_{MP}}^t\big(t_0, P_{MP}(z_0)x_0\big), \tag{41a}$$

$$P_{MP}^{\perp}\big(t, \Phi_F^t(t_0, x_0), \dot{\Phi}_F^t(t_0, x_0)\big)\, \Phi_F^t(t_0, x_0) = g_{MP}\big(t, \Phi_{h_{MP}}^t(t_0, P_{MP}(z_0)x_0)\big). \tag{41b}$$

Proof. By definition, the function $\Phi_F^{(\cdot)}(t_0, x_0)$ agrees with the unique solution of the IVP (13) for every $(t_0, x_0) \in \mathcal{C}_{F,\mu+1}$, cp. Corollary 3.1. Using formula (38), we have verified the representation (40).
As a consequence of the construction of h_{MP}, on $[t_0, \hat{t}_0^+)$ the associated flow $\Phi_{h_{MP}}$ satisfies $\Phi_{h_{MP}}^t(t_0, P_{MP}(z_0)x_0) \in \operatorname{coker}\big(\hat{F}_{\dot{x}}\big(t, \Phi_F^t(t_0, x_0), \dot{\Phi}_F^t(t_0, x_0)\big)\big)$, cp. [2, Theorem 3.2.2]. Hence,

$$P_{MP}^{\perp}\big(t, \Phi_F^t(t_0, x_0), \dot{\Phi}_F^t(t_0, x_0)\big)\ \Phi_{h_{MP}}^t(t_0, x_0) = 0, \tag{42}$$

on $[t_0, \hat{t}_0^+)$. Similarly, $g_{MP}(t, x_d) \in \ker(\hat{F}_{\dot{x}}(t, x, \dot{x}))$, cp. [3, Thm. 3.1], implying that

$$P_{MP}\big(t, \Phi_F^t(t_0, x_0), \dot{\Phi}_F^t(t_0, x_0)\big)\, g_{MP}\big(t, \Phi_h^t(t_0, P_{MP}(z_0)x_0)\big) = 0, \tag{43}$$

on $[t_0, \hat{t}_0^+)$. From (42) and (43), we conclude that $\Phi_F^{(\cdot)}(t_0, x_0)$ satisfies (41) on $[t_0, \hat{t}_0^+)$. □

The flow formula (40) reflects the two flavors of a DAE: Parts of the solution are evolved by a flow, while the other part is coupled to this evolution via an algebraic relation. For the overall solution, this results in a dynamic evolution which is constrained to a flat subset in the state space. Locally, this constraint can be represented as a time-varying manifold.

Lemma 3.3. *Consider the IVP* (13) *and let* $\hat{F} = [\hat{F}_1^T, \hat{F}_2^T]^T \in C^2([t_0, \hat{t}_0^+) \times \hat{\mathcal{U}}(x) \times \hat{\mathcal{U}}(\dot{x}), \mathbb{R}^n)$ *be an s-free remodeling. For every* $(t_0, x_0) \in \mathcal{C}_{F,\mu+1}$*, the following assertions are true.*

(i) The set $\mathbb{M}_F(t_0, x_0) := \hat{F}_2^{-1}(0)$ *is a time-varying, embedded* C^2*-submanifold with* $\dim(\mathbb{M}_F(t_0, x_0)) = d$*.*

(ii) The set $\mathbb{M}_F(t_0, x_0)$ *is independent of the chosen remodeling* $\hat{F}$*.*

(iii) The set of consistent initial values satisfies $\big(([t_0, \hat{t}_0^+) \times \hat{\mathcal{U}}(x)) \cap \mathcal{C}_{F,\mu+1}\big) \subset \mathbb{M}_F(t_0, x_0)$*.*

(iv) The flow Φ_F *satisfies* $\Phi_F^t(t_0, x_0) \in \mathbb{M}_F(t_0, x_0)(t)$ *for* $t \in [t_0, \hat{t}_0^+)$*.*

Proof. (i), (iv) For the remodeling $\hat{F} \in C^2([t_0, \hat{t}_0^+) \times \hat{\mathcal{U}}(x) \times \hat{\mathcal{U}}(\dot{x}), \mathbb{R}^n)$, we have that $\mathrm{rank}(D\hat{F}_2) = \mathrm{rank}(\hat{F}_{2,x}) = n - d$ on $\mathcal{C}_{\hat{F}_2} \cap (\mathcal{I}_0 \times \mathcal{U}(x_0))$ as $\hat{F}$ is s-free, cp. Hypothesis 3.1. Hence, the algebraic solution set $\hat{F}_2^{-1}(0)$ a time-varying submanifold embedded in $\mathbb{R}^n$ with $\dim(\hat{F}_2^{-1}(0)) = d$, cp. Lemma 2.1. As the function $\Phi_F^{(\cdot)}(t_0, x_0)$ solves the s-free remodeling, it follows that $\Phi_F^t(t_0, x_0) \in \mathbb{M}_F(t_0, x_0)(t)$ on $[t_0, \hat{t}_0^+)$.
(ii) Given μ and an initial value $(t_0, x_0) \in \mathcal{C}_{F,\mu+1}$, the remodeling $\hat{F}_2$ is specified up to nonsingular transformations of the matrix Z_2, cp. Remark 3.2. These transformations do not alter the solution set $\hat{F}_2^{-1}(0)$, hence $\mathbb{M}_F(t_0, x_0)$ is independent of the choice of $\hat{F}$.
(iii) The remodeling $\hat{F}$ serves as s-free remodeling for every initial condition $(\tilde{t}_0, \tilde{x}_0) \in ([t_0, \hat{t}_0^+) \times \hat{\mathcal{U}}(x) \times \hat{\mathcal{U}}(\dot{x})) \cap \mathcal{L}_{F,\mu+1}$. Hence, $([t_0, \hat{t}_0^+) \times \hat{\mathcal{U}}(x)) \cap \mathcal{C}_{F,\mu+1} \subset \mathbb{M}_F(t_0, x_0)$. □

The projection properties (41) allow to access the differential and algebraic solution components x_d and x_a by projecting with $P_{MP}(t, x, \dot{x})$ and $P_{MP}^{\perp}(t, x, \dot{x})$, respectively. Analyzing system properties like stability or positivity, this allows to specify the condition on the differential and algebraic solution components, cp. [2, ch. 5].
The representation (40) is uniquely defined by (13a) as the Moore-Penrose projection P_{MP} and remodeling g_{MP}, h_{MP} are independent of the chosen remodeling $\hat{F}$.

Remark 3.5. *The non-autonomous DAE* (1) *can be autonomized by setting*

$$F_{aut}(z, \dot{z}) = 0, \tag{44a}$$

$$F_{aut}(z, \dot{z}) := \begin{bmatrix} \dot{t} - 1 \\ f(t, x, \dot{x}) \end{bmatrix}, \quad z := \begin{bmatrix} t \\ x \end{bmatrix}, \quad \dot{z} := \begin{bmatrix} \dot{t} \\ \dot{x} \end{bmatrix}, \tag{44b}$$

cp. [29, p. 159]. If $F \in C^{\mu+1}_{\mu,d,a}(\mathbb{D}, \mathbb{R}^n)$, then $F_{aut} \in C^{\mu+1}_{\mu,d+1,a}(\Omega_x \times \Omega_{\dot{x}}, \mathbb{R}^{n+1})$, cp. [29, p. 159], and the flows Φ_F and $\Phi_{F_{aut}}$ associated with F and F_{aut} are related by

$$\Phi^t_{F_{aut}}(t_0) = \begin{bmatrix} t \\ \Phi_F^t(t_0, x_0) \end{bmatrix}. \tag{45}$$

3.2 Explicit remodeling using constant Moore-Penrose projections

The decoupled system (19) is constructed by decomposing the variables along the solution x, yielding a smooth decomposition of the differential and algebraic components on the full interval of existence. To explicitly compute the remodeling (19) and the flow formula (40), however, we need to consider the explicit variable decomposition

$$x_d = P_{MP}(t_0, x_0, v_0)x, \quad x_a = P_{MP}^{\perp}(t_0, x_0, v_0)x \tag{46}$$

induced by evaluating the Moore-Penrose projections in a consistent initialization $(t_0, x_0, v_0) \in \mathcal{L}_{\mu+1}$.

Lemma 3.4. *Consider the IVP* (13). *Let $\hat{F} \in C^2([t_0, \hat{t}_0^+) \times \hat{\mathcal{U}}(x) \times \hat{\mathcal{U}}(\dot{x}), \mathbb{R}^n)$ be an s-free remodeling and $P_{MP} \in C^1([t_0, \hat{t}_0^+) \times \mathcal{U}_{P_{MP}}(x) \times \mathcal{U}_{P_{MP}}(\dot{x}), \mathbb{R}^n)$ the associated Moore-Penrose projection. Let $v_0 \in \mathbb{R}^n$ be such that $z_0 := (t_0, x_0, v_0) \in \mathcal{L}_{F,\mu+1}$ and consider the variable decomposition* (46). *Then, there exists $t_1 \in (t_0, t_0^+)$, such that, on $\mathcal{I}_0 := [t_0, t_1)$, the function $x \in C^1(\mathcal{I}_0, \mathbb{R}^n)$ solves* (13) *if and only if its components x_d, x_a solve the decoupled system*

$$\dot{x}_d = h_{MP,0}(t, x_d), \quad x_d(t_0) = P_{MP}(z_0)x_0, \tag{47a}$$

$$x_a = g_{MP,0}(t, x_d). \tag{47b}$$

The functions $g_{MP,0} \in C^1(\mathcal{I}_0 \times \mathcal{U}(x_{d,0}), \mathcal{U}(x_{a,0}))$ *and* $h_{MP,0} \in C^1(\mathcal{I}_0 \times \mathcal{U}(x_{d,0}), \mathbb{R}^n)$ *are uniquely defined by* (13) *as the implicit solutions of*

$$\hat{F}_1\big(t, x_d + x_a, h_{MP,0}(t, x_d) + \dot{g}_{MP,0}(t_0, x_{d,0})\big) = 0, \tag{48a}$$

$$\hat{F}_2\big(t, x_d + g_{MP,0}(t, x_d)\big) = 0 \tag{48b}$$

and satisfy $g_{MP,0}(t, x_d) \in \ker(P_{MP}(z_0))$ *and* $h_{MP,0}(t, x_d) \in \ker(P_{MP}^{\perp}(z_0)) + (\dot{P}_{MP}P)(z_0)$.

Proof. The assertion follows using similar arguments proving Theorem 3.3. If $z_0 := (t_0, x_0, v_0) \in [t_0, \hat{t}_0^+) \times \mathcal{U}_{P_{MP}}(x) \times \mathcal{U}_{P_{MP}}(\dot{x}) \cap \mathcal{L}_{F,\mu+1}$, we have shown that $P(z_0), \hat{F}_2(z_0)$ satisfy condition (22). By the projection-based Implicit Function Theorem [3, Thm. 3.1], then there exist neighborhoods $\mathcal{I}_0, \mathcal{U}(x_{d,0}), \mathcal{U}(x_{a,0})$ and a function $g_{MP,0} \in C^1(\mathcal{I}_0 \times \mathcal{U}(x_{d,0}), \mathcal{U}(x_{a,0}))$ such that (t, x) solves $\hat{F}_2(t, x) = 0$ if and only if $(t, x_d) \in \tilde{I}_0 \times \tilde{\mathcal{U}}(x_0)$ and $x_a = g_{MP,0}(t, x_d)$, where $x_d = P_{MP}(z_0)x$ and $x_a = P_{MP}^{\perp}(z_0)x$. In particular, $g_{MP,0}(t, x_d) \in \ker(P_{MP}(z_0))$. Similarly, condition (24) is satisfied in z_0, and following the steps in the proof of Theorem 3.3, we can construct a function $h_{MP,0} \in C^1(\mathcal{I}_0 \times \mathcal{U}(x_{d,0}), \mathbb{R}^n)$ such that $(t, x, \dot{x})$ solves $\hat{F}_1(t, x, \dot{x}) = 0$ if and only if $\dot{x}_d = h_{MP,0}(t, x_d)$. In particular, $h_{MP,0}(t, x_d) \in \ker(P_{MP}^{\perp}(z_0)) + (\dot{P}_{MP}P)(z_0)$f. Hence, for $t_1 \in (t_0, t_0^+)$ sufficiently small, the function $x \in C^1(\mathcal{I}_0, \mathbb{R}^n)$ solves (13) on $\mathcal{I}_0 := [t_0, t_1)$ if and only if its components x_d, x_a solve the decoupled system (47). □

Using the local (in time) Moore-Penrose remodeling (47), we can explicitly compute the solution x of (13) and hence the flow by proceeding piecewise along x.

Corollary 3.2. *Consider the DAE* (13a). *For every* $(t_0, x_0) \in \mathcal{C}_{F,\mu+1}$ *and* $z_0 := (t_0, x_0, v_0) \in \mathcal{L}_{F,\mu+1}$, *there exists* $t_1 \in (t_0, t_0^+)$, *such that, on* $\mathcal{I}_0 := [t_0, t_1)$,

$$\Phi_F^t(t_0, x_0) = \Phi_{h_{MP,0}}^t\big(t_0, P_{MP}(z_0)x_0\big) + g_{MP,0}\big(t, \Phi_{h_{MP,0}}^t\big(t_0, P_{MP}(z_0)x_0\big)\big), \tag{49}$$

where $g_{MP,0} \in C^1(\mathcal{I}_0 \times \mathcal{U}(x_{d,0}), \mathcal{U}(x_{a,0}))$ *and* $h_{MP,0} \in C^1(\mathcal{I}_0 \times \mathcal{U}(x_{d,0}), \mathbb{R}^n)$ *are induced by* $P_{MP}(z_0)$ *and* $\Phi_{h_{MP,0}}$ *is the flow associated with* $h_{MP,0}$.
On $\mathcal{I}_0$, *the flow* Φ_F *satisfies*

$$P_{MP}(z_0)\Phi_F^t(t_0, x_0) = \Phi_{h_{MP,0}}^t(t_0, P_{MP}(z_0)x_0), \tag{50a}$$

$$P_{MP}^{\perp}(z_0)\Phi_F^t(t_0, x_0) = g_{MP,0}(t, \Phi_{h_{MP,0}}^t(t_0, P_{MP}(z_0)x_0)). \tag{50b}$$

In a numerical solution, the projection properties (50) allow to check the consistency of the numerical solution x_Δ by projecting with $P_{MP}(z_0)$ and $P_{MP}^{\perp}(z_0)$, respectively, and checkin the relation $x_{\Delta,N,a} = g(t_N, x_{\Delta,N,d})$. As the projections $P_{MP}(z_0), P_{MP}^{\perp}(z_0)$ are constant, this test is independent of the numerical solution.

3.3 Linear differential-algebraic equations

For linear systems

$$F_{E,A,b}(t, x, \dot{x}) := E(t)\dot{x} - A(t)x - b(t) = 0 \tag{51}$$

with $E, A \in C^\ell(\mathcal{I}, \mathbb{R}^{n\times n})$ and $b \in C^\ell(\mathcal{I}, \mathbb{R}^n)$, the derivative array (7) is linear in the state z_ℓ and the block matrices M_ℓ, N_ℓ are defined globally on $\mathcal{I} \times \mathbb{R}^n$ independent of a particular initial value

(t_0, x_0), cp. [29, p. 81]. For sufficiently smooth functions $F_{E,A,b}$, the assertions of Hypothesis 3.1 are satisfied globally on $\mathbb{R}^n$, cp. [29, p. 108], i.e., $F_{E,A,b} \in C^{\ell}_{\mu,d,a,\mathrm{reg}}(\mathbb{D}, \mathbb{R}^n)$ if $F_{E,A,b} \in C^{\ell}(\mathbb{D}, \mathbb{R}^n)$, where $\mathbb{D} = \mathcal{I} \times \mathbb{R}^n \times \mathbb{R}^n$. If $F_{E,A,b} \in C^{\mu+1}_{\mu,d,a,\mathrm{reg}}(\mathbb{D}, \mathbb{R}^n)$, then (51) is uniquely solvable for every initial value $(t_0, x_0) \in \mathcal{C}_{E,A,b,\mu}$, where $\mathcal{C}_{E,A,b,\mu} := \mathcal{C}_{F,\mu}$, and the solution is defined on the full interval $\mathcal{I}$. The s-free formulation $\hat{F}_{\hat{E},\hat{A},\hat{b}}$ of (51) is globally defined on $\mathbb{D}$ and independent of the initial value, cp. [29, p. 109, 111]. A function $x \in C^1(\mathcal{I}, \mathbb{R}^n)$ solves (51) on $\mathcal{I}$ if and only if x solves

$$\hat{F}_{\hat{E},\hat{A},\hat{b}}(t, x, \dot{x}) = \begin{bmatrix} \hat{E}_1 \\ 0 \end{bmatrix} \dot{x} - \begin{bmatrix} \hat{A}_1 \\ \hat{A}_2 \end{bmatrix} x - \begin{bmatrix} \hat{b}_1 \\ \hat{b}_2 \end{bmatrix} = 0. \tag{52}$$

The Jacobian $\hat{F}_{\dot{x}} = \hat{E}$ is independent of the state x and hence the Moore-Penrose projection $P_{MP}(t) = (\hat{E}^+\hat{E})(t)$ is globally defined on $\mathbb{D}$. The remodeling h_{MP}, g_{MP} is explicitly given as affine linear transformations that are globally defined on $\mathcal{I} \times \mathbb{R}^n$.

Theorem 3.5. *Consider the DAE* (51) *with* $F_{E,A,b} \in C^{\mu+1}_{\mu,d,a,reg}(\mathbb{D}, \mathbb{R}^n)$. *Let* $\hat{F}_{\hat{E},\hat{A},\hat{b}} \in C^1_{0,d,a,reg}(\mathbb{D}, \mathbb{R}^n)$ *be an s-free remodeling and* $P_{MP} \in C^1(\mathcal{I}, \mathbb{R}^{n\times n})$ *the associated Moore-Penrose projection. A function* $x \in C^1(\mathcal{I}, \mathbb{R}^n)$ *solves* (51) *with* $x(t_0) = x_0$, $(t_0, x_0) \in \mathcal{C}_{E,A,b,\mu}$, *if and only if the components* x_d, x_a *solve*

$$\dot{x}_d = h_{MP}(t, x), \tag{53a}$$

$$x_a = g_{MP}(t, x_d), \tag{53b}$$

where $h_{MP}(t, x) := D_d(t)x + b_d(t)$ *and* $g_{MP}(t, x_d) := D_a(t)x_d - b_a(t)$ *with*

$$D_d := (\hat{E}^+\hat{A} + \dot{P}_{MP})\mathcal{P}_{MP}, \qquad b_d := \hat{E}^+\hat{b} - (\hat{E}^+\hat{A} + \dot{P}_{MP})b_a, \tag{54a}$$

$$D_a := (\hat{A}_2 P^{\perp}_{MP})^+ \hat{A}_2 P_{MP}, \qquad b_a := (\hat{A}_2 P^{\perp}_{MP})^+ \hat{b}. \tag{54b}$$

In particular, $h_{MP} \in C(\mathcal{I} \times \operatorname{coker}(\hat{E}(\cdot)), \mathbb{R}^n)$ *and* $g_{MP} \in C^1(\mathcal{I} \times \operatorname{coker}(\hat{E}(\cdot)), \ker(\hat{E}(\cdot)))$.

Exploiting the linearity, we can specify the solution formula (38) and construct a globally defined representation of the flow $\Phi_{E,A,b} := \Phi_{F_{E,A,b}}$.

Theorem 3.6. *Consider the DAE* (51) *with* $F_{E,A,b} \in C^{\mu+1}_{\mu,d,a,reg}(\mathbb{D}, \mathbb{R}^n)$. *Let* P_{MP} *and* D_d, D_a, b_d, b_a *be the associated Moore-Penrose projection and remodeling and* $\mathcal{P}_{MP} = (I_n - D_a)P_{MP}$. *On* $\mathcal{C}_{E,A,b,\mu} \times \mathcal{I}$, *the flow* $\Phi_{E,A,b}$ *is given by*

$$\Phi^t_{E,A,b}(t_0, x_0) = \Phi^t_{E,A}(t_0, x_0) + \int_{t_0}^{t} \big(\Phi^t_{E,A} b_d\big)(s)\, ds - b_a(t), \tag{55}$$

with the homogeneous flow $\Phi^t_{E,A}(t_0) = \mathcal{P}_{MP}(t)\big(\Phi^t_{D_d} P_{MP}\big)(t_0)$.
For every $t \in \mathcal{I}$, *the homogeneous flow* $\Phi_{E,A}$ *possesses the semi inverse* $(\Phi^t_{E,A}(t_0))^{ginv} = \Phi^{t_0}_{E,A}(t)$ *satisfying* $\Phi^{t_0}_{E,A}(t)\Phi^t_{E,A}(t_0) = \mathcal{P}_{MP}(t_0)$ *and* $\Phi^t_{E,A}(t_0)\Phi^{t_0}_{E,A}(t) = \mathcal{P}_{MP}(t)$.

Like for ODEs, the flow $\Phi_{F_{E,A,b}}$ is an affine linear transformation composed of the homogeneous flow $\Phi_{E,A}$ and an inhomogeneous part induced by b. For constrained systems, however, only the parts of the initial value and the inhomogeneity lying in $\operatorname{coker}(E(\cdot))$ are dynamically evolved, while the components in $\ker(E(\cdot))$ are fixed by an algebraic relation. Formula (55) generalizes

Duhamel's formula to linear constrained systems with sufficiently smooth coefficients.
As the projections $P_{MP}, P_{MP}^{\perp}$ are linear in the state x, the projection properties (41) allow to access the differential and the algebraic solution components independently of a a given solution. Thus, we can check the consistency of the dynamic and algebraic approximations $x_{d,\Delta}$ and $x_{a,\Delta}$ of a numerical solution $x_\Delta \approx x$ *exactly* by projecting onto $\operatorname{coker}(\hat{E})$ and $\operatorname{coker}(\hat{E})$, respectively. The semi-inverse $(\Phi_{E,A}^t(t_0))^{ginv} = \Phi_{E,A}^{t_0}(t)$ allows to recover the initial value x_0 from a given solution $\Phi_{E,A}^t(t_0)$ for every time $t \in \mathcal{I}$.
For linear problems, the solution manifold is a time-varying linear subspace that coincides with the set of consistent initial values $\mathcal{C}_{E,A,b,\mu}$.

Lemma 3.5. *Consider the DAE* (51) *with* $F_{E,A,b} \in C^{\mu+1}_{\mu,d,a,reg}(\mathbb{D}, \mathbb{R}^n)$. *The set of consistent initial values* $\mathcal{C}_{E,A,b,\mu}$ *is a time-varying, affine linear* C^1*-subspace on* $\mathcal{I}$ *and* $\mathbb{M}_{E,A,b}(t_0, x_0) = \mathcal{C}_{E,A,b,\mu}$ *for every* $(t_0, x_0) \in \mathcal{C}_{E,A,b,\mu}$. *The function* $\mathcal{P}_{MP,-b_a}(t, x) = \mathcal{P}_{MP}(t)\, x - b_a(t)$, *is an affine linear projection onto* $\mathcal{C}_{E,A,b,\mu}$.

Proof. An initial value $(t_0, x_0) \in \mathcal{I} \times \mathbb{R}^n$ is consistent if and only if $\hat{F}_{\hat{E},\hat{A},\hat{b};2}(t_0, x_0) = \hat{A}_2(t_0)x_0 - \hat{b}_2(t_0) = 0$, cp. [29, p. 111]. Hence, $\mathcal{C}_{E,A,b,\mu} = \hat{F}^{-1}_{\hat{E},\hat{A},\hat{b};2}(0)$, implying that $\mathcal{C}_{E,A,b,\mu} = \mathbb{M}_{E,A,b}(t_0, x_0)$. In particular, as $\hat{F}_{\hat{E},\hat{A},\hat{b};2}$ is an affine linear function, its algebraic solution set is a time-varying, affine linear subspace on $\mathcal{I}$, cp. [3, Lem. 2.6] and [2, Rem. 2.4.3].
By construction of the function g_{MP}, $\mathcal{C}_{E,A,b,\mu} = \hat{F}^{-1}_{\hat{E},\hat{A},\hat{b};2}(0)$ further implies that $(t_0, x_0) \in \mathcal{C}_{E,A,b,\mu}$ if and only if $P_{MP}^{\perp}(t_0)x_0 = (D_a P_{MP})(t_0)x_0 - b_a(t_0)$, i.e., if and only if $x_0 = \mathcal{P}_{MP}(t_0) - b_a(t_0)$. Hence, $\mathcal{C}_{E,A,b,\mu} = \operatorname{range}(\mathcal{P}_{MP,-b_a})$. Noting that $P_{MP}D_a = 0$ and $D_a P_{MP} = D_a$, cp. (54), we verify that $\mathcal{P}_{MP} = (I_n - D_a)P_{MP}$ is idempotent and hence $\mathcal{P}_{MP,-b_a}(t, x)$ is an affine projection onto $\mathcal{C}_{E,A,b,\mu}$. ☐

Hence, we can validate the consistency of a numerical solution x_Δ using the projection $\mathcal{P}_{MP,-b_a}$.

Remark 3.6. *For constant coefficients* $E, A \in \mathbb{R}^{n\times n}$ *and* $b \in C^{\mu+1}(\mathcal{I}, \mathbb{R}^n)$, *the Moore-Penrose remodeling is given by* $D_d := \hat{E}^+\hat{A}\mathcal{P}_{MP}, D_a := (\hat{A}_2 P_{MP}^{\perp})^+\hat{A}_2, b_d := \hat{E}^+(\hat{b} - \hat{A}b_a), b_a := (\hat{A}_2 P_{MP}^{\perp})^+\hat{b}_2$. *The homogeneous flow* $\Phi_{E,A}$ *reads* $\Phi_{E,A}^t(t_0) := \mathcal{P}_{MP}\, e^{D_d(t-t_0)} P_{MP}$.

3.4 Linearization of the flow

To study properties of the DAE (13a) like invariant sets, stability or positivity, we need the linearization of its solutions. For the ODE (2), the linearization of a solution x in a point (t_0, x_0) is explicitly given by the function f, i.e., $x(t) = x_0 + (t - t_0)f(t_0, x_0) + \mathcal{O}((t - t_0)^2)$ if $f \in C^1(\mathcal{I} \times \Omega, \mathbb{R}^n)$. For the DAE (13a), the derivative $\dot{x}$ of a solution is specified implicitly only. Having a flow Φ_F, however, that coincides with the solutions, we can define a vector field $\mathcal{T}_F\colon \mathcal{C}_{F,\mu+1} \to \mathbb{R}^n$ that assigns the derivative $\dot{\Phi}_F^t(t_0, x_0)$ to every $(t_0, x_0) \in \mathcal{C}_{F,\mu+1}$, i.e.,

$$\mathcal{T}_F(t_0, x_0) := \dot{\Phi}_F^t(t_0, x_0).$$

For $F \in C^{\mu+2}(\mathbb{D}, \mathbb{R}^n)$, the linearization of the solution in (t_0, x_0) is given by $x(t) = x_0 + (t - t_0)\mathcal{T}_F(t_0, x_0) + \mathcal{O}((t - t_0)^2)$. We call $\mathcal{T}_F(t_0, x_0)$ the *tangent field of* Φ_F. Using the explicit representation (47) of the flow Φ_F, we can explicitly compute $\mathcal{T}_F$.

Lemma 3.6. *Consider the DAE* (13a) *with flow* Φ_F. *For* $(t_0, x_0) \in \mathcal{C}_{F,\mu+1}$, *let* $v_0 \in \mathbb{R}^n$ *be such that* $z_0 = (t_0, x_0, v_0) \in \mathcal{L}_{F,\mu+1}$. *Let* $P_{MP} \in C^1([t_0, \hat{t}_0^+) \times \mathcal{U}_{P_{MP}}(x) \times \mathcal{U}_{P_{MP}}(\dot{x}), \mathbb{R}^n)$ *be the Moore-Penrose projection induced by* (13) *and* $h_{MP,0} \in C^1(\mathcal{I}_0 \times \mathcal{U}(x_{d,0}), \mathbb{R}^n)$, $g_{MP,0} \in C^1(\mathcal{I}_0 \times \mathcal{U}(x_{d,0}), \mathcal{U}(x_{a,0}))$ *the Moore-Penrose remodeling obtained using* $P_{MP}(z_0)$. *Then, the tangent field* $\mathcal{T}_F$ *is given by*

$$\mathcal{T}_F(t_0, x_0) = h_{MP,0}\big(t_0, P_{MP}(z_0)x_0\big) + \dot{g}_{MP,0}\big(t_0, P_{MP}(z_0)x_0\big). \tag{56}$$

Proof. For $(t_0, x_0) \in \mathcal{C}_{F,\mu+1}$ and $v_0 \in \mathbb{R}^n$ such that $z_0 = (t_0, x_0, v_0) \in \mathcal{L}_{F,\mu+1}$, there exists an interval $\mathcal{I}_0$ on which the solution is represented using the Moore-Penrose remodeling $g_{MP,0} \in C^1(\mathcal{I}_0 \times \mathcal{U}(x_{d,0}), \mathcal{U}(x_{a,0}))$ and $h_{MP,0} \in C^1(\mathcal{I}_0 \times \mathcal{U}(x_{d,0}), \mathbb{R}^n)$ induced by $P_{MP}(z_0)$, cp. Lemma 3.2. Considering the time derivative of formula (49) and evaluating in $t = t_0$, we obtain formula (56). □

For linear problems, using the flow formula (55), formula (56) can be specified as affine linear transformation.

Corollary 3.3. *Consider the DAE* (51) *with* $F_{E,A,b} \in C^{\mu+1}_{\mu,d,a,reg}(\mathbb{D}, \mathbb{R}^n)$. *Let* P *and* D_d, D_a, b_d, b_a *be the Moore-Penrose projection and remodeling induced by* (51) *and* $\mathcal{P}_{MP} = (I_n - D_a)P_{MP}$. *On* $\mathcal{I} \times \mathbb{R}^n$, *the tangent field of* (51) *is given by*

$$\mathcal{T}_{E,A,b}(t, x) = \mathcal{T}_{E,A}(t)\, x + \big(\mathcal{P}_{MP} b_d - \dot{b}_a\big)(t), \tag{57}$$

where the homogeneous tangent field $\mathcal{T}_{E,A}$ *is given by* $\mathcal{T}_{E,A}(t) = \big((\dot{\mathcal{P}}_{MP} + \mathcal{P}_{MP} D_d)P_{MP}\big)(t)$.

Remark 3.7. *For constant coefficients* $E, A \in \mathbb{R}^{n\times n}$ *and* $b \in C^{\mu+1}(\mathcal{I}, \mathbb{R}^n)$, *the tangent field is given by* $\mathcal{T}_{E,A,b}(t, x) = \mathcal{T}_{E,A}(t)x + \big(\mathcal{P}_{MP} b_d - \dot{b}_a\big)(t)$ *on* $\mathcal{C}_{E,A,b,\mu}$, *with the homogeneous tangent field* $\mathcal{T}_{E,A}(t) = (\mathcal{P}_{MP}\hat{E}^+\hat{A}P_{MP})(t)$.

If $F(t, x, \dot{x}) = \dot{x} - f(t, x)$ with $f \in C^{\mathrm{Lip}}_{\mathrm{loc}}(\mathcal{I} \times \Omega_x, \mathbb{R}^n)$, then $\Phi_F = \Phi_f$ and $\dot{\Phi}^t_F(t, x) = f(t, x)$ on $\mathcal{I} \times \Omega$. Thus, the tangent field $\mathcal{T}_F$ coincides with the system function f.
We use the tangent field to study properties like flow invariance, stability and positivity of the DAE (13a) in [2].

4 Examples

We illustrate the remodeling by the Moore-Penrose projection and the computation of the flow for a linear and a nonlinear DAE. For details of the computation, see [2, Ex. 4.5.1, Ex. 4.5.2].

Example 4.1. *We consider* $F_{E,A,b} \in C^1(\mathcal{I}, \mathbb{R}^n)$ *with*

$$E = \begin{bmatrix} \frac{t+1}{t+2} & \frac{\sqrt{t+1}}{t+2} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} \frac{t}{2(t+2)^2} & \frac{t}{2(t+2)^2\sqrt{t+1}} & 0 \\ -\frac{1}{\sqrt{t+2}} & \frac{\sqrt{t+1}}{\sqrt{t+2}} & 0 \\ -1 & -\frac{1}{\sqrt{t+1}} & 1 \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix},$$

$\mathcal{I} := (-1, \infty)$ *and* $b_1, b_2, b_3 \in \mathbb{R}$. *We first show that* $F_{E,A,b}$ *is s-free and already in the remodeled form* (52) *with* $\hat{E}_1 = e_1^T$, $\hat{A}_1 = e_1^T A$, $\hat{A}_2 = [e_2, e_3]^T A$ *and* $\hat{f}_1 = e_1^T b$, $\hat{f}_2 = [e_2, e_3]^T b$. *On* $\mathcal{I}$,

$\operatorname{rank}(E) = 1$ *and* $\operatorname{rank}([e_2, e_3]^T A) = 2$*, so it remains to prove condition* (22)*, cp. Remark 3.4. With*

$$E^+ = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{\sqrt{t+1}} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

the Moore-Penrose projections are given by

$$P_{MP} = \begin{bmatrix} \frac{t+1}{t+2} & \frac{\sqrt{t+1}}{t+2} & 0 \\ \frac{\sqrt{t+1}}{t+2} & \frac{1}{t+2} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad P_{MP}^{\perp} = \begin{bmatrix} \frac{1}{t+2} & -\frac{\sqrt{t+1}}{t+2} & 0 \\ -\frac{\sqrt{t+1}}{t+2} & \frac{t+1}{t+2} & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{58}$$

Then,

$$\hat{A}_2 P_{MP}^{\perp} = \begin{bmatrix} -\frac{1}{\sqrt{t+2}} A_{22} & \frac{\sqrt{t+1}}{\sqrt{t+2}} A_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad (\hat{A}_2 P_{MP}^{\perp})^+ = \begin{bmatrix} 0 & -\frac{1}{\sqrt{t+2}} & 0 \\ 0 & \frac{\sqrt{t+1}}{\sqrt{t+2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and we verify by direct computation that P_{MP} *and* A *satisfy condition* (22)*.*
Now, we compute the Moore-Penrose remodeling g_{MP} *and* h_{MP} *with*

$$D_a = \begin{bmatrix} \frac{1}{t+2} & -\frac{\sqrt{t+1}}{t+2} & 0 \\ -\frac{\sqrt{t+1}}{t+2} & \frac{t+1}{t+2} & 0 \\ -1 & -\frac{1}{\sqrt{t+1}} & 1 \end{bmatrix}, \quad b_a = \begin{bmatrix} \frac{b_2}{\sqrt{t+2}} \\ \frac{b_2\sqrt{t+1}}{\sqrt{t+2}} \\ b_3 \end{bmatrix},$$

$$D_d = \begin{bmatrix} \frac{t+1}{2(t+2)^2} & \frac{\sqrt{t+1}}{2(t+2)^2} & 0 \\ -\frac{\sqrt{t+1}}{2(t+2)^2(t+1)} & -\frac{1}{2(t+2)^2(t+1)} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad b_d = (b_1 + \frac{b_2}{2(t+2)^{3/2}}) \begin{bmatrix} 1 \\ \frac{1}{\sqrt{t+1}} \\ 0 \end{bmatrix}.$$

The variables are partitioned according to

$$x_d = \begin{bmatrix} \frac{\sqrt{t+1}(\sqrt{t+1}x_1 + x_2)}{t+2} \\ \frac{\sqrt{t+1}x_1 + x_2}{t+2} \\ 0 \end{bmatrix}, \quad x_a = \begin{bmatrix} \frac{x_1 - \sqrt{t+1}x_2}{t+2} \\ -\frac{\sqrt{t+1}(x_1 - \sqrt{t+1}x_2)}{t+2} \\ x_3 \end{bmatrix}. \tag{59}$$

In conclusion, the Moore-Penrose remodeling (53) *consists of the ODE*

$$\begin{bmatrix} \dot{x}_{d,1} \\ \dot{x}_{d,2} \\ \dot{x}_{d,3} \end{bmatrix} = \begin{bmatrix} \frac{t+1}{2(t+2)^2} & \frac{\sqrt{t+1}}{2(t+2)^2} & 0 \\ -\frac{\sqrt{t+1}}{2(t+2)^2(t+1)} & -\frac{1}{2(t+2)^2(t+1)} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_{d,1} \\ x_{d,2} \\ x_{d,3} \end{bmatrix} + (b_1 + \frac{b_2}{2(t+2)^{3/2}}) \begin{bmatrix} 1 \\ \frac{1}{\sqrt{t+1}} \\ 0 \end{bmatrix} \tag{60a}$$

and the algebraic relation

$$\begin{bmatrix} x_{a,1} \\ x_{a,2} \\ x_{a,3} \end{bmatrix} = -\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & -\frac{1}{\sqrt{t+1}} & 0 \end{bmatrix} \begin{bmatrix} x_{d,1} \\ x_{d,2} \\ x_{d,3} \end{bmatrix} - \begin{bmatrix} \frac{b_2}{\sqrt{t+2}} \\ \frac{b_2\sqrt{t+1}}{\sqrt{t+2}} \\ b_3 \end{bmatrix}. \tag{60b}$$

To compute the flow Φ_{D_d,b_d}*, we note that* $[e_1, e_2, 0]^T \mathcal{P}_{MP} = [e_1, e_2, 0]^T P_{MP}$ *and* $[0, 0, e_3]^T (E^+ A + \dot{P}_{MP}) = 0$ *on* $\mathcal{I}$*, such that we can simplify the system matrix according to* $D_d = \mathcal{P}_{MP}(E^+ A + \dot{P}_{MP})\mathcal{P}_{MP} = (E^+ A + \dot{P}_{MP})P_{MP}$*. Thus, ODE* (60a) *according to*

$$D_d = \mathcal{P}_{MP}(E^+A + \dot{P}_{MP})\mathcal{P}_{MP} = (E^+A + \dot{P}_{MP})P_{MP} = \begin{bmatrix} \frac{1}{2(t+2)} & 0 & 0 \\ 0 & -\frac{1}{2(t+2)(t+1)} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Noting that $\int_{t_0}^{t} \frac{1}{2(s+2)}\, ds = \frac{1}{2}\ln[\frac{t+2}{t_0+2}]$ *and* $\int_{t_0}^{t} -\frac{1}{2(s+2)(s+1)}\, ds = \frac{1}{2}\ln[\frac{(t+1)(t_0+2)}{(t+2)(t_0+1)}]$*, cp. [38, p. 96, 98], we get that* $exp(\int_{t_0}^{t} \frac{1}{2(s+2)}\, ds) = \frac{\sqrt{t+2}}{\sqrt{t_0+2}}$ *and* $exp(\int_{t_0}^{t} -\frac{1}{2(s+2)(s+1)}\, ds) = \frac{\sqrt{t+1}\sqrt{t_0+2}}{\sqrt{t+2}\sqrt{t_0+1}}$ *and the flow* Φ_{D_d} *restricted to* $\mathcal{C}_{E,A}$ *is given by*

$$(\Phi^t_{D_d} P_{MP})(t_0) = \begin{bmatrix} \frac{\sqrt{t+2}}{\sqrt{t_0+2}} & 0 & 0 \\ & \frac{\sqrt{t+1}\sqrt{t_0+2}}{\sqrt{t+2}\sqrt{t_0+1}} & 0 \\ 0 & 0 & 1 \end{bmatrix} P_{MP}(t_0).$$

With $\Phi^t_{E,A}(t_0) = \mathcal{P}_{MP}(t)(\Phi^t_{D_d} P)(t_0)$*, then the homogeneous flow of the DAE is given by*

$$\Phi^t_{E,A}(t_0) = \begin{bmatrix} \frac{(t+1)(t_0+1)}{\sqrt{t+2}(t_0+2)^{3/2}} + \frac{t+1}{(t+2)^{3/2}\sqrt{t_0+2}} & \frac{(t+1)\sqrt{t_0+1}}{\sqrt{t+2}(t_0+2)^{3/2}} + \frac{t+1}{(t+2)^{3/2}\sqrt{t_0+1}\sqrt{t_0+2}} & 0 \\ \frac{\sqrt{t+1}(t_0+1)}{\sqrt{t+2}(t_0+2)^{3/2}} + \frac{\sqrt{t+1}}{(t+2)^{3/2}\sqrt{t_0+2}} & \frac{\sqrt{t+1}\sqrt{t_0+1}}{\sqrt{t+2}(t_0+2)^{3/2}} + \frac{\sqrt{t+1}}{(t+2)^{3/2}\sqrt{t_0+1}\sqrt{t_0+2}} & 0 \\ \frac{\sqrt{t+2}(t_0+1)}{(t_0+2)^{3/2}} + \frac{1}{\sqrt{t+2}\sqrt{t_0+2}} & \frac{\sqrt{t+2}\sqrt{t_0+1}}{(t_0+2)^{3/2}} + \frac{1}{\sqrt{t+2}\sqrt{t_0+1}\sqrt{t_0+2}} & 0 \end{bmatrix}.$$

From this formula, we can compute inhomogeneous flow according to

$$\Phi^t_{E,A,f}(t_0, x_0) = \Phi^t_{E,A}(t_0)x_0 + \int_{t_0}^{t} (\Phi^t_{E,A} b_d)(s)\, ds - b_a(t).$$

To compute the set of consistent initial values $\mathcal{C}_{E,A,b,\mu}$*, we compute the projection*

$$\mathcal{P}_{MP} = \begin{bmatrix} \frac{t+1}{t+2} & \frac{\sqrt{t+1}}{t+2} & 0 \\ \frac{\sqrt{t+1}}{t+2} & \frac{1}{t+2} & 0 \\ 1 & \frac{1}{\sqrt{t+1}} & 0 \end{bmatrix},$$

and find that, cp. Lemma (3.5)*,*

$$\mathcal{C}_{E,A,b,\mu} = \left\{ x_0 \in \mathbb{R}^3 \middle| \begin{bmatrix} \frac{x_{1,0} - \sqrt{t_0+1}x_{2,0}}{t_0+2} \\ -\frac{\sqrt{t_0+1}(x_{1,0} - \sqrt{t_0+1}x_{2,0})}{t_0+2} \\ -\frac{\sqrt{t_0+1}x_{1,0} + x_{2,0}}{\sqrt{t_0+1}} + x_{0,3} \end{bmatrix} = \begin{bmatrix} \frac{b_2}{\sqrt{t+2}} \\ \frac{b_2\sqrt{t+1}}{\sqrt{t+2}} \\ b_3 \end{bmatrix} \right\}.$$

Example 4.2. *Consider* $F \in C^\infty(\mathbb{D}, \mathbb{R}^3)$*,* $\mathbb{D} = (-1, \infty) \times \mathbb{R}^3 \times \mathbb{R}^3$*, with*

$$F(t, x, \dot{x}) = \begin{bmatrix} \frac{x_1}{2(t+2)\sqrt{t+1}} + \frac{\sqrt{t+1}\dot{x}_1 + \dot{x}_2}{t+2} - \frac{\sqrt{t+1}x_1 + x_2}{(t+2)^2} + \frac{(\sqrt{t+1}x_1 + x_2)^2}{(t+2)^2} \\ \frac{(x_1 - \sqrt{t+1}x_2)^2}{(t+2)^2} - 2 \\ x_3^2 - \frac{\sqrt{t+1}x_1 + x_2}{t+2} - 1 \end{bmatrix}.$$

The algebraic solution set is given by

$$F^{-1}(0) = \Big\{(t,x,v) \in \mathbb{D} \,|\, x_1 = \sqrt{t+1}x_2 + \sqrt{2}(t+2),\ x_3 = \sqrt{1 + x_2 + \sqrt{2(t+1)}},$$
$$\tfrac{\sqrt{t+1}v_1+v_2}{t+2} = \big(1 - 2\sqrt{2(t+1)} - \tfrac{1}{2(t+2)}\big)x_2 - x_2^2 + \tfrac{\sqrt{2(t+1)}}{t+2} - 2(t+1)\Big\}.$$

To prove that F *is s-free and already in the remodeled form* (15) *with* $\hat{F}_1 = e_1^T F$, $\hat{F}_2 = [e_2, e_3]^T F$, *we note that the Jacobians*

$$F_{\dot{x}}(t,x,\dot{x}) = \begin{bmatrix} \frac{\sqrt{t+1}}{t+2} & \frac{1}{t+2} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$
$$F_x(t,x,\dot{x}) = \begin{bmatrix} \frac{1}{2(t+2)\sqrt{t+1}} - \frac{\sqrt{t+1}}{(t+2)^2} + \frac{2(\sqrt{t+1}x_1+x_2)\sqrt{t+1}}{(t+2)^2} & -\frac{1}{(t+2)^2} + \frac{2(\sqrt{t+1}x_1+x_2)}{(t+2)^2} & 0 \\ \frac{x_1-\sqrt{t+1}x_2}{(t+2)^2} & -\frac{(x_1-\sqrt{t+1}x_2)\sqrt{t+1}}{(t+2)^2} & 0 \\ -\frac{\sqrt{t+1}}{t+2} & -\frac{1}{t+2} & 2x_3 \end{bmatrix},$$

satisfy $\operatorname{rank}(\hat{F}_{1,\dot{x}}(t,x,\dot{x})) = 1$, $\operatorname{rank}(\hat{F}_{2,x}(t,x,\dot{x})) = 2$ *on* $F^{-1}(0)$ *and the solvability condition* (22)*, cp. Remark 3.4. Comparing* $F_{\dot{x}}$ *and the matrix* E *of Example 4.1, we find that the Moore-Penrose projections induced by* F *are given by* (58)*. Noting that*

$$\hat{F}_{2,x}P_{MP}^{\perp} = \begin{bmatrix} \frac{x_1-\sqrt{t+1}x_2}{(t+2)^2} & -\frac{(x_1-\sqrt{t+1}x_2)\sqrt{t+1}}{(t+2)^2} & 0 \\ 0 & 0 & 2x_3 \end{bmatrix}, \quad (\hat{F}_{2,x}P^c)^+ = \begin{bmatrix} \frac{t+2}{x_1-\sqrt{t+1}x_2} & 0 \\ -\frac{\sqrt{t+1}(t+2)}{x_1-\sqrt{t+1}x_2} & 0 \\ -\frac{(\sqrt{t+1}x_1+x_2)}{2x_3(x_1-\sqrt{t+1}x_2)} & \frac{1}{2x_3} \end{bmatrix},$$

we verify that $\hat{F}_{2,x}$, P^c *satisfy condition* (22)*. The variables* x_d, x_a *are given as in* (59)*. To compute the Moore-Penrose remodeling* g_{MP}*, we solve*

$$\hat{F}_2(t, x_d + x_a, \dot{x}_d + \dot{x}_a) = \begin{bmatrix} x_{a1}^2 - 2 \\ x_{a3}^2 - x_{d,2} - 1 \end{bmatrix} = 0$$

for x_a *and noting that* $x_{a2} = -\sqrt{t+1}x_{a1}$*, we obtain that* $g_{MP} \in C^\infty(\mathcal{I} \times \mathcal{U}_{g_{MP}}, \mathbb{R}^3)$*, where* $\mathcal{U}_{g_{MP}} = \{x \in \mathbb{R}^3 \,|\, x_{d,2} > -1\}$ *and*

$$g_{MP}(t,x_d) = \begin{bmatrix} \sqrt{2} & -\sqrt{2(t+1)} & \sqrt{1+x_{d,2}} \end{bmatrix}^T.$$

Then, the set of consistent initial values is given by

$$\mathcal{C}_F = \Big\{(t,x) \in \mathcal{I} \times \mathbb{R}^3 \,|\, x_1 = \sqrt{t+1}x_2 + \sqrt{2}(t+2),\ x_3 = \sqrt{1 + x_2 + \sqrt{2(t+1)}}\Big\}.$$

To compute the function h_{MP}*, we solve*

$$\hat{F}_1(t, x_d + x_a, \dot{x}_d + \dot{x}_a) = \dot{x}_{d,2} + x_{d,2}^2 = 0$$

for $\dot{x}_d$*, and noting that* $x_{d1} = \sqrt{t+1}x_{d2}$*, we get that* $h_{MP} \in C^\infty(\mathcal{I} \times \mathbb{R}^3, \mathbb{R}^3)$ *with*

$$h_{MP}(t,x_d) = \begin{bmatrix} -\frac{x_{d,1}^2}{\sqrt{t+1}} + \frac{x_{d,1}}{2(t+1)} & -x_{d,2}^2 & 0 \end{bmatrix}^T.$$

On $\mathcal{I}\times\mathcal{U}_{\mathcal{P}_{MP}}$, $\mathcal{U}_{\mathcal{P}_{MP}}=\{x\in\mathbb{R}^3 \mid \frac{\sqrt{t+1}x_1+x_2}{t+2}>-1\}$, *then the Moore-Penrose remodeling given by*

$$\dot{x}_d=\begin{bmatrix}\frac{x_{d,1}^2}{\sqrt{t+1}}+\frac{x_{d,1}}{2(t+1)}\\ x_{d,2}^2\\ 0\end{bmatrix},\quad x_a=\begin{bmatrix}\sqrt{2},\\ -\sqrt{2(t+1)}\\ \sqrt{1+x_{d,2}}\end{bmatrix}. \tag{61}$$

On $\mathcal{I}\times\mathbb{R}^3$, *the flow* $\Phi_{h_{MP}}$ *is given by*

$$\Phi^t_{h_{MP}}(t_0,x_{d0})=\begin{bmatrix}-\frac{t+2}{2t^2\sqrt{t+1}}+\frac{t_0+2}{2t_0^2\sqrt{t_0+1}}+x_{d,1,0}\\ \frac{1}{t}-\frac{1}{t_0}+x_{d,2,0}\\ 0\end{bmatrix},$$

and with $x_{d,1,0}=\sqrt{t_0+1}x_{d,2,0}$ *and* $x_{d,2,0}=\frac{\sqrt{t_0+1}x_{1,0}+x_{2,0}}{t_0+2}$, *we get the DAE flow*

$$\Phi^t_F(t_0,x_0)=\begin{bmatrix}-\frac{t+2}{2t^2\sqrt{t+1}}+\frac{t_0+2}{2t_0^2\sqrt{t_0+1}}+\frac{\sqrt{t_0+1}(\sqrt{t_0+1}x_{1,0}+x_{2,0})}{t_0+2}+\sqrt{2}\\ \frac{1}{t}-\frac{1}{t_0}+\frac{\sqrt{t_0+1}x_{1,0}+x_{2,0}}{t_0+2}-\sqrt{2(t+1)}\\ \sqrt{1+\frac{1}{t}-\frac{1}{t_0}+\frac{\sqrt{t_0+1}x_{1,0}+x_{2,0}}{t_0+2}}\end{bmatrix}.$$

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