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# Model reduction of descriptor systems


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**Abstract**

Model reduction is of fundamental importance in many control applications. We consider model reduction methods for linear continuous-time descriptor systems. The methods are based on balanced truncation techniques and closely related to the controllability and observability Gramians and Hankel singular values of descriptor systems. The Gramians can be computed by solving the generalized Lyapunov equations with special right-hand sides. The numerical solution of generalized Lyapunov equations is also discussed. A numerical example is given.

**Key words:** descriptor systems, controllability and observability Gramian, Hankel singular values, model reduction.

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# 1 Introduction

Consider a linear continuous-time system

$$\begin{aligned} E\,\dot{x}(t) &= A\,x(t) + B\,u(t), \quad x(0) = x^0, \\ y(t) &= C\,x(t), \end{aligned} \tag{1}$$

where $E,\ A \in \mathbb{R}^{n,n}$, $B \in \mathbb{R}^{n,m}$, $C \in \mathbb{R}^{p,n}$, $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the control input, $y(t) \in \mathbb{R}^p$ is the output and $x^0 \in \mathbb{R}^n$ is the initial value. The number of state variables $n$ is called the *order* of system (1). If $I = E$, then (1) is a *standard state space system*. Otherwise, (1) is a *descriptor system* or *generalized state space system*. Such systems arise naturally in many applications such as multibody dynamics [6, 14], electrical circuits [8, 21], semidiscretization of partial differential equations [43] and they may have very large order $n$.

We will assume throughout the paper that the pencil $\lambda E - A$ is *regular*, i.e., $\det(\lambda E - A) \neq 0$ for some $\lambda \in \mathbb{C}$. The pencil $\lambda E - A$ is called *c-stable*, if it is regular and all finite eigenvalues of $\lambda E - A$ lie in the open left half-plane.

The model reduction problem consists in an approximation of the descriptor system (1) by a reduced order system

$$\begin{aligned} \widetilde{E}\,\dot{\widetilde{x}}(t) &= \widetilde{A}\,\widetilde{x}(t) + \widetilde{B}\,u(t), \quad \widetilde{x}(0) = \widetilde{x}^0, \\ \widetilde{y}(t) &= \widetilde{C}\,\widetilde{x}(t), \end{aligned} \tag{2}$$

where $\widetilde{E},\ \widetilde{A} \in \mathbb{R}^{\ell,\ell}$, $\widetilde{B} \in \mathbb{R}^{\ell,m}$, $\widetilde{C} \in \mathbb{R}^{p,\ell}$ and $\ell \ll n$. Note that systems (1) and (2) have the same input $u(t)$. We require for the approximate system (2) to preserve properties of the original system (1) like regularity and stability. The descriptor system (1) consists of differential equations that describe the dynamic behavior of the system as well as algebraic equations characterizing a constraint manifold for the solution. Therefore, it is natural to require for the reduced order system to have the same algebraic constraints as the original one. Clearly, it is also desirable that the approximation error is small. Moreover, the computation of the reduced order system should be numerically stable and efficient.

There exist various model reduction approaches for standard state space systems such as balanced truncation [29, 31, 34, 35, 40, 41], moment matching approximation [16, 20], singular perturbation approximation [27, 30] or optimal Hankel norm approximation [17]. Surveys on system approximation and model reduction can be found in [2, 15]. One of most effective and well studied model reduction techniques is balanced truncation which is closely related to the two Lyapunov equations

$$A\mathcal{P} + \mathcal{P}A^T = -BB^T, \qquad A^T\mathcal{Q} + \mathcal{Q}A = -C^TC.$$

The solutions $\mathcal{P}$ and $\mathcal{Q}$ of these equations are called the *controllability* and *observability Gramians*, respectively. The balanced truncation approach consists in transforming the state space system to a balanced form whose the controllability and observability Gramians become diagonal and equal together with a truncation of states that are both difficult to reach and to observe, see [31] for details.

In this paper we generalize controllability and observability Gramians as well as Hankel singular values for descriptor systems (Section 2). In Section 3 we present an extension of balanced truncation algorithms [29, 40, 41] to descriptor systems. These algorithms are based on splitting system (1) into its dynamic and algebraic parts and then reducing the order only for the dynamic part via a standard model reduction method. Section 4 contains a numerical example.

## 2 Descriptor systems

Consider the descriptor system (1). If the pencil $\lambda E - A$ is regular, then it can be reduced to the Weierstrass canonical form [37], that is, there exist nonsingular matrices $W$ and $T$ such that

$$E = W \begin{pmatrix} I_{n_f} & 0 \\ 0 & N \end{pmatrix} T, \qquad A = W \begin{pmatrix} J & 0 \\ 0 & I_{n_\infty} \end{pmatrix} T, \tag{3}$$

where $I_k$ is the identity matrix of order $k$ and $N$ is nilpotent with index of nilpotency $\nu$. The number $\nu$ is the *index* of the pencil $\lambda E - A$. Representation (3) defines the decomposition of $\mathbb{R}^n$ into complementary deflating subspaces of dimensions $n_f$ and $n_\infty$ corresponding to the finite and infinite eigenvalues of the pencil $\lambda E - A$, respectively. The matrices

$$P_r = T^{-1} \begin{pmatrix} I_{n_f} & 0 \\ 0 & 0 \end{pmatrix} T \qquad \text{and} \qquad P_l = W \begin{pmatrix} I_{n_f} & 0 \\ 0 & 0 \end{pmatrix} W^{-1} \tag{4}$$

are the *spectral projections* onto the right and left deflating subspaces of the pencil $\lambda E - A$ corresponding to the finite eigenvalues.

It is well known that if the pencil $\lambda E - A$ is regular, $u(t)$ is $\nu$ times continuously differentiable and $x^0$ is consistent, i.e., it belongs to the *set of consistent initial conditions*

$$\mathcal{X}_0 = \left\{ x^0 \in \mathbb{R}^n \quad : \quad (I - P_r)x^0 = \sum_{k=0}^{\nu-1} F_{-k-1} B u^{(k)}(0) \right\},$$

then the descriptor system (1) has a unique continuously differentiable solution $x(t)$, see [8, 11], that is given by

$$x(t) = \mathcal{F}(t)Ex^0 + \int_0^t \mathcal{F}(t-\tau)Bu(\tau)\,d\tau + \sum_{k=0}^{\nu-1} F_{-k-1}Bu^{(k)}(t).$$

Here

$$\mathcal{F}(t) = T^{-1} \begin{pmatrix} e^{tJ} & 0 \\ 0 & 0 \end{pmatrix} W^{-1} \tag{5}$$

is a *fundamental solution matrix* of the descriptor system (1), and the matrices $F_k$ have the form

$$F_k = T^{-1} \begin{pmatrix} 0 & 0 \\ 0 & -N^{-k-1} \end{pmatrix} W^{-1}, \qquad k = -1, -2, \ldots.$$

Clearly, $F_k = 0$ for $k < -\nu$.

If the initial condition $x^0$ is inconsistent or the input $u(t)$ is not sufficiently smooth (for example, in most control problems u(t) is only piecewise continuous), then the solution of the descriptor system (1) may have impulsive modes [10, 11].

The rational matrix-valued function $\mathbf{G}(s) := C(sE - A)^{-1}B$ is called the *transfer function* of the descriptor system (1). The transfer function $\mathbf{G}(s)$ is called *proper* if $\lim\limits_{s\to\infty} \mathbf{G}(s) < \infty$. A quadruple of matrices $[E, A, B, C]$ is called a *realization* of $\mathbf{G}(s)$. We will also often denote a realization of $\mathbf{G}(s)$ by

$$\left[\begin{array}{c|c} sE - A & B \\ \hline C & 0 \end{array}\right].$$

Two realizations $[E, A, B, C]$ and $[\check{E}, \check{A}, \check{B}, \check{C}]$ are *restricted system equivalent* if there exist nonsingular matrices $\check{W}$ and $\check{T}$ such that

$$\left[\begin{array}{c|c} s\check{E} - \check{A} & \check{B} \\ \hline \check{C} & 0 \end{array}\right] = \left[\begin{array}{c|c} s\check{W}E\check{T} - \check{W}A\check{T} & \check{W}B \\ \hline C\check{T} & 0 \end{array}\right].$$

A pair $(\check{W}, \check{T})$ is called *system equivalence transformation.* A characteristic quantity of system (1) is *system invariant* if it is preserved under a system equivalence transformation. The transfer function $\mathbf{G}(s)$ is system invariant, since

$$\mathbf{G}(s) = C(sE - A)^{-1}B = \check{C}\check{T}^{-1}\check{T}(s\check{E} - \check{A})^{-1}\check{W}\check{W}^{-1}\check{B} = \check{C}(s\check{E} - \check{A})^{-1}\check{B}.$$

Other important results from the theory of rational functions and realization theory may be found in [11, 23].

## 2.1 Controllability and observability

For descriptor systems there are various concepts of controllability and observability, e.g., [7, 11, 44].

**Definition 2.1** System (1) and the triplet $(E, A, B)$ are called *controllable on the reachable set* (*R-controllable*) if

$$\operatorname{rank}[\lambda E - A,\ B] = n \quad \text{for all finite } \lambda \in \mathbb{C}. \tag{6}$$

System (1) and the triplet $(E, A, B)$ are called *controllable at infinity* (*I-controllable*) if

$$\operatorname{rank}[E,\ AK_E,\ B] = n, \ \text{ where the columns of } K_E \text{ span} \ker E. \tag{7}$$

System (1) and the triplet $(E, A, B)$ are called *strongly controllable* (*S-controllable*) if (6) and (7) hold.
System (1) and the triplet $(E, A, B)$ are called *completely controllable* (*C-controllable*) if (6) holds and

$$\operatorname{rank}[E,\ B] = n. \tag{8}$$

Observability is a dual property of controllability.

**Definition 2.2** System (1) and the triplet $(E, A, C)$ are called *observable on the reachable set* (*R-observable*) if

$$\operatorname{rank}\begin{bmatrix} \lambda E - A \\ C \end{bmatrix} = n \quad \text{for all finite } \lambda \in \mathbb{C}. \tag{9}$$

System (1) and the triplet $(E, A, C)$ are called *observable at infinity* (*I-observable*) if

$$\operatorname{rank}\begin{bmatrix} E \\ K_{E^T}^T A \\ C \end{bmatrix} = n, \quad \text{where the columns of } K_{E^T} \text{ span} \ker E^T. \tag{10}$$

System (1) and the triplet $(E, A, C)$ are called *strongly observable* (*S-observable*) if (9) and (10) hold.
System (1) and the triplet $(E, A, C)$ are called *completely observable* (*C-observable*) if (9) holds and

$$\operatorname{rank}\begin{bmatrix} E \\ C \end{bmatrix} = n. \tag{11}$$

Clearly, conditions (7) and (10) are weaker than (8) and (11), respectively. Equivalent algebraic characterizations of various concepts of controllability and observability for descriptor systems are presented in [11, 44].

## 2.2 Controllability and observability Gramians

If we assume that the pencil $\lambda E - A$ is *c-stable*, then the integrals

$$\mathcal{G}_{pc} = \int_0^\infty \mathcal{F}(t) B B^T \mathcal{F}^T(t)\, dt \tag{12}$$

and

$$\mathcal{G}_{po} = \int_0^\infty \mathcal{F}^T(t) C^T C \mathcal{F}(t)\, dt \tag{13}$$

exist, where $\mathcal{F}(t)$ is as in (5). The matrix $\mathcal{G}_{pc}$ is called the *proper controllability Gramian* and the matrix $\mathcal{G}_{po}$ is called the *proper observability Gramian* of the continuous-time descriptor system (1), see [4, 39]. The *improper controllability Gramian* of system (1) is defined by

$$\mathcal{G}_{ic} = -\sum_{k=-\nu}^{-1} F_k B B^T F_k^T,$$

and the *improper observability Gramian* of system (1) is defined by

$$\mathcal{G}_{io} = -\sum_{k=-\nu}^{-1} F_k^T C^T C F_k.$$

In summary, the *controllability Gramian* of the descriptor system (1) is given by

$$\mathcal{G}_c = \mathcal{G}_{pc} + \mathcal{G}_{ic}$$

and the *observability Gramian* of the descriptor system (1) is given by

$$\mathcal{G}_o = \mathcal{G}_{po} + \mathcal{G}_{io}.$$

If $E = I$, then $\mathcal{G}_{pc} = \mathcal{G}_c$ and $\mathcal{G}_{po} = \mathcal{G}_o$ are the usual controllability and observability Gramians for the standard state space system [45].

The proper controllability and observability Gramians are the unique symmetric, positive semidefinite solutions of the *projected generalized continuous-time Lyapunov equations*

$$\begin{aligned} &E\mathcal{G}_{pc}A^T + A\mathcal{G}_{pc}E^T = -P_l B B^T P_l^T, \\ &\mathcal{G}_{pc} = P_r \mathcal{G}_{pc} \end{aligned} \tag{14}$$

and

$$\begin{aligned} &E^T\mathcal{G}_{po}A + A^T\mathcal{G}_{po}E = -P_r^TC^TCP_r,\\ &\mathcal{G}_{po} = \mathcal{G}_{po}P_l, \end{aligned} \tag{15}$$

respectively, where $P_l$ and $P_r$ are given by (4), see [39]. If $\lambda E - A$ is in Weierstrass canonical form (3) and if the matrices

$$W^{-1}B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \qquad \text{and} \qquad CT^{-1} = [\,C_1,\ \ C_2]$$

are partitioned in blocks conformally $E$ and $A$, then we can show that

$$\mathcal{G}_{pc} = T^{-1}\begin{pmatrix} G_{1c} & 0 \\ 0 & 0 \end{pmatrix}T^{-T}, \qquad \mathcal{G}_{po} = W^{-T}\begin{pmatrix} G_{1o} & 0 \\ 0 & 0 \end{pmatrix}W^{-1}, \tag{16}$$

where $G_{1c}$ and $G_{1o}$ satisfy the standard continuous-time Lyapunov equations

$$\begin{aligned} JG_{1c} + G_{1c}J^T &= -B_1B_1^T,\\ J^TG_{1o} + G_{1o}J &= -C_1^TC_1. \end{aligned}$$

The improper controllability and observability Gramians are the unique symmetric, negative semidefinite solutions of the *projected generalized discrete-time Lyapunov equations*

$$\begin{aligned} &A\mathcal{G}_{ic}A^T - E\mathcal{G}_{ic}E^T = -(I-P_l)BB^T(I-P_l)^T,\\ &P_r\mathcal{G}_{ic} = 0 \end{aligned}$$

and

$$\begin{aligned} &A^T\mathcal{G}_{io}A - E^T\mathcal{G}_{io}E = -(I-P_r)^TC^TC(I-P_r),\\ &\mathcal{G}_{io}P_l = 0, \end{aligned}$$

respectively [39]. They can be represented as

$$\mathcal{G}_{ic} = T^{-1}\begin{pmatrix} 0 & 0 \\ 0 & G_{2c} \end{pmatrix}T^{-T}, \qquad \mathcal{G}_{io} = W^{-T}\begin{pmatrix} 0 & 0 \\ 0 & G_{2o} \end{pmatrix}W^{-1}, \tag{17}$$

where $G_{2c}$ and $G_{2o}$ satisfy the standard discrete-time Lyapunov equations

$$\begin{aligned} G_{2c} - NG_{2c}N^T &= -B_2B_2^T,\\ G_{2o} - N^TG_{2o}N &= -C_2^TC_2. \end{aligned}$$

Unfortunately, we do not know how to express the controllability and observability Gramians $\mathcal{G}_c$ and $\mathcal{G}_o$ for the descriptor system (1) via the solutions of single Lyapunov equations.

The controllability and observability Gramians can be used to characterize controllability and observability properties of system (1).

**Theorem 2.3** [4, 39] *Consider the descriptor system* (1)*. Assume that $\lambda E - A$ is c-stable.*

1. *System* (1) *is R-controllable if and only if the proper controllability Gramian $\mathcal{G}_{pc}$ is positive definite on the subspace* $\operatorname{im} P_r^T$*.*

2. *System* (1) *is I-controllable if the improper controllability Gramian $\mathcal{G}_{ic}$ is negative definite on the subspace* $\ker P_r^T$*.*

3. *System* (1) *is C-controllable if and only if the controllability Gramian* $\mathcal{G}_c$ *is positive definite on* $\operatorname{im} P_r^T$ *and negative definite on* $\ker P_r^T$.

4. *System* (1) *is R-observable if and only if the proper observability Gramian* $\mathcal{G}_{po}$ *is positive definite on the subspace* $\operatorname{im} P_l$.

5. *System* (1) *is I-observable if the improper observability Gramian* $\mathcal{G}_{io}$ *is negative definite on the subspace* $\ker P_l$.

6. *System* (1) *is C-observable if and only if the observability Gramian* $\mathcal{G}_o$ *is positive definite on* $\operatorname{im} P_l$ *and negative definite on* $\ker P_l$.

Note that the I-controllability (I-observability) of (1) does not imply that the improper controllability (observability) Gramian is negative definite on $\ker P_r^T$ (on $\ker P_l$).

**Example 2.4** The descriptor system (1) with

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad C = (\,1,\ \ 0\,)$$

is I-controllable and I-observable. We have $\mathcal{G}_{ic} = \mathcal{G}_{io} = 0$ and $P_r^T = P_l$, i.e., neither $\mathcal{G}_{ic}$ nor $\mathcal{G}_{io}$ are negative definite on $\ker P_r^T = \ker P_l$.

**Corollary 2.5** *Consider the descriptor system* (1). *Assume that* $\lambda E - A$ *is c-stable.*

1. *System* (1) *is R-controllable and R-observable if and only if*

$$\operatorname{rank}(\mathcal{G}_{pc}) = \operatorname{rank}(\mathcal{G}_{po}) = \operatorname{rank}(\mathcal{G}_{pc}E^T\mathcal{G}_{po}E) = n_f.$$

2. *System* (1) *is I-controllable and I-observable if*

$$\operatorname{rank}(\mathcal{G}_{ic}) = \operatorname{rank}(\mathcal{G}_{io}) = \operatorname{rank}(\mathcal{G}_{ic}A^T\mathcal{G}_{io}A) = n_\infty.$$

3. *System* (1) *is C-controllable and C-observable if and only if*

$$\operatorname{rank}(\mathcal{G}_{c}) = \operatorname{rank}(\mathcal{G}_{o}) = \operatorname{rank}(\mathcal{G}_{pc}E^T\mathcal{G}_{po}E + \mathcal{G}_{ic}A^T\mathcal{G}_{io}A) = n.$$

PROOF. The result follows from Theorem 2.3 and representations (3), (16) and (17). □

## 2.3 Hankel singular values

The proper controllability and observability Gramians $\mathcal{G}_{pc}$ and $\mathcal{G}_{po}$ as well as the improper controllability and observability Gramians $\mathcal{G}_{ic}$ and $\mathcal{G}_{io}$ are not system invariant. Indeed, under a system equivalence transformation $(\check{W}, \check{T})$ the proper and improper controllability Gramians $\mathcal{G}_{pc}$ and $\mathcal{G}_{ic}$ are transformed to $\check{\mathcal{G}}_{pc} = \check{T}^{-1}\mathcal{G}_{pc}\check{T}^{-T}$ and $\check{\mathcal{G}}_{ic} = \check{T}^{-1}\mathcal{G}_{ic}\check{T}^{-T}$, respectively, whereas the proper and improper observability Gramians $\mathcal{G}_{po}$ and $\mathcal{G}_{io}$ are transformed to $\check{\mathcal{G}}_{po} = \check{W}^{-T}\mathcal{G}_{po}\check{W}^{-1}$ and $\check{\mathcal{G}}_{io} = \check{W}^{-T}\mathcal{G}_{io}\check{W}^{-1}$, respectively. Then

$$\begin{aligned} \check{\mathcal{G}}_{pc}\check{E}^T\check{\mathcal{G}}_{po}\check{E} &= \check{T}^{-1}\mathcal{G}_{pc}E^T\mathcal{G}_{po}E\check{T}, \\ \check{\mathcal{G}}_{ic}\check{A}^T\check{\mathcal{G}}_{io}\check{A} &= \check{T}^{-1}\mathcal{G}_{ic}A^T\mathcal{G}_{io}A\check{T}. \end{aligned}$$

We see from these formulas that the spectra of the matrices $\mathcal{G}_{pc}E^T\mathcal{G}_{po}E$ and $\mathcal{G}_{ic}A^T\mathcal{G}_{io}A$ are system invariant. These matrices play the same role for descriptor systems as the product of the controllability and observability Gramians for standard state space systems [45]. We have the following result.

**Theorem 2.6** *Let $\lambda E - A$ be c-stable. Then the matrices $\mathcal{G}_{pc}E^T\mathcal{G}_{po}E$ and $\mathcal{G}_{ic}A^T\mathcal{G}_{io}A$ have real and non-negative eigenvalues.*

Proof. It follows from (12) and (13) that $\mathcal{G}_{pc}$ and $E^T\mathcal{G}_{po}E$ are symmetric and positive semidefinite. In this case there exists a nonsingular matrix $\check{T}$ such that

$$\check{T}^{-1}\mathcal{G}_{pc}\check{T}^{-T} = \begin{pmatrix} \Sigma_1 & & & 0 \\ & \Sigma_2 & & \\ & & 0 & \\ 0 & & & 0 \end{pmatrix}, \qquad \check{T}^T E^T\mathcal{G}_{po}E\check{T} = \begin{pmatrix} \Sigma_1 & & & 0 \\ & 0 & & \\ & & \Sigma_3 & \\ 0 & & & 0 \end{pmatrix},$$

where $\Sigma_1$, $\Sigma_2$ and $\Sigma_3$ are diagonal matrices with positive diagonal elements [45, p.76]. Then we get

$$\check{T}^{-1}\mathcal{G}_{pc}E^T\mathcal{G}_{po}E\check{T} = \begin{pmatrix} \Sigma_1^2 & 0 \\ 0 & 0 \end{pmatrix}.$$

Hence, $\mathcal{G}_{pc}E^T\mathcal{G}_{po}E$ is diagonalizable and it has real and non-negative eigenvalues.

Similarly, we can show that eigenvalues of $\mathcal{G}_{ic}A^T\mathcal{G}_{io}A$ are real and non-negative. □

**Definition 2.7** Let $n_f$ and $n_\infty$ be the dimensions of the deflating subspaces of the pencil $\lambda E - A$ corresponding to the finite and infinite eigenvalues, respectively. The square roots of the largest $n_f$ eigenvalues of the matrix $\mathcal{G}_{pc}E^T\mathcal{G}_{po}E$ denoted by $\varsigma_j$ are called the *proper Hankel singular values* of the c-stable continuous-time descriptor system (1). The square roots of the largest $n_\infty$ eigenvalues of the matrix $\mathcal{G}_{ic}A^T\mathcal{G}_{io}A$ denoted by $\vartheta_j$ are called the *improper Hankel singular values* of system (1).

We will assume that the proper and improper Hankel singular values are ordered decreasingly, i.e.,

$$\varsigma_1 \ge \varsigma_2 \ge \ldots \ge \varsigma_{n_f} \ge 0, \qquad \vartheta_1 \ge \vartheta_2 \ge \ldots \ge \vartheta_{n_\infty} \ge 0.$$

The proper and improper Hankel singular values put together the set of the Hankel singular values of the continuous-time descriptor system (1). For $E = I$, the proper Hankel singular values are the classical Hankel singular values of the standard state space system [17].

Since the proper (improper) controllability and observability Gramians are symmetric and positive (negative) semidefinite, there exist Cholesky factorizations

$$\mathcal{G}_{pc} = RR^T, \qquad \mathcal{G}_{po} = L^T L \tag{18}$$

and

$$\mathcal{G}_{ic} = -R'(R')^T, \qquad \mathcal{G}_{io} = -(L')^T L', \tag{19}$$

where the matrices $R$, $L$, $R'$, $L' \in \mathbb{R}^{n,n}$ are the upper triangular Cholesky factors [28]. The following lemma gives a connection between the proper and improper Hankel singular values of system (1) and the standard singular values of the matrices $LER$ and $L'AR'$.

**Lemma 2.8** *Assume that the descriptor system* (1) *is c-stable. Consider the Cholesky factorizations* (18) *and* (19) *of the proper and improper Gramians of* (1)*. Then the proper Hankel singular values of system* (1) *are the* $n_f$ *largest singular values of the matrix* $LER$*, and the improper Hankel singular values of system* (1) *are the* $n_\infty$ *largest singular values of the matrix* $L'AR'$*.*

PROOF. We have

$$\varsigma_j^2 = \lambda_j(\mathcal{G}_{pc}E^T\mathcal{G}_{po}E) = \lambda_j(RR^TE^TL^TLE) = \lambda_j(R^TE^TL^TLER) = \sigma_j^2(LER),$$
$$\vartheta_j^2 = \lambda_j(\mathcal{G}_{ic}A^T\mathcal{G}_{io}A) = \lambda_j(R'(R')^TA^T(L')^TL'A) = \lambda_j((R')^TA^T(L')^TL'AR') = \sigma_j^2(L'AR'),$$

where $\lambda_j(\cdot)$ and $\sigma_j(\cdot)$ denote, respectively, the eigenvalues and the singular values ordered decreasingly. □

# 3 Model reduction

In this section we consider the problem of reducing the order of the descriptor system (1).

## 3.1 Balanced reduction

For a given transfer function $\mathbf{G}(s)$, there are many different realizations [11]. Here we are interesting only in particular realizations that are useful in applications.

**Definition 3.1** A realization $[\,E, A, B, C\,]$ of the transfer function $\mathbf{G}(s)$ is called *R-minimal* if the triplet $(E, A, B)$ is R-controllable and the triplet $(E, A, C)$ is R-observable.

**Definition 3.2** A realization $[\,E, A, B, C\,]$ of the c-stable transfer function $\mathbf{G}(s)$ is called *proper balanced* if the proper controllability and observability Gramians $\mathcal{G}_{pc}$ and $\mathcal{G}_{po}$ are equal and diagonal.

We will show that for a R-minimal realization $[\,E, A, B, C\,]$ of the c-stable transfer function $\mathbf{G}(s)$, there exists a system equivalence transformation $(W_b^T, T_b)$ such that the realization

$$[\,W_b^TET_b,\ W_b^TAT_b,\ W_b^TB,\ CT_b\,] \tag{20}$$

is proper balanced.

If $(E, A, B)$ is R-controllable and $(E, A, C)$ is R-observable, then by Corollary 2.5 we have $\mathrm{rank}(\mathcal{G}_{pc}) = \mathrm{rank}(\mathcal{G}_{po}) = \mathrm{rank}(\mathcal{G}_{pc}E^T\mathcal{G}_{po}E) = n_f$. Consider the Cholesky factorizations (18) of the proper controllability and observability Gramians. We may assume without loss of generality that the Cholesky factors $R$, $L^T \in \mathbb{R}^{n,n_f}$ have full column rank. It follows from Lemma 2.8 and Corollary 2.5 that $\varsigma_j = \sigma_j(LER) > 0$ for $j = 1, \ldots, n_f$ and, hence, the matrix $LER \in \mathbb{R}^{n_f,n_f}$ is nonsingular.

Let

$$LER = U\Sigma V^T \tag{21}$$

be a singular value decomposition of $LER$, where $U$ and $V$ are orthogonal matrices and $\Sigma = \mathrm{diag}(\varsigma_1, \ldots, \varsigma_{n_f})$ is nonsingular. Consider the matrices

$$W_b = \left[\,L^TU\Sigma^{-1/2},\ \ W_\infty\,\right], \qquad W_b' = \left[\,ERV\Sigma^{-1/2},\ \ W_\infty'\,\right] \tag{22}$$

and

$$T_b = \Big[ RV\Sigma^{-1/2},\ \ T_\infty \Big], \qquad T_b' = \Big[ E^TL^TU\Sigma^{-1/2},\ \ T_\infty' \Big]. \tag{23}$$

Here the columns of matrices $W_\infty$ and $T_\infty$ span, respectively, the left and right deflating subspaces of the pencil $\lambda E - A$ corresponding to the infinite eigenvalues, and matrices $W_\infty'$ and $T_\infty'$ satisfy $W_\infty^T W_\infty' = (T_\infty')^T T_\infty = I_{n_\infty}$. Clearly, for $P_r$ and $P_l$ as in (4), we have $I - P_r = T_\infty (T_\infty')^T$ and $I - P_l = W_\infty' W_\infty^T$. Since

$$\begin{aligned} (I-P_r)RR^T(I-P_r)^T &= (I-P_r)\mathcal{G}_{pc}(I-P_r)^T = 0, \\ (I-P_l)^TL^TL(I-P_l) &= (I-P_l)^T\mathcal{G}_{po}(I-P_l) = 0, \end{aligned}$$

we obtain that

$$R^TT_\infty' = 0 \qquad \text{and} \qquad LW_\infty' = 0. \tag{24}$$

Then

$$(T_b')^TT_b = \begin{pmatrix} \Sigma^{-1/2}U^TLERV\Sigma^{-1/2} & \Sigma^{-1/2}U^TLET_\infty \\ (T_\infty')^TRV\Sigma^{-1/2} & (T_\infty')^TT_\infty \end{pmatrix} = I_n,$$

i.e., the matrices $T_b$ and $T_b'$ are nonsingular and $(T_b')^T = T_b^{-1}$. Similarly, we can show that the matrices $W_b$ and $W_b'$ are also nonsingular and $(W_b')^T = W_b^{-1}$.

Using (18) and (21)-(24), we obtain that the proper controllability and observability Gramians of the transformed system (20) have the form

$$T_b^{-1}\mathcal{G}_{pc}T_b^{-T} = \begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix} = W_b^{-1}\mathcal{G}_{po}W_b^{-T},$$

where $\Sigma = \operatorname{diag}(\varsigma_1, \ldots, \varsigma_{n_f})$ with the proper Hankel singular values $\varsigma_j$. Thus, $(W_b^T, T_b)$ with $W_b$ and $T_b$ as in (22) and (23), respectively, is the balancing transformation and realization (20) is proper balanced.

Just as for standard state space systems [17, 31], the balancing transformation for descriptor systems is not unique.

**Remark 3.3** Note that the pencil $\lambda E_b - A_b = W_b^T(\lambda E - A)T_b$ is in Weierstrass-like canonical form. Indeed, from (21)-(23) we have

$$\begin{aligned} E_b &= \begin{pmatrix} \Sigma^{-1/2}U^TLERV\Sigma^{-1/2} & \Sigma^{-1/2}U^TLET_\infty \\ W_\infty^TERV\Sigma^{-1/2} & W_\infty^TET_\infty \end{pmatrix} = \begin{pmatrix} I_{n_f} & 0 \\ 0 & E_\infty \end{pmatrix}, \\ A_b &= \begin{pmatrix} \Sigma^{-1/2}U^TLARV\Sigma^{-1/2} & \Sigma^{-1/2}U^TLAT_\infty \\ W_\infty^TARV\Sigma^{-1/2} & W_\infty^TAT_\infty \end{pmatrix} = \begin{pmatrix} A_1 & 0 \\ 0 & A_\infty \end{pmatrix}, \end{aligned}$$

where $A_1 = \Sigma^{-1/2}U^TLARV\Sigma^{-1/2}$, $E_\infty = W_\infty^TET_\infty$ is nilpotent and $A_\infty = W_\infty^TAT_\infty$ is nonsingular. Clearly, the pencil $\lambda E_b - A_b$ is regular, c-stable and has the same index as $\lambda E - A$.

## 3.2 Balanced truncation

In the previous subsection we have considered a reduction of an R-minimal realization to proper balanced form. However, computing the proper balanced realization may be ill-conditioned as soon as $\Sigma$ in (21) has small singular values. In addition, if the realization is not R-minimal, then the matrix $\Sigma$ is singular. In the similar situation for standard state space systems one performs a model reduction by truncating the state components corresponding to the zero and small Hankel singular values without significant changes of the system properties, see, e.g., [31, 40]. This procedure is known as *projection of dynamics* or *balanced truncation*. It can be also applied to the descriptor system (1).

The proper controllability and observability Gramians can be used to describe the future output energy

$$\mathbf{E}_y := \int_0^\infty y^T(t)y(t)\,dt$$

and the minimal past proper input energy

$$\mathbf{E}_u := \min_{u\in\mathbb{L}_2^m(\mathbb{R}^-)} \int_{-\infty}^0 u^T(t)u(t)\,dt \tag{25}$$

that is needed to reach from $x(-\infty) = 0$ the state $x(0) = x^0 \in \operatorname{im} P_r$. Here $\mathbb{R}^- = (-\infty, 0)$ and $\mathbb{L}_2^m(\mathbb{R}^-)$ is the Hilbert space of all square integrable functions $f : \mathbb{R} \longrightarrow \mathbb{R}^m$ such that $f(t) = 0$ for $t \geq 0$.

**Theorem 3.4** *Consider a descriptor system* (1) *that is c-stable and R-minimal. Let* $\mathcal{G}_{pc}$ *and* $\mathcal{G}_{po}$ *be the proper controllability and observability Gramians of* (1)*. If* $x^0 \in \operatorname{im} P_r$ *and* $u(t) = 0$ *for* $t \geq 0$*, then*

$$\mathbf{E}_y = (x^0)^T E^T \mathcal{G}_{po} E x^0.$$

*Moreover, for* $u_{opt}(t) = B^T\mathcal{F}(-t)\mathcal{G}_{pc}^- x^0$*, we have*

$$\mathbf{E}_{u_{opt}} = (x^0)^T \mathcal{G}_{pc}^- x^0,$$

*where* $\mathcal{G}_{pc}^-$ *is the unique solution of*

$$\begin{array}{c} \mathcal{G}_{pc}\mathcal{G}_{pc}^-\mathcal{G}_{pc} = \mathcal{G}_{pc}, \\ P_r^T\mathcal{G}_{pc}^-P_r = \mathcal{G}_{pc}^-. \end{array} \tag{26}$$

Proof. System (1) with $x^0 \in \operatorname{im} P_r$ and $u(t) = 0$ for $t \geq 0$ has a unique solution given by $x(t) = \mathcal{F}(t)Ex^0$. Then $y(t) = C\mathcal{F}(t)Ex^0$ for $t \geq 0$ and, hence,

$$\mathbf{E}_y = \int_0^\infty y^T(t)y(t)\,dt = \int_0^\infty (x^0)^T E^T \mathcal{F}^T(t) C^T C \mathcal{F}(t) E x^0\,dt = (x^0)^T E^T \mathcal{G}_{po} E x^0.$$

Consider now the minimization problem (25) subject to the constraint for the initial conditions

$$x^0 = \int_{-\infty}^0 \mathcal{F}(-t)Bu(t)\,dt. \tag{27}$$

Let $\mu \in \mathbb{R}^n$ be a Lagrange multiplier vector and let

$$L(u(t),\mu) = \int_{-\infty}^0 u^T(t)u(t)\,dt + \mu^T\left(x^0 - \int_{-\infty}^0 \mathcal{F}(-t)Bu(t)\,dt\right)$$

be the Lagrange function. For any variations $\Delta u(t)$ and $\Delta\mu$ we have that

$$\begin{array}{rcl} \Delta L(u(t),\mu) & = & 2\displaystyle\int_{-\infty}^0 u^T(t)\Delta u(t)\,dt - \mu^T \int_{-\infty}^0 \mathcal{F}(-t)B\Delta u(t)\,dt \\ & + & \Delta\mu^T\left(x^0 - \displaystyle\int_{-\infty}^0 \mathcal{F}(-t)Bu(t)\,dt\right) = 0 \end{array}$$

if and only if (27) holds and

$$u^T(t) = \frac{1}{2}\mu^T \mathcal{F}(-t)B = \frac{1}{2}\mu^T P_r \mathcal{F}(-t)B. \tag{28}$$

Substitution of (28) in (27) gives

$$x^0 = \frac{1}{2}\int_{-\infty}^{0} \mathcal{F}(-t)BB^T\mathcal{F}^T(-t)\mu\, dt = \frac{1}{2}\int_{0}^{\infty} \mathcal{F}(t)BB^T\mathcal{F}^T(t)\mu\, dt = \frac{1}{2}\mathcal{G}_{pc}\mu. \tag{29}$$

Using the representation for $\mathcal{G}_{pc}$ as in (16), where $G_{1c}$ is symmetric and positive definite (since $(E, A, B)$ is R-controllable), we obtain that (26) has a unique solution $\mathcal{G}_{pc}^-$ given by

$$\mathcal{G}_{pc}^- = T^T \begin{pmatrix} G_{1c}^{-1} & 0 \\ 0 & 0 \end{pmatrix} T. \tag{30}$$

It follows from (29) that $2\mathcal{G}_{pc}^- x^0 = \mathcal{G}_{pc}^-\mathcal{G}_{pc}\mu = P_r^T\mu$. Hence, for the optimal input

$$u_{opt}(t) = B^T\mathcal{F}^T(-t)\mathcal{G}_{pc}^- x^0,$$

we have that

$$\begin{aligned} \mathbf{E}_{u_{opt}} &= \int_{-\infty}^{0} u_{opt}^T(t)u_{opt}(t)\, dt = \int_{-\infty}^{0} (x^0)^T(\mathcal{G}_{pc}^-)^T\mathcal{F}(-t)BB^T\mathcal{F}^T(-t)\mathcal{G}_{pc}^- x^0\, dt \\ &= (x^0)^T(\mathcal{G}_{pc}^-)^T\left(\int_{0}^{\infty} \mathcal{F}(t)BB^T\mathcal{F}^T(t)\, dt\right)\mathcal{G}_{pc}^- x^0 = (x^0)^T\mathcal{G}_{pc}^- x^0. \end{aligned}$$ □

**Remark 3.5** Using (16) and (30) we obtain the relationships

$$\mathcal{G}_{pc}\mathcal{G}_{pc}^- = P_r, \qquad \mathcal{G}_{pc}^-\mathcal{G}_{pc} = P_r^T, \qquad \mathcal{G}_{pc}^-\mathcal{G}_{pc}\mathcal{G}_{pc}^- = \mathcal{G}_{pc}^-$$

which, together with the first equation in (26), imply that $\mathcal{G}_{pc}^-$ is, in general, a $(1,2)$-pseudo-inverse of $\mathcal{G}_{pc}$, see [9]. However, if $P_r^T = P_r$, then $\mathcal{G}_{pc}^-$ is the Moore-Penrose inverse [9] of $\mathcal{G}_{pc}$.

Theorem 3.4 shows that a large input energy $\mathbf{E}_u$ is required to reach from $x(-\infty) = 0$ the state $x(0) = P_r x^0$ which lies in an invariant subspace of the proper controllability Gramian $\mathcal{G}_{pc}$ corresponding to its small non-zero eigenvalues. Moreover, if $x^0$ is contained in an invariant subspace of the matrix $E^T\mathcal{G}_{po}E$ corresponding to its small non-zero eigenvalues, then the initial value $x(0) = P_r x^0$ has a small effect on the output energy $\mathbf{E}_y$. For the proper balanced system, $\mathcal{G}_{pc}$ and $E^T\mathcal{G}_{po}E$ are equal and, hence, they have the same invariant subspaces. In this case the truncation of the states related to the small Hankel singular values does not change system properties significantly.

Let $[\,E, A, B, C\,]$ be a realization (not necessarily R-minimal) of the c-stable transfer function $\mathbf{G}(s)$. Consider the full rank Cholesky factorizations (18), where the matrices $R \in \mathbb{R}^{n,r_c}$, $L^T \in \mathbb{R}^{n,r_o}$ have full column rank and $r_c = \operatorname{rank}(\mathcal{G}_{pc}) \le n_f$, $r_o = \operatorname{rank}(\mathcal{G}_{po}) \le n_f$. Let

$$LER = [\,U_1,\, U_0\,]\begin{pmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_0 \end{pmatrix}[\,V_1,\, V_0\,]^T \tag{31}$$

be an "economy size" singular value decomposition of $LER \in \mathbb{R}^{r_o,r_c}$, where $[\,U_1,\, U_0\,] \in \mathbb{R}^{r_o,r}$ and $[\,V_1,\, V_0\,] \in \mathbb{R}^{r_c,r}$ have orthogonal columns,

$$\Sigma_1 = \operatorname{diag}(\varsigma_1, \ldots, \varsigma_{\ell_f}) \qquad \text{and} \qquad \Sigma_0 = \operatorname{diag}(\varsigma_{\ell_f+1}, \ldots, \varsigma_r)$$

with $\varsigma_1 \geq \varsigma_2 \geq \ldots \geq \varsigma_{\ell_f} > \varsigma_{\ell_f+1} \geq \ldots \geq \varsigma_r > 0$ and $r = \operatorname{rank}(\mathcal{G}_{pc}E^T\mathcal{G}_{po}E) \leq \min(r_c, r_o)$. Then the reduced order realization can be computed as

$$\left[\begin{array}{c|c} s\widetilde{E} - \widetilde{A} & \widetilde{B} \\ \hline \widetilde{C} & 0 \end{array}\right] = \left[\begin{array}{c|c} W_\ell^T(sE - A)T_\ell & W_\ell^T B \\ \hline CT_\ell & 0 \end{array}\right], \tag{32}$$

where

$$W_\ell = \left[\, L^T U_1 \Sigma_1^{-1/2}, \;\; W_\infty \,\right] \in \mathbb{R}^{n,\ell}, \qquad T_\ell = \left[\, RV_1 \Sigma_1^{-1/2}, \;\; T_\infty \,\right] \in \mathbb{R}^{n,\ell} \tag{33}$$

and $\ell = \ell_f + n_\infty$. Here $W_\infty$ and $T_\infty$ form the bases of the left and right deflating subspaces, respectively, corresponding to the infinite eigenvalues of $\lambda E - A$.

Note that computing the reduced order descriptor system can be interpreted as performing a system equivalence transformation $(\check{W}, \check{T})$ such that

$$\left[\begin{array}{c|c} \check{W}(sE - A)\check{T} & \check{W}B \\ \hline C\check{T} & 0 \end{array}\right] = \left[\begin{array}{cc|c} sE_f - A_f & & B_f \\ & sE_\infty - A_\infty & B_\infty \\ \hline C_f & C_\infty & 0 \end{array}\right],$$

where the pencil $\lambda E_f - A_f$ has the finite eigenvalues only, all eigenvalues of $\lambda E_\infty - A_\infty$ are infinite, and then reducing the order of the subsystem $[\,E_f,\, A_f,\, B_f,\, C_f\,]$ with nonsingular $E_f$. Clearly, the reduced order system (32) is c-stable and R-minimal.

The described decoupling of system matrices is equivalent to the additive decomposition of the transfer function as $\mathbf{G}(s) = \mathbf{G}_p(s) + \mathbf{P}(s)$, where $\mathbf{G}_p(s) = C_f(sE_f - A_f)^{-1}B_f$ is the proper part and $\mathbf{P}(s) = C_\infty(sE_\infty - A_\infty)^{-1}B_\infty$ is the polynomial part of $\mathbf{G}(s)$. The transfer function of the reduced system has the form $\widetilde{\mathbf{G}}(s) = \widetilde{\mathbf{G}}_p(s) + \mathbf{P}(s)$, where $\widetilde{\mathbf{G}}_p(s) = \widetilde{C}_f(s\widetilde{E}_f - \widetilde{A}_f)^{-1}\widetilde{B}_f$ is the reduced subsystem. In this case the difference $\mathbf{G}(s) - \widetilde{\mathbf{G}}(s) = \mathbf{G}_p(s) - \widetilde{\mathbf{G}}_p(s)$ is a proper rational function, and we have the following upper bound for the $\mathbb{H}_\infty$-norm of the error system

$$\|\mathbf{G}(s) - \widetilde{\mathbf{G}}(s)\|_{\mathbb{H}_\infty} := \sup_{\omega \in \mathbb{R}} \|\mathbf{G}(i\omega) - \widetilde{\mathbf{G}}(i\omega)\| \leq 2(\varsigma_{\ell_f+1} + \ldots + \varsigma_{n_f}) \tag{34}$$

that has been derived in [17]. Here $\|\cdot\|$ denotes the spectral matrix norm.

### 3.3 Numerical aspects

To reduce the order of the descriptor system (1) we have to compute the full rank Cholesky factors $R$ and $L$ of the proper controllability and observability Gramians that satisfy the projected generalized Lyapunov equations (14) and (15). We need also the matrices $W_\infty$ and $T_\infty$ whose columns span the left and right deflating subspaces, respectively, corresponding to the infinite eigenvalues of $\lambda E - A$.

The classical numerical methods for (generalized) Lyapunov equations are (generalized) Bartels-Stewart and Hammarling methods [3, 22, 32] based on the preliminary reduction of the matrix (matrix pencil) to the (generalized) Schur form [18], calculation of the solution of a reduced system and back transformation. To solve numerically the projected generalized Lyapunov equations (14) and (15) for the full rank Cholesky factors, we can use the generalized Schur-Hammarling method proposed in [38]. Simultaneously, this method produces the matrices $W_\infty$ and $T_\infty$.

**Algorithm 3.1** *Solution of the projected generalized Lyapunov equations* (14) *and* (15)
**Input:** *System* $[E, A, B, C]$ *such that* $\lambda E - A$ *is c-stable.*
**Output:** *Full rank Cholesky factors* $R$ *and* $L$ *of the proper controllability and observability Gramians* $\mathcal{G}_{pc} = RR^T$ *and* $\mathcal{G}_{po} = L^T L$ *and the matrices* $W_\infty$ *and* $T_\infty$ *that form the bases of the left and right deflating subspaces corresponding to the infinite eigenvalues of* $\lambda E - A$.
**Step 1.** *Use the GUPTRI algorithm* [12, 13] *to compute orthogonal transformation matrices* $U$ *and* $V$ *such that*

$$V^T E U = \begin{pmatrix} E_f & E_u \\ 0 & E_\infty \end{pmatrix} \qquad \textit{and} \qquad V^T A U = \begin{pmatrix} A_f & A_u \\ 0 & A_\infty \end{pmatrix}, \tag{35}$$

*where* $E_f$ *is upper triangular, nonsingular and* $E_\infty$ *is upper triangular with zeros on the diagonal,* $A_f$ *is upper quasi-triangular and* $A_\infty$ *is upper triangular, nonsingular.*
**Step 2.** *Use the generalized Schur method* [25, 26] *or the recursive blocked algorithm* [24] *to compute the solution of the generalized Sylvester equation*

$$\begin{aligned} E_f Y - Z E_\infty &= -E_u, \\ A_f Y - Z A_\infty &= -A_u. \end{aligned}$$

**Step 3.** *Compute the matrices*

$$V^T B = \begin{bmatrix} B_1 \\ B_\infty \end{bmatrix}, \qquad CU = [\, C_f, \;\; C_2 \,].$$

**Step 4.** *Use the generalized Hammarling method* [22, 32] *to compute the Cholesky factors* $R_f$ *and* $L_f$ *of the solutions* $X_c = R_f R_f^T$ *and* $X_o = L_f^T L_f$ *of the generalized Lyapunov equations*

$$\begin{aligned} &E_f X_c A_f^T + A_f X_c E_f^T = -(B_1 - Z B_\infty)(B_1 - Z B_\infty)^T, \\ &E_f^T X_o A_f + A_f^T X_o E_f = -C_f^T C_f. \end{aligned}$$

**Step 5.** *If* $\operatorname{rank}(R_f) < n_f$, *then compute the full column rank matrix* $R_1$ *from the QR decomposition*

$$R_f^T = Q_R \begin{bmatrix} R_1^T \\ 0 \end{bmatrix}.$$

*Otherwise,* $R_1 := R_f$.
**Step 6.** *If* $\operatorname{rank}(L_f) < n_f$, *then compute the full row rank matrix* $L_1$ *from the QR decomposition*

$$L_f = Q_L \begin{bmatrix} L_1 \\ 0 \end{bmatrix}.$$

*Otherwise,* $L_1 := L_f$.
**Step 7.** *Compute the full rank Cholesky factors*

$$R = U \begin{bmatrix} R_1 \\ 0 \end{bmatrix}, \qquad L = [\, L_1, \;\; -L_1 Z \,]\, V^T. \tag{36}$$

**Step 8.** *Compute the matrices*

$$W_\infty = V \begin{bmatrix} 0 \\ I_{n_\infty} \end{bmatrix} \qquad \textit{and} \qquad T_\infty = U \begin{bmatrix} Y \\ I_{n_\infty} \end{bmatrix}. \tag{37}$$

The generalized Schur-Hammarling method costs $O(n^3)$ flops and can be used, unfortunately, only for problems of small and medium size. Moreover, it does not take into account the sparsity or any structure of the system and is not attractive for parallelization. Recently, iterative methods related to the ADI method and the Smith method have been proposed to compute the low rank approximation of the solutions of standard large-scale sparse Lyapunov equations [1, 33]. It is important to extend these methods for projected generalized Lyapunov equations. This topic is currently under investigations.

The following algorithm is a generalization of the *square root balanced truncation method* [29, 40] for the descriptor system (1).

**Algorithm 3.2** *Generalized Square Root (GSR) method.*
**Input:** *A realization* $[\,E,\,A,\,B,\,C\,]$ *such that* $\lambda E - A$ *is c-stable.*
**Output:** *A reduced order system* $[\,\widetilde{E},\,\widetilde{A},\,\widetilde{B},\,\widetilde{C}\,]$.
**Step 1.** *Use* Algorithm 3.1 *to compute the full rank factors* $R$ *and* $L$ *of the proper controllability and observability Gramians* $\mathcal{G}_{pc} = R^TR$ *and* $\mathcal{G}_{po} = LL^T$ *as well as the bases* $W_\infty$ *and* $T_\infty$ *of the left and right deflating subspaces of* $\lambda E - A$ *corresponding to the infinite eigenvalues.*
**Step 2.** *Compute the "economy size" singular value decomposition* (31).
**Step 3.** *Compute the matrices* $W_\ell = [\,L^TU_1\Sigma_1^{-1/2},\ \ W_\infty\,]$ *and* $T_\ell = [\,RV_1\Sigma_1^{-1/2},\ \ T_\infty\,]$.
**Step 4.** *Compute the reduced order system* $[\,\widetilde{E},\,\widetilde{A},\,\widetilde{B},\,\widetilde{C}\,] = [\,W_\ell^TET_\ell,\,W_\ell^TAT_\ell,\,W_\ell^TB,\,CT_\ell\,]$.

If the original system (1) is highly unbalanced, then the matrices $W_\ell$ and $T_\ell$ are ill-conditioned. To avoid accuracy loss in the reduced system, a *square root balancing free* method has been proposed for standard state space systems in [41]. This approach can be generalized for descriptor systems as follows.

**Algorithm 3.3** *Generalized Square Root Balancing Free (GSRBF) method.*
**Input:** *A realization* $[\,E,\,A,\,B,\,C\,]$ *such that* $\lambda E - A$ *is c-stable.*
**Output:** *A reduced order system* $[\,\widetilde{E},\,\widetilde{A},\,\widetilde{B},\,\widetilde{C}\,]$.
**Step 1.** *Use* Algorithm 3.1 *to compute the full rank factors* $R$ *and* $L$ *of the proper controllability and observability Gramians* $\mathcal{G}_{pc} = R^TR$ *and* $\mathcal{G}_{po} = LL^T$ *as well as the bases* $W_\infty$ *and* $T_\infty$ *of the left and right deflating subspaces of* $\lambda E - A$ *corresponding to the infinite eigenvalues.*
**Step 2.** *Compute the "economy size" singular value decomposition* (31).
**Step 3.** *Compute the "economy size" QR decompositions*

$$RV_1 = Q_cR_0, \qquad\qquad L^TU_1 = Q_oL_0,$$

*where* $Q_c,\ Q_o \in \mathbb{R}^{n,\ell_f}$ *have orthogonal columns and* $R_0,\ L_0 \in \mathbb{R}^{\ell_f,\ell_f}$ *are upper triangular, nonsingular.*
**Step 4.** *Compute the reduced order system* $[\,\widetilde{E},\,\widetilde{A},\,\widetilde{B},\,\widetilde{C}\,] = [\,W_\ell^TET_\ell,\,W_\ell^TAT_\ell,\,W_\ell^TB,\,CT_\ell\,]$, *where* $W_\ell = [\,Q_o,\ \ W_\infty\,]$ *and* $T_\ell = [\,Q_c,\ \ T_\infty\,]$.

The GSR and GSRBF methods are mathematically equivalent in the sense that they deliver a reduced system with the same transfer function. But the matrices $W_\ell$ and $T_\ell$ computed by the GSRBF method are often significantly better conditioned than those computed via the GSR method.

**Remark 3.6** In fact, we do not need to compute the full rank Cholesky factors $R$ and $L$ in Step 7 and the matrices $W_\infty$ and $T_\infty$ in Step 8 of Algorithm 3.1. From (35) and (37) we have $W_\infty^T E T_\infty = E_\infty$, $W_\infty^T A T_\infty = A_\infty$, $W_\infty^T B = B_\infty$ and $C T_\infty = C_f Y + C_2 = C_\infty$. Moreover, it follows from (35) and (36) that $LER = L_1 E_f R_1$. Thus, computation of the proper Hankel singular values in Step 2 of Algorithms 3.2 and 3.3 can be performed working only with the matrices $L_1$, $E_f$ and $R_1$. This reduces the computational cost and the memory requirement. Note that the singular value decomposition of $L_1 E_f R_1$ may be computed without forming this product explicitly, see [19] for details.

# 4 Numerical example

Consider the holonomically constrained planar model of a truck [36]. The linearized equation of motion has the form

$$\begin{aligned} \dot{\mathbf{p}}(t) &= \mathbf{v}(t), \\ M\dot{\mathbf{v}}(t) &= K\mathbf{p}(t) + D\mathbf{v}(t) - G^T \boldsymbol{\lambda}(t) + B_2 u(t), \\ 0 &= G\mathbf{p}(t), \end{aligned} \tag{38}$$

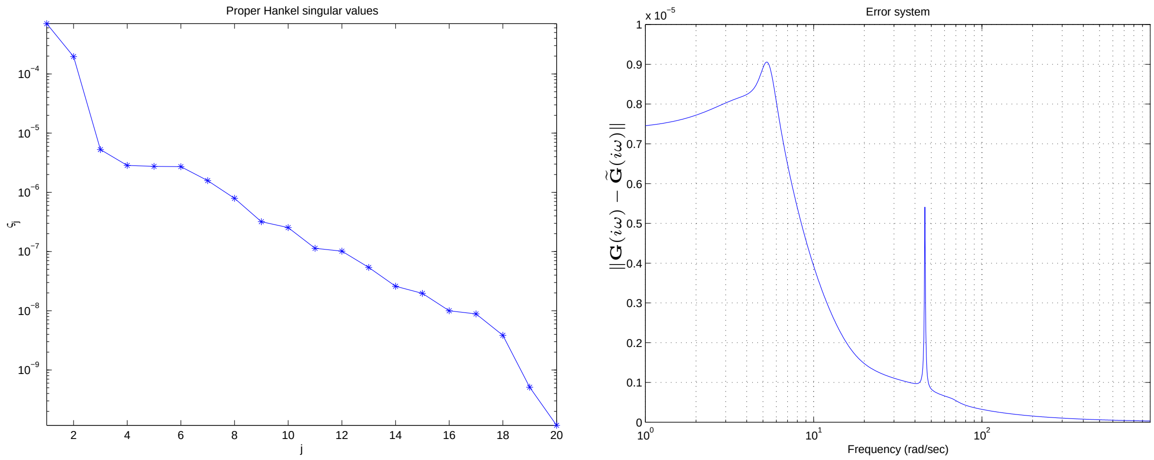
where $\mathbf{p}(t) \in \mathbb{R}^{11}$ is the position vector, $\mathbf{v}(t) \in \mathbb{R}^{11}$ is the velocity vector, $\boldsymbol{\lambda}(t) \in \mathbb{R}$ is the Lagrange multiplier, $M$ is the positive definite mass matrix, $K$ is the stiffness matrix, $D$ is the damping matrix, $G$ is the constraint matrix and $B_2$ is the input matrix. System (38) together with the output equation $y(t) = \mathbf{p}(t)$ forms a descriptor system of order $n = 23$ with $m = 1$ input and $p = 11$ outputs. The dimension of the deflating subspace corresponding to the finite eigenvalues is $n_f = 20$. This toy-example is presented to illustrate the reliability of the proposed model reduction methods for descriptor systems.

All of the following results were obtained on an IBM RS 6000 44P Model 270 with relative machine precision $\epsilon = 2.22 \times 10^{-16}$ using the MATLAB mex-functions based on the GUPTRI routine [12, 13] and the SLICOT library routines [5, 42].

Figure 1(a) shows the proper Hankel singular values $\varsigma_j$, $j = 1, \ldots, 20$. We approximate system (38) by a model of order $\ell = 5$. Note that the Bode plots of the original and reduced systems are not presented, since they were impossible to distinguish. Figure 1(b) illustrate how accurate the reduced order model approximate the original one. We display the amplitude Bode plot of the error system $\mathbf{G}(i\omega) - \widetilde{\mathbf{G}}(i\omega)$ for a frequency range $\omega \in [1, 10^3]$. Comparison of this error with the upper bound $2(\varsigma_3 + \ldots + \varsigma_{20}) = 1.69 \times 10^{-5}$ shows that the error estimate (34) is tight.

# 5 Conclusion

We have generalized the controllability and observability Gramians as well as Hankel singular values for descriptor systems and studied their important features. Model reduction methods for descriptor systems have been presented. These methods are based on the balanced truncation technique and deliver reduced order systems that preserve the regularity and stability properties of the original system. Moreover, for these methods the approximation error bound is available.




(a) (b)

Figure 1: Proper Hankel singular values of the linearized truck model (a) and Bode plot of the error system (b)

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