Introduction

A bounded selfadjoint operator A in a Krein space $(\mathcal{H}, [\cdot,\cdot])$ is called *definitizable* if there exists a polynomial $p \neq 0$ such that $[p(A)x, x] \geq 0$ for all $x \in \mathcal{H}$. A definitizable operator A has real spectrum with the possible exception of a finite number of nonreal eigenvalues, and it has a spectral function defined for all real intervals the boundary points of which do not belong to some finite set of real points, the so-called critical points ([18], see [20] and also [6]).

With the help of the spectral function E the real spectral points of A can be classified in points of positive and negative type and critical points: If a point $\lambda \in \sigma(A) \cap \mathbf{R}$ is contained in some open interval δ such that $E(\delta)$ is defined and $(E(\delta)\mathcal{H}, [\cdot,\cdot])$ (resp. $(E(\delta)\mathcal{H}, -[\cdot,\cdot])$) is a Hilbert space, it is called of *positive* (resp. *negative*) *type*. A point $\lambda \in \sigma(A) \cap \mathbf{R}$ which is neither of positive nor of negative type is called a *critical point*.

In [19] H. Langer studied a class of compact perturbations of fundamentally reducible selfadjoint operators in a Krein space. It was proved in that paper that the restrictions of the perturbed operator to the spectral subspaces corresponding to those open intervals which contain no critical points of the unperturbed operator, are definitizable. Such locally definitizable operators have been studied in connection with perturbation problems in [7], [8], [9], [13]. For locally definitizable operators, due to the finite order growth of the resolvent near to some open subset of the real axis, a local variant of the functional calculus for generalized spectral operators (see [1]) can be established. In [7] and [8] spectral points of positive and negative type are introduced with the help of this functional calculus or by making use of some properties of the resolvent, and a local spectral function is constructed.

In [17], for a bounded selfadjoint operator A in a Krein space, the points of positive and negative type were introduced with the help of approximate eigenvector sequences (in [17] these points are called of plus type and of minus type). In [21] H. Langer, A. Markus and V. Matsaev, leaning on that definition, constructed a local spectral function which is defined for all real intervals which do not contain accumulation points of the nonreal spectrum, and the spectral points of which are of positive type (or of negative type). In the same paper this approach was applied to investigate the behaviour of sign types under perturbations.

The main objective of the present paper is to prove that the sign type definitions of [7], [8] and [21], and some variations of them, are equivalent. All definitions and the results will be given for selfadjoint linear relations in a Krein space. At the same time we give the versions for unitary operators, which are connected with selfadjoint linear relations by the Cayley transform. For the problems we are dealing with, it is often convenient to prove the unitary versions and translate the results to the selfadjoint case.

The construction of a functional calculus suitable for sign type definitions will be included in this paper. In view of an application in a forthcoming paper we will give this functional calculus in a more general setting: the resolvent is replaced by an operator function with similar properties.

The second objective of this article is to give necessary and sufficient conditions for definitizability and local definitizability, which is closely connected with the description of sign types.

In Section 1 we give the definition of the spectra of positive and negative type for a selfadjoint linear relation and a unitary operator in a Krein space with the help of approximate eigenvector sequences; and we show that these parts of the spectra behave covariantly with respect to the elementary functional calculus. In Section 2, after some preliminaries on the extension of functional calculi (Sections 2.1 and 2.2), we recall the definition of open sets of positive and negative type with respect to an operator function and an operator, and we describe these sets in different equivalent ways (Section 2.3). Local spectral functions are introduced in Section 2.4; we recall an extension procedure for such spectral functions. In Theorem 2.15 we characterize open sets of positive and negative type in different ways. In Section 3 the properties of local definitizability for operator functions and operators are introduced. The definitions in the present paper slightly differ from the definitions in [7] and [8]. In Theorem 3.7 it is shown that locally definitizable relations can be characterized by (spectral) decompositions into two relations one of which is definitizable.

1. The spectra of positive and negative type

Let $(\mathcal{H}, [\cdot,\cdot])$ be a Krein space. Recall that a closed linear relation T in $\mathcal{H}$ is a closed linear subspace of $\mathcal{H}^2$; a closed linear operator in $\mathcal{H}$ is viewed as a closed linear relation via its graph in $\mathcal{H}^2$. For the usual definitions of the linear operations with closed linear relations and the inverse we refer to [2]. The linear span of two linear subspaces of $\mathcal{H}^2$ will be denoted by $\dotplus$.

The *resolvent set* $\rho(T)$ of a linear relation T is the set of all $z \in \mathbf{C}$ such that $(T - z)^{-1} \in \mathcal{L}(\mathcal{H})$, the *spectrum* $\sigma(T)$ of T is the complement of $\rho(T)$ in $\mathbf{C}$. The *point spectrum* $\sigma_p(T)$ of T is the set of all $z \in \mathbf{C}$ such that

$\binom{f}{zf} \in T$ for some $f \neq 0$. We define

$$\begin{array}{llll} \tilde{\rho}(T) := \rho(T) \cup \{\infty\} & \text{if} \quad 0 \in \rho(T^{-1}), & \tilde{\rho}(T) := \rho(T) & \text{if} \quad 0 \notin \rho(T^{-1}), \\ \tilde{\sigma}(T) := \sigma(T) \cup \{\infty\} & \text{if} \quad 0 \in \sigma(T^{-1}), & \tilde{\sigma}(T) := \sigma(T) & \text{if} \quad 0 \notin \sigma(T^{-1}), \\ \tilde{\sigma}_p(T) := \sigma_p(T) \cup \{\infty\} & \text{if} \quad 0 \in \sigma_p(T^{-1}), & \tilde{\sigma}_p(T) := \sigma_p(T) & \text{if} \quad 0 \notin \sigma_p(T^{-1}). \end{array}$$

The following definition was introduced in [17] for the case of a bounded operator T.

Definition 1.1. We say that $\lambda \in \mathbf{C}$ belongs to the *approximate point spectrum* of T, denoted by $\sigma_{ap}(T)$, if there exists a sequence $\binom{x_n}{y_n} \in T - \lambda$, $n = 1, 2, \ldots$, such that $\|x_n\| = 1$, $\lim_{n\to\infty} \|y_n\| = 0$. We define the *extended approximate point spectrum* $\tilde{\sigma}_{ap}(T)$ of T by $\tilde{\sigma}_{ap}(T) := \sigma_{ap}(T) \cup \{\infty\}$ if $0 \in \sigma_{ap}(T^{-1})$, and $\tilde{\sigma}_{ap}(T) := \sigma_{ap}(T)$ if $0 \notin \sigma_{ap}(T^{-1})$.

Definition 1.2. A point $\lambda \in \sigma_{ap}(T)$, is said to be of *positive type* (*negative type*) with respect to T, if for every sequence $\binom{x_n}{y_n} \in T - \lambda$, $n = 1, 2, \ldots$, with $\|x_n\| = 1$, $\lim_{n\to\infty} \|y_n\| = 0$ we have

$$\liminf_{n\to\infty} [x_n, x_n] > 0 \quad (\text{resp. } \limsup_{n\to\infty} [x_n, x_n] < 0).$$

If $\infty \in \tilde{\sigma}_{ap}(T)$, ∞ is said to be of *positive type* (*negative type*) with respect to T if for every sequence $\binom{x_n}{y_n} \in T$, $n = 1, 2, \ldots$, with $\lim_{n\to\infty} \|x_n\| = 0$, $\|y_n\| = 1$ we have

$$\liminf_{n\to\infty} [y_n, y_n] > 0 \quad (\text{resp. } \limsup_{n\to\infty} [y_n, y_n] < 0).$$

The set of all points of positive type (negative type) with respect to T will be denoted by $\sigma_{++}(T)$ (resp. $\sigma_{--}(T)$).

In the following lemmas it is convenient to make use of a so called transformer of a linear relation (see [23], [3]): If $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ is a matrix with complex entries and T a linear relation, we define the relation MT by

$$MT = \left\{ \begin{pmatrix} \alpha x + \beta y \\ \gamma x + \delta y \end{pmatrix} : \begin{pmatrix} x \\ y \end{pmatrix} \in T \right\}.$$

Evidently,

$$\begin{pmatrix} 1 & 0 \\ \gamma & \delta \end{pmatrix} T = \delta T + \gamma, \qquad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} T = T^{-1}$$

and, for two 2×2 matrices M_1 and M_2, it holds

$$M_1(M_2 T) = (M_1 M_2) T$$

(see [3]).

We assign to every matrix $M = \left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right)$ with $\det M \neq 0$ the fractional linear mapping Φ_M of $\overline{\mathbf{C}}$ onto itself defined by

$$\Phi_M(z) = \frac{\delta z + \gamma}{\beta z + \alpha}.$$

If $\beta \neq 0$ and $-\alpha\beta^{-1} \in \rho(T)$, then Φ_M is locally holomorphic on $\widetilde{\sigma}(T)$ and MT coincides with the operator $\Phi_M(T)$ defined by extension of the Riesz-Dunford-Taylor functional calculus to closed linear relations (see [3, Section 3]). If M_1 and M_2 are two regular matrices, we have $\Phi_{M_1M_2} = \Phi_{M_1} \circ \Phi_{M_2}$.

Lemma 1.3. *Let T be a closed linear relation and M a regular 2×2 matrix. Then*

$$\sigma_{ap}(MT) = \Phi_M(\sigma_{ap}(T)). \tag{1.1}$$

Proof. It is sufficient to prove that

$$\Phi_M(\sigma_{ap}(T)) \subset \sigma_{ap}(MT), \tag{1.2}$$

since (1.2) and the analogous relation for M replaced by M^{-1} implies (1.1). Moreover we have to verify (1.2) only for matrices M of the form $\left(\begin{smallmatrix}1 & 0\\ b & a\end{smallmatrix}\right)$, $a \neq 0$, and $\left(\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right)$, since every 2×2 matrix can be written as a product of matrices of that form and regular diagonal matrices.

Let $\lambda \in \sigma_{ap}(T)$, $\lambda \neq \infty$. Then there exists a sequence

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} \in T \quad \text{with} \quad \|x_n\| = 1,\ \|y_n - \lambda x_n\| \to 0. \tag{1.3}$$

If $M = \left(\begin{smallmatrix}1 & 0\\ b & a\end{smallmatrix}\right)$, $a \neq 0$, then $MT = aT + b$ and $\Phi_M(\lambda) = a\lambda + b$. By

$$aT + b - (a\lambda + b) = \left\{ \begin{pmatrix} x \\ a(y - \lambda x) \end{pmatrix} : \begin{pmatrix} x \\ y \end{pmatrix} \in T \right\}$$

and (1.3) we have $a\lambda + b \in \sigma_{ap}(aT + b)$.

If $M = \left(\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right)$, then $MT = T^{-1}$ and $\Phi_M(\lambda) = \lambda^{-1}$. For $\lambda \neq 0$, we have

$$T^{-1} - \lambda^{-1} = \left\{ \begin{pmatrix} y \\ x - \lambda^{-1} y \end{pmatrix} : \begin{pmatrix} x \\ y \end{pmatrix} \in T \right\} = \left\{ \begin{pmatrix} y - \lambda x + \lambda x \\ -\lambda^{-1}(y - \lambda x) \end{pmatrix} : \begin{pmatrix} x \\ y \end{pmatrix} \in T \right\}$$

which along with (1.3) gives $\lambda^{-1} \in \sigma_{ap}(T^{-1})$. If $\lambda = 0$, then $\infty \in \sigma_{ap}(T^{-1})$.

Let now $\infty \in \sigma_{ap}(T)$. Then there exists a sequence

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} \in T \quad \text{with} \quad \|y_n\| = 1, \ \|x_n\| \to 0.$$

We have

$$aT + b = \left\{ \begin{pmatrix} x \\ ay + bx \end{pmatrix} : \begin{pmatrix} x \\ y \end{pmatrix} \in T \right\}$$

which shows that $\infty \in \sigma_{ap}(aT + b)$. Furthermore, $0 \in \sigma_{ap}(T^{-1})$, and the lemma is proved.

Lemma 1.4. *Let T be a closed linear relation and M a regular 2×2 matrix. Then*

$$\sigma_{++}(MT) = \Phi_M(\sigma_{++}(T)), \qquad \sigma_{--}(MT) = \Phi_M(\sigma_{--}(T)).$$

Proof. We shall prove only the first relation. The second one can be proved in an analogous way. It is sufficient to prove

$$\Phi_M(\sigma_{++}(T)) \subset \sigma_{++}(MT)$$

for $M = \begin{pmatrix} 1 & 0 \\ b & a \end{pmatrix}$, $a \neq 0$, and $M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, similarly to the proof of Lemma 1.3.

Let $\lambda \in \sigma_{++}(T)$, $\lambda \neq \infty$, and let $\binom{x_n}{y_n}$ be a sequence in $aT + b - (a\lambda + b) = a(T - \lambda)$ with $\|x_n\| = 1$, $\|y_n\| \to 0$. Then, since $\binom{x_n}{a^{-1}y_n} \in T - \lambda$, we have $\liminf_{n\to 0} [x_n, x_n] > 0$, i.e. $a\lambda + b \in \sigma_{++}(aT + b)$.

If, in addition, $\lambda \neq 0$ and if $\binom{x_n}{y_n}$ is a sequence in $T^{-1} - \lambda^{-1}$ with $\|x_n\| = 1$, $\|y_n\| \to 0$, then $\binom{y_n + \lambda^{-1}x_n}{-\lambda y_n} \in T - \lambda$ which implies $\liminf_{n\to 0} [x_n, x_n] > 0$, i.e. $\lambda^{-1} \in \sigma_{++}(T^{-1})$. If $\lambda = 0$ and $\binom{x_n}{y_n}$ is a sequence in T^{-1} with $\|y_n\| = 1$, $\|x_n\| \to 0$, then $\binom{y_n}{x_n} \in T$ and $\liminf_{n\to 0} [y_n, y_n] > 0$, i.e. $\infty \in \sigma_{++}(T^{-1})$.

Now let $\infty \in \sigma_{++}(T)$. If $\binom{x_n}{y_n}$ is a sequence in $aT + b$ with $\|y_n\| = 1$, $\|x_n\| \to 0$, then $\binom{x_n}{a^{-1}(y_n - bx_n)} \in T$ and $\liminf_{n\to 0} [y_n, y_n] > 0$, i.e. $\infty \in \sigma_{++}(aT + b)$.

If $\binom{x_n}{y_n}$ is a sequence in T^{-1} with $\|x_n\| = 1$, $\|y_n\| \to 0$, then $\binom{y_n}{x_n} \in T$ and $\liminf_{n\to 0} [x_n, x_n] > 0$, i.e. $0 \in \sigma_{++}(T^{-1})$. This proves Lemma 1.4.

2. Locally definite operators in Krein spaces

2.1. First we introduce the main objects of our considerations and some notation. We denote by $\mathbf{C}^+$ and $\mathbf{C}^-$ the open upper and the open lower half plane, respectively. $\mathbf{D}$ denotes the open unit disc and $\mathbf{T}$ denotes the unit circle. For every subset M of $\overline{\mathbf{C}}$ we set $M^* := \{\bar{\lambda} : \lambda \in M\}$ and $\hat{M} := \{\bar{\lambda}^{-1} : \lambda \in M\}$. For a function f defined on a set $M \subset \overline{\mathbf{C}}$ with $M = M^*$ ($M = \hat{M}$) we set $f^*(\lambda) := \overline{f(\bar{\lambda})}$ (resp. $\hat{f}(\lambda) := \overline{f(\bar{\lambda}^{-1})}$).

Let in this and the following sections Ω be a domain in $\overline{\mathbf{C}}$ which is symmetric with respect to $\mathbf{R}$, $\Omega = \Omega^*$, such that $\Omega \cap \overline{\mathbf{R}} \neq \emptyset$, and $\Omega \cap \mathbf{C}^+$ and $\Omega \cap \mathbf{C}^-$ are simply connected.

Let A be a selfadjoint linear relation in the Krein space $(\mathcal{H}, [\cdot,\cdot])$ such that $\sigma(A) \cap (\Omega \setminus \overline{\mathbf{R}})$ consists of isolated points which are poles of the resolvent of A, and no point of $\Omega \cap \overline{\mathbf{R}}$ is an accumulation point of the nonreal spectrum $\sigma(A) \setminus \mathbf{R}$ of A.

Let $\lambda_0 \in \Omega \cap \rho(A) \cap \mathbf{C}^+$. Then the function ψ, $\psi(\lambda) := -(\lambda - \lambda_0)(\lambda - \overline{\lambda}_0)^{-1}$ is locally holomorphic on $\widetilde{\sigma}(A)$, and the operator

$$U := \psi(A) = -1 + (\lambda_0 - \overline{\lambda}_0)(A - \overline{\lambda}_0)^{-1}$$

is a unitary operator in $(\mathcal{H}, [\cdot,\cdot])$. Evidently, the domain $\psi(\Omega)$ is symmetric with respect to $\mathbf{T}$, $\psi(\Omega) \cap \mathbf{T} \neq \emptyset$, and $\psi(\Omega) \cap \mathbf{D}$ and $\psi(\Omega) \cap \hat{\mathbf{D}}$ are simply connected domains of $\overline{\mathbf{C}}$. By the spectral mapping theorem for closed linear relations (see e.g. [3, Section 3]) $\sigma(U) \cap (\psi(\Omega) \setminus \mathbf{T})$ consists of isolated points which are poles of the resolvent of U, and no point of $\psi(\Omega) \cap \mathbf{T}$ is an accumulation point of $\sigma(U) \setminus \mathbf{T}$.

Evidently, a function g is locally holomorphic on $\sigma(U)$ if and only if $g \circ \psi$ is locally holomorphic on $\widetilde{\sigma}(A)$. The Riesz-Dunford-Taylor functional calculi for A (see e.g. [3, Section 3]) and U are connected by

$$(g \circ \psi)(A) = g(U). \tag{2.1}$$

Let Δ be an open subset of $\Omega \cap \overline{\mathbf{R}}$. We shall say that A belongs to the class $S^m(\Delta)$, $m \geq 1$, if for every closed subset Δ' of Δ there exist a constant M and an open neighbourhood $\mathcal{U}$ of Δ' in $\overline{\mathbf{C}}$ such that

$$\|(A - \lambda)^{-1}\| \leq M(|\lambda| + 1)^{2m-2} |\mathrm{Im}\,\lambda|^{-m} \tag{2.2}$$

for all $\lambda \in \mathcal{U} \setminus \overline{\mathbf{R}}$.

We shall say that A belongs to the class $S^\infty(\Delta)$ (cf. [8, Section 1.2]), if for every closed subset Δ' of Δ there exist $m \geq 1$, $M > 0$ and an open neighbourhood $\mathcal{U}$ of Δ' in $\overline{\mathbf{C}}$ such that (2.2) holds for all $\lambda \in \mathcal{U} \setminus \overline{\mathbf{R}}$.

Let Γ be an open subset of $\psi(\Omega) \cap \mathbf{T}$. We shall say that U belongs to the class $S^m(\Gamma)$ (cf. [7, Section 1.2]), if for every closed subset Γ' of Γ there exist a constant M and an $r_0 \in (0,1)$ such that

$$\|(U - re^{i\Theta})^{-1}\| \leq M|1 - |r||^{-m} \tag{2.3}$$

for all $e^{i\Theta} \in \Gamma'$ and $r \in [r_0, 1) \cup (1, r_0^{-1}]$. We shall say that U belongs to the class $S^\infty(\Gamma)$, if for every closed subset Γ' of Γ there exist $m \geq 1$, $M > 0$ and $r_0 \in (0,1)$ such that (2.3) holds for all $e^{i\Theta} \in \Gamma'$ and $r \in [r_0, 1) \cup (1, r_0^{-1}]$.

With the help of the relation

$$2i(\operatorname{Im}\lambda_0)\psi(\lambda)(U - \psi(\lambda))^{-1} = \lambda - \lambda_0 + (\lambda - \lambda_0)(\lambda - \bar{\lambda}_0)(A - \lambda)^{-1} \tag{2.4}$$

and the fact that, for all λ outside of a neighbourhood of $\{\lambda_0, \bar{\lambda}_0\}$, we have

$$m'|\operatorname{Im}\lambda|(1 + |\lambda|)^{-2} \leq ||\psi(\lambda)| - 1| \leq M'|\operatorname{Im}\lambda|(1 + |\lambda|)^{-2} \tag{2.5}$$

with some positive constants m', M', one easily verifies that

$$A \in S^m(\Delta) \Longleftrightarrow U \in S^m(\psi(\Delta)),\ m = 1, 2, \ldots, \infty. \tag{2.6}$$

If A and U fulfil the conditions (2.2) and (2.3), respectively, the Riesz-Dunford-Taylor functional calculi for A and U can be extended by continuity to some classes of functions which are not locally holomorphic on the spectrum. We recall this fact in the following subsection within a more general setting: we replace the resolvent by a holomorphic operator function.

2.2. *Extensions of the functional calculi of A and U, and extensions of some analytic functionals connected with operator functions.* Let G be an $\mathcal{L}(\mathcal{H})$-valued meromorphic function in $\Omega \setminus \overline{\mathbf{R}}$ which is symmetric with respect to the real line, that is

$$G(\bar{\lambda}) = G(\lambda)^+ \tag{2.7}$$

for all points λ of holomorphy of G, such that no point of $\Omega \cap \overline{\mathbf{R}}$ is an accumulation point of nonreal poles of G. Here $G(\lambda)^+$ denotes the Krein space adjoint of $G(\lambda)$. Assume that $\lambda_0 \in \Omega \cap \mathbf{C}^+$ is a point of holomorphy of G.

In the following, if M is a closed subset of $\overline{\mathbf{C}}$, the linear space of all locally holomorphic functions on M will be denoted by $H(M)$.

Let $\mathcal{O}^+$ be a bounded C^∞ domain (not necessarily simply connected) with $\overline{\mathcal{O}^+} \subset \Omega \cap \mathbf{C}^+$ and $\lambda_0 \in \mathcal{O}^+$ such that $\overline{\mathcal{O}^+}$ is contained in the domain of holomorphy of G. Then by (2.7) G is also locally holomorphic on $\overline{\mathcal{O}^-}$, $\mathcal{O}^- := (\mathcal{O}^+)^*$. For every $g \in H(\overline{\mathbf{C}} \setminus (\mathcal{O}^+ \cup \mathcal{O}^-))$ we define

$$(2.8) \qquad S_G.g := 2i\,(\operatorname{Im}\lambda_0) \int_{\mathcal{C}} G(\lambda)g(\lambda)(\lambda-\lambda_0)^{-1}(\lambda-\overline{\lambda_0})^{-1}\,d\lambda,$$

where $\mathcal{C} = \partial\mathcal{O}^+ \cup \partial\mathcal{O}^-$. Evidently, for every function g locally holomorphic on

$$(\overline{\mathbf{C}} \setminus \Omega) \cup \overline{\mathbf{R}} \cup \{\text{poles of } G\}$$

we may find some domain $\mathcal{O}^+$ as above and such that $g \in H(\overline{\mathbf{C}} \setminus (\mathcal{O}^+ \cup \mathcal{O}^-))$. Then the operator $S_G.g$ is defined, and it does not depend on the choice of $\mathcal{O}^+$. It is easy to see that $S_G.g^* = (S_G.g)^+$.

Let F be the $\mathcal{L}(\mathcal{H})$-valued meromorphic function in $\psi(\Omega \setminus \overline{\mathbf{R}}) = \psi(\Omega) \setminus \mathbf{T}$, $\psi(\lambda) :== -(\lambda-\lambda_0)(\lambda-\bar{\lambda}_0)^{-1}$, defined by

$$(2.9) \qquad F(\psi(\lambda)) = -iG(\lambda).$$

The function F is skew symmetric with respect to the unit circle $\mathbf{T}$:

$$F(\bar{z}^{-1}) = -F(z)^+$$

for all points z of holomorphy of F. No point of $\psi(\Omega) \cap \mathbf{T}$ is an accumulation point of non-unimodular poles of F. Moreover, F is holomorphic at 0 and ∞. If $\mathcal{O}^+$ is as above, then for every $f \in H(\overline{\mathbf{C}} \setminus \psi(\mathcal{O}^+ \cup \mathcal{O}^-))$ we define

$$T_F.f := \int_{\psi(\mathcal{C})} F(z)f(z)(iz)^{-1}dz,$$

where $\psi(\mathcal{C}) = \partial\psi(\mathcal{O}^+) \cup \partial\psi(\mathcal{O}^-)$. We have $T_F.\hat{f} = (T_F.f)^+$.

Similarly to the functional S_G, the functional T_F is defined for every function f locally holomorphic on

$$(2.10) \qquad (\overline{\mathbf{C}} \setminus \psi(\Omega)) \cup \mathbf{T} \cup \{\text{poles of } F\}.$$

A simple calculation shows that

$$(2.11) \qquad T_F.f = S_G.(f \circ \psi)$$

for every function f locally holomorphic on (2.10).

If $\lambda \in \Omega \setminus \overline{\mathbf{R}}$ and $\zeta \in \psi(\Omega) \setminus \mathbf{T}$ are points of holomorphy of G and F, respectively, and if we set

$$g_\lambda(w) := (4\pi)^{-1}(\operatorname{Im}\lambda_0)^{-1}\left(\lambda - \operatorname{Re}\lambda_0 + (\lambda-\lambda_0)(\lambda-\bar{\lambda}_0)(w-\lambda)^{-1}\right),$$
$$(2.12) \qquad h_\zeta(z) := (4\pi)^{-1}(z+\zeta)(z-\zeta)^{-1},$$

then
(2.13)
$$G(\lambda) = S_G.g_\lambda + \tfrac{1}{2}(G(\lambda_0) + G(\lambda_0)^+), \quad F(\zeta) = T_F.h_\zeta + \tfrac{1}{2}(F(0) - F(0)^+).$$

If U and A are as in Section 2.1, we define

$$F_U(z) := (4\pi)^{-1}(U+z)(U-z)^{-1}, \qquad z \in \rho(U), \tag{2.14}$$

$$\begin{aligned} G_A(\lambda) &:= \\ &= (4\pi)^{-1}(\operatorname{Im}\lambda_0)^{-1}\{(\lambda - \operatorname{Re}\lambda_0) + (\lambda-\lambda_0)(\lambda-\overline{\lambda_0})(A-\lambda)^{-1}\}, \ \lambda \in \rho(A), \end{aligned} \tag{2.15}$$

then we have

$$F_U(\psi(\lambda)) = -iG_A(\lambda), \ \lambda \in \rho(A), \tag{2.16}$$

and a little calculation shows that $f \mapsto T_{F_U}.f$ and $g \mapsto S_{G_A}.g$ are the Riesz-Dunford-Taylor functional calculi for U and A, respectively, that is, $T_{F_U}.f = f(U)$, $S_{G_A}.g = g(A)$. In this case the relation (2.11) follows from the functional calculus of A: $f(U) = f(\psi(A)) = (f \circ \psi)(A)$.

Now let again G be as at the beginning of Section 2.2 and F as in (2.9). Let Δ be an open subset of $\Omega \cap \overline{\mathbf{R}}$, and let $m \geq 1$. We shall say that the *order of growth of* G *near* Δ *is* $\leq m$, if for every closed subset Δ' of Δ there exists a constant M and an open neighbourhood $\mathcal{U}$ of Δ' in $\overline{\mathbf{C}}$ such that

$$\|G(\lambda)\| \leq M(1+|\lambda|)^{2m}|\operatorname{Im}\lambda|^{-m}$$

for all $\lambda \in \mathcal{U} \setminus \overline{\mathbf{R}}$. We do not exclude the case when $\Omega = \overline{\mathbf{C}}$ and $\Delta = \overline{\mathbf{R}}$.

Analogously, if $\Gamma = \psi(\Delta)$ we shall say that the *order of growth of* F *near* Γ *is* $\leq m$, if for every closed subset Γ' of Γ there exists a constant M and an $r_0 \in (0,1)$ such that

$$\|F(re^{i\Theta})\| \leq M|1-|r||^{-m}$$

for all $e^{i\Theta} \in \Gamma'$ and $r \in [r_0, 1) \cup (1, r_0^{-1}]$. It is easy to verify that the order of growth of G near Δ is $\leq m$ if and only if the order of growth of F near Γ is $\leq m$.

We have, on account of (2.6) and (2.16),

$$\begin{aligned} A \in S^m(\Delta) &\Longleftrightarrow \text{the order of growth of } G_A \text{ near } \Delta \text{ is} \leq m \Longleftrightarrow \\ U \in S^m(\psi(\Delta)) &\Longleftrightarrow \text{the order of growth of } F_U \text{ near } \psi(\Delta) \text{ is} \leq m\ . \end{aligned} \tag{2.17}$$

With the help of the topology which we introduce now the functional T_F will be extended in Theorem 2.1 below. Let Γ_0 be the union of a finite number of pairwise disjoint open arcs of $\mathbf{T}$, $\Gamma_0 \neq \mathbf{T}$, and let $\delta_0 \in (0,1)$ be such that for

$$Q_0 := \{re^{i\Theta} : e^{i\Theta} \in \Gamma_0,\, r \in (\delta_0, 1) \cup (1, \delta_0^{-1})\} \tag{2.18}$$

the function F is locally holomorphic on $\overline{Q_0} \setminus \overline{\Gamma_0}$.

We denote by $D^{(p)}(\overline{\mathbf{C}} \setminus Q_0)$, p nonnegative integer, the linear space of all continuous complex functions f on $\overline{\mathbf{C}} \setminus Q_0$ such that f is locally holomorphic on $\overline{\mathbf{C}} \setminus (Q_0 \cup \Gamma_0)$ and the restriction $f|\mathbf{T}$ is a C^p function. We introduce a locally convex topology on $D^{(p)}(\overline{\mathbf{C}} \setminus Q_0)$: Let ϵ_0, $0 < \epsilon_0 < 1 - \delta_0$, be such that for $0 < \epsilon < \epsilon_0$ every component of Γ_0 contains a point of

$$\Gamma_\epsilon := \{e^{i\Theta} \in \Gamma_0 : \operatorname{dist}(e^{i\Theta}, \mathbf{T} \setminus \Gamma_0) > \epsilon\} \subset \Gamma_0,$$

and set

$$Q_\epsilon := \{re^{i\Theta} : e^{i\Theta} \in \Gamma_\epsilon,\, r \in (\delta_0 + \epsilon, 1) \cup (1, (\delta_0 + \epsilon)^{-1})\}.$$

Let $(\epsilon_n) \subset (0, \epsilon_0)$ be a decreasing null sequence and let $D_n^{(p)}$ be the subspace of $D^{(p)}(\overline{\mathbf{C}} \setminus Q_0)$ of all $f \in D^{(p)}(\overline{\mathbf{C}} \setminus Q_0)$ which can analytically be continued to $\overline{\mathbf{C}} \setminus \overline{(Q_{\epsilon_n} \cup \Gamma_{\epsilon_n})}$ such that f is continuous on $\overline{\mathbf{C}} \setminus (Q_{\epsilon_n} \cup \Gamma_{\epsilon_n})$. We have $D^{(p)}(\overline{\mathbf{C}} \setminus Q_0) = \bigcup_{n=1}^{\infty} D_n^{(p)}$. On the space $D_n^{(p)}$ we consider the norm

$$\begin{aligned} \|f\|_n^{(p)} &:= \sup \left\{ |f(z)| : z \in \overline{\mathbf{C}} \setminus \overline{(Q_{\epsilon_n} \cup \Gamma_{\epsilon_n})} \right\} + \\ &+ \sup \left\{ \left| \tfrac{d^\nu}{d\Theta^\nu} f(e^{i\Theta}) \right| : e^{i\Theta} \in \Gamma_0,\, 0 \le \nu \le p \right\},\ f \in D_n^{(p)}. \end{aligned}$$

$(D_n^{(p)}, \|f\|_n^{(p)})$ is a Banach space. On the space $D^{(p)}(\overline{\mathbf{C}} \setminus Q_0)$ we consider the topology of the inductive limit of the spaces $D_n^{(p)}$, $n = 1, 2, \ldots$. One verifies as in [16, §27,4.(2)] that this topology is separated. By well-known properties of the Abel-Poisson integral, $H(\overline{\mathbf{C}} \setminus Q_0)$ is dense in $D^{(p)}(\overline{\mathbf{C}} \setminus Q_0)$.

The following theorem shows that under the growth condition on F considered above, the functional T_F can be extended by continuity to a larger class of functions (see also [11, Proposition 1.1]).

Theorem 2.1. *Let F be as in* (2.9) *and let Γ be an open subset of $\psi(\Omega) \cap \mathbf{T}$. Assume that the order of growth of F near Γ is $\le m$, m some positive integer. Let Γ_0 be the union of a finite number of open arcs of $\mathbf{T}$ such that $\overline{\Gamma}_0 \subset \Gamma$, $\Gamma_0 \neq \mathbf{T}$, and let Q_0 be as in* (2.18).

Then the functional T_F is a continuous linear mapping of $H(\overline{\mathbf{C}} \setminus Q_0)$ equipped with the topology of $D^{(m+1)}(\overline{\mathbf{C}} \setminus Q_0)$ into $\mathcal{L}(\mathcal{H})$. Therefore, T_F can *be extended by continuity to* $D^{(m+1)}(\overline{\mathbf{C}} \setminus Q_0)$*. Moreover, if the support of* $f \in D^{(m+1)}(\overline{\mathbf{C}} \setminus Q_0)$ *is in* Γ_0*, we have*

$$T_F.f = \lim_{r \uparrow 1} \int_{S_0} f(e^{i\Theta})\{F(re^{i\Theta}) - F(r^{-1}e^{i\Theta})\}\, d\Theta, \tag{2.19}$$

where $S_0 := \{\Theta \in [0, 2\pi) : e^{i\Theta} \in \Gamma_0\}$*.*

Proof. Let H be a locally holomorphic function in Q_0 such that $H^{[m+1]}(z) = F(z)$, where

$$H^{[0]}(z) := H(z), \quad H^{[j]}(z) := iz(dH^{[j-1]}/dz)(z), \quad j = 1, 2, \ldots. \tag{2.20}$$

For every component $Q_{0,k}$ of Q_0 we have

$$H^{[j-1]}(z) = H_{kj} + \int_{z_k}^{z} H^{[j]}(\zeta)(i\zeta)^{-1} d\zeta, \quad j = 1, 2, \ldots, m+1, \quad z \in Q_{0,k},$$

where $z_k \in Q_{0,k}$ and $H_{kj} \in \mathcal{L}(\mathcal{H})$. By these relations and the assumption on F, H has continuous boundary values on Γ_0.

Let $f \in D^{(m+1)}(\overline{\mathbf{C}} \setminus Q_0)$. There exists an n_0 such that $f \in D_n^{(m+1)}$ for all $n \geq n_0$. We set $\mathcal{C}_n^+ := \partial(Q_{\epsilon_n} \cap \mathbf{D})$, $\mathcal{C}_n^- := \partial(Q_{\epsilon_n} \cap \hat{\mathbf{D}})$, $n \geq n_0 + 1$, and define

$$\tilde{T}(f) := \int_{\mathcal{C}_n^+ \cup \mathcal{C}_n^-} (-1)^{m+1} H(z) f^{[m+1]}(z)(iz)^{-1} dz. \tag{2.21}$$

The right hand side of (2.21) does not depend on $n \geq n_0 + 1$. Evidently, the restriction of $\tilde{T}$ to $D_{n_0}^{(m+1)}$ is continuous with respect to $\|\cdot\|_{n_0}^{(m+1)}$. Therefore, $\tilde{T}$ is continuous with respect to the topology of inductive limit of $D^{(m+1)}(\overline{\mathbf{C}} \setminus Q_0)$.

If $f \in H(\overline{\mathbf{C}} \setminus Q_0)$, integration by parts applied to the right hand side of (2.21) shows that $\tilde{T}(f) = T_F.f$, and the first assertion of Theorem 2.1 is proved.

If $f \in D_{n_0}^{(m+1)}$ and $\operatorname{supp} f \subset \Gamma_{\epsilon_{n_0}}$, then (2.21) gives

$$\tilde{T}(f) = (-1)^{m+1} \lim_{r \uparrow 1} \int_{\Gamma_{\epsilon_{n_0}}} (H(rz) - H(r^{-1}z)) f^{[m+1]}(z)(iz)^{-1} dz.$$

By integration by parts we obtain

$$\tilde{T}(f) = \lim_{r \uparrow 1} \int_{\Gamma_{\epsilon_{n_0}}} (F(rz) - F(r^{-1}z)) f(z)(iz)^{-1} dz.$$

which proves (2.19).

Remark 2.2. The topology with respect to which T_F is extended in Theorem 2.1 is finer than the topology considered in [11, Section 1.1]. The extension of T_F with respect to the latter topology contains the extension considered here.

The following theorem, which is a variant of Theorem 2.1 for the case $\Omega = \psi(\Omega) = \overline{\mathbf{C}}$, $\Gamma = \mathbf{T}$, is well known ([15]). For completeness we shall give a proof here.

Theorem 2.3. *Let F be a meromorphic function in $\overline{\mathbf{C}} \setminus \mathbf{T}$, $F = \hat{F}$. Assume that the set P of all poles of F in $\overline{\mathbf{C}} \setminus \mathbf{T}$ is finite, $0, \infty \notin P$, and that the order of growth of F near $\mathbf{T}$ is $\leq m$, m some positive integer.*

Then $T_F \in \mathcal{L}(H(\mathbf{T} \cup P), \mathcal{L}(\mathcal{H}))$ is continuous with respect to the topology of $C^{m+1}(\mathbf{T}) \times H(P)$. Therefore, T_F can be extended by continuity to $C^{m+1}(\mathbf{T}) \times H(P)$. Moreover, if $f \in C^{m+1}(\mathbf{T}) \times \{0\}$ then

$$T_F.f = \lim_{r \uparrow 1} \int_0^{2\pi} (F(re^{i\theta}) - F(r^{-1}e^{i\theta}))f(e^{i\theta})d\theta.$$

Proof. Let $\chi_0 \in H(\mathbf{T} \cup P)$ be equal to one (zero) in some neighbourhood of P (resp. $\mathbf{T}$). For $\zeta \in \overline{\mathbf{C}} \setminus (\mathbf{T} \cup P)$ we define with h_ζ as in (2.12)

$$\begin{aligned} F_0(\zeta) &:= T_F.\chi_0 h_\zeta + \tfrac{1}{2}(F(0) - F(0)^+), \\ F_{(0)}(\zeta) &:= T_F.(1-\chi_0)h_\zeta, \quad \widetilde{F}(\zeta) := F_{(0)}(\zeta) - F_{(0)}(\infty). \end{aligned}$$

It is easy to see that F_0 is locally holomorphic on $\overline{\mathbf{C}} \setminus P$ and that $F_{(0)}$ and $\widetilde{F}$ are locally holomorphic on $\overline{\mathbf{C}} \setminus \mathbf{T}$. By (2.13) we have

$$F(\zeta) = F_0(\zeta) + F_{(0)}(\zeta) = F_0(\zeta) + F_{(0)}(\infty) + \widetilde{F}(\zeta). \tag{2.22}$$

Define

$$\widetilde{F}^{[-1]}(\zeta) := \begin{cases} \int_0^\zeta \widetilde{F}(z)(iz)^{-1}dz & \text{if} \quad |\zeta| < 1 \\ \int_\infty^\zeta \widetilde{F}(z)(iz)^{-1}dz & \text{if} \quad |\zeta| > 1. \end{cases}$$

Repeating this construction m times we obtain a function $H := \widetilde{F}^{[-m-1]}$ locally holomorphic on $\overline{\mathbf{C}} \setminus \mathbf{T}$ such that $H^{[m+1]}(\zeta) = \widetilde{F}(\zeta)$. Since the order of growth of F near $\mathbf{T}$ is $\leq m$, H has continuous boundary values on $\mathbf{T}$:

$$H_i(e^{i\theta}) := \lim_{r \uparrow 1} H_i(re^{i\theta}), \quad H_a(e^{i\theta}) := \lim_{r \uparrow 1} H_i(r^{-1}e^{i\theta}).$$

Let $f \in H(\mathbf{T} \cup P)$ and assume that f is zero on some neighbourhood of P. If f is holomorphic on the closure of $\mathcal{A}_r := \{z : \ r < |z| < r^{-1}\}$ we have

$$T_F.f = -\int_{\partial \mathcal{A}_r} F(z) f(z)(iz)^{-1} dz$$

and, therefore,

$$\begin{aligned} T_F.f &= -\int_{\partial \mathcal{A}_r} \widetilde{F}(z) f(z)(iz)^{-1} dz = \\ &= -\int_{\partial \mathcal{A}_r} H^{[m+1]}(z) f(z)(iz)^{-1} dz = \\ &= -(-1)^{m+1} \int_{\partial \mathcal{A}_r} H(z) f^{[m+1]}(z)(iz)^{-1} dz = \\ &= (-1)^{m+1} \int_{\mathbf{T}} (H_i(z) - H_a(z)) \, f^{[m+1]}(z)(iz)^{-1} dz. \end{aligned} \tag{2.23}$$

This implies the continuity statement of Theorem 2.3. Moreover, if $f \in C^{m+1}(\mathbf{T}) \times \{0\}$ then, by (2.23),

$$\begin{aligned} T_F.f &= (-1)^{m+1} \lim_{r \uparrow 1} \int_{\mathbf{T}} (H(rz) - H(r^{-1}z)) f^{[m+1]}(z)(iz)^{-1} dz = \\ &= \lim_{r \uparrow 1} \int_{\mathbf{T}} (\widetilde{F}(rz) - \widetilde{F}(r^{-1}z)) f(z)(iz)^{-1} dz = \\ &= \lim_{r \uparrow 1} \int_0^{2\pi} (F(re^{i\theta}) - F(r^{-1}e^{i\theta})) f(e^{i\theta}) d\theta, \end{aligned}$$

which proves Theorem 2.3.

In the following, when we consider a functional T_F as in Theorem 2.1 and Γ and m are as in that theorem, the linear space of all functions f defined on $\mathbf{T} \cup \mathcal{U}_f$, where $\mathcal{U}_f$ is some neighbourhood of $(\overline{\mathbf{C}} \setminus \psi(\Omega)) \cup \{\text{poles of } F \text{ in } \psi(\Omega) \setminus \mathbf{T}\} \cup (\mathbf{T} \setminus \Gamma)$, such that $f|\mathcal{U}_f = 0$ and $f|\Gamma$ belongs to $C_0^k(\Gamma)$, $k \geq m+1$, will be denoted by $C_0^k(\Gamma)$, for the simplicity of notation.

The following lemma, which will be used below, is a consequence of Theorem 2.1.

Lemma 2.4. *Let F be as in* (2.9) *and let Γ_1 and Γ be open subsets of $\psi(\Omega) \cap \mathbf{T}$ such that $\overline{\Gamma_1} \subset \Gamma$. Assume that the order of growth of F near Γ is $\leq m$.*

Let $\chi \in C_0^\infty(\Gamma)$ be equal to one on a neighbourhood of $\overline{\Gamma}_1$ and let

$$\begin{aligned} F_1(z) &:= T_F.\chi h_z + \tfrac{1}{2}(F(0) - F(0)^+), \\ F_2(z) &:= T_F.(1-\chi)h_z \end{aligned}$$

(see (2.12)). Then F_1 and F_2 are $\mathcal{L}(\mathcal{H})$-valued meromorphic functions in $\psi(\Omega) \setminus \mathbf{T}$ with

$$F_j(\overline{z}^{-1}) = -F_j(z)^+, \qquad z \in \psi(\Omega) \setminus \mathbf{T}, \qquad j = 1,2,$$

and $F = F_1 + F_2$. F_1 is locally holomorphic on $\overline{\mathbf{C}} \setminus \Gamma$, F_2 is locally holomorphic on $\overline{\Gamma}_1$ and the order of growth of F_1 near $\mathbf{T}$ is $\leq m+2$.

Proof. By the continuity properties of T_F proved in Theorem 2.1 F_1 is complex differentiable in some neighbourhood of any point of $\mathbf{C} \setminus \operatorname{supp} \chi$. Similarly, F_2 is locally holomorphic on $\overline{\Gamma}_1$. Therefore, F_1 and F_2 are meromorphic functions in $\psi(\Omega) \setminus \mathbf{T}$. $F = F_1 + F_2$ follows from the second relation of (2.13). By Theorem 2.1, (2.19), we have

$$\begin{aligned} F_1(z) = (4\pi)^{-1} \lim_{r \uparrow 1} \int_0^{2\pi} \chi(e^{i\Theta}) \frac{e^{i\Theta} + z}{e^{i\Theta} - z} \{F(re^{i\Theta}) - F(r^{-1}e^{i\Theta})\} d\Theta \\ + \tfrac{1}{2}(F(0) - F(0)^+). \end{aligned}$$

This implies that F_1 is skew-symmetric with respect to $\mathbf{T}$. Then the same is true for F_2.

Let K be a compact subset of $\mathbf{C} \setminus \{0\}$. Then by the definition of F_1 and the local C^{m+1}-continuity of T_F there exist constants M and M' such that $z \in K \setminus \mathbf{T}$ implies

$$\begin{aligned} \|F_1(z)\| &\leq M \sup \{|\frac{d^k}{d\Theta^k} h_z(e^{i\Theta})| : \Theta \in [0, 2\pi], \ k = 0, \ldots, m+1\} \\ &\leq M' |1 - |z||^{m+2}. \end{aligned}$$

That is, the order of growth of F_1 near $\mathbf{T}$ is $\leq m+2$.

If we set $F(z) = F_U(z)$, Theorems 2.1 and 2.3 imply the following

Corollary 2.5. *Let Γ be an open subset of $\psi(\Omega) \cap \mathbf{T}$, and assume that $U \in S^m(\Gamma)$, m some positive integer.*

Let Γ_0, $\Gamma_0 \neq \mathbf{T}$, be the union of a finite number of open arcs of $\mathbf{T}$ such that $\overline{\Gamma}_0 \subset \Gamma$, and let Q_0 and S_0 be as in (2.18) and (2.19) with $\overline{Q}_0 \setminus \overline{\Gamma}_0 \subset \rho(U)$. Then the Riesz-Dunford functional calculus of U can be *extended by continuity to* $D^{(m+1)}(\overline{\mathbf{C}}\backslash Q_0)$*. If the support of* $f \in D^{(m+1)}(\overline{\mathbf{C}}\backslash Q_0)$ *is in* Γ_0*, we have*

$$[f(U)x,y] = \lim_{r\uparrow 1}\int_{S_0} f(e^{i\Theta})[\{F_U(re^{i\Theta}) - F_U(r^{-1}e^{i\Theta})\}x,y]\,d\Theta =$$

$$= (2\pi)^{-1}\lim_{r\uparrow 1}\int_{S_0} f(e^{i\Theta})[U\{U - re^{i\Theta})^{-1} - (U - r^{-1}e^{i\Theta})^{-1}\}x,y]\,d\Theta. \tag{2.24}$$

If $\Omega = \overline{\mathbf{C}}$ *and* $U \in S^m(\mathbf{T})$*, then the Riesz-Dunford functional calculus of* U *is continuous with respect to the topology of* $C^{m+1}(\mathbf{T}) \times H(\sigma(U)\setminus\mathbf{T})$ *and can, therefore, be extended to this space. For* $f \in C^{m+1}(\mathbf{T}) \times \{0\}$ (2.24) *holds with* $S_0 = [0, 2\pi)$*.*

Remark 2.6. Compared with the functional calculus of [7] the domain of this functional calculus is smaller. On the other hand it is sufficient for the characterization of sign types.

If the order of growth of G near Δ, Δ open subset of $\Omega \cap \overline{\mathbf{R}}$, is $\leq m$, we define the extension of the functional S_G by $S_G.(f \circ \psi) := T_F.f$ (see (2.11)), where f belongs to the extended domain of T_F. If we regard $\overline{\mathbf{R}}$ as a real-analytic manifold in the usual way, then the restriction of ψ to $\overline{\mathbf{R}}$ is a real-analytic diffeomorphism of $\overline{\mathbf{R}}$ onto $\mathbf{T}$, and therefore, $f \circ \psi$ is C^m, $m = 0, 1, \ldots, \infty$, on the open subset Δ of $\Omega \cap \overline{\mathbf{R}}$ if and only if f is C^m on $\psi(\Delta)$. In connection with the functional S_G we will use the notation $C_0^m(\Delta)$, $\Delta \subset \Omega \cap \overline{\mathbf{R}}$, in a way analogous to the notation $C_0^m(\Gamma)$ introduced above.

2.3. *Open sets of positive and negative type with respect to operator functions.* Let, as at the beginning of Section 2.2, G be an $\mathcal{L}(\mathcal{H})$-valued meromorphic function in $\Omega \setminus \overline{\mathbf{R}}$ with $G(\bar{\lambda}) = G(\lambda)^+$ such that no point of $\Omega \cap \overline{\mathbf{R}}$ is an accumulation point of nonreal poles of G and G is holomorphic at λ_0. Now we take over a definition from [14, Section 3.1] to our slightly more general situation. Observe that if $\lambda \in \Omega \cap \mathbf{C}^+$ is a point of holomorphy of G and if $x \in \mathcal{H}$, we have

$$-i\,[(G(\lambda) - G(\bar{\lambda}))x, x] = 2\,\mathrm{Im}\,[G(\lambda)x, x].$$

Definition 2.7. An open subset $\Delta \subset \Omega \cap \overline{\mathbf{R}}$ is said to be of *positive type with respect to* G, if the following conditions are fulfilled for every $x \in \mathcal{H}$.

(i) $\liminf_{\epsilon\downarrow 0}\{-i[(G(t+i\epsilon) - G(t-i\epsilon))x,x]\} \geq 0$ for almost every $t \in \Delta \setminus \{\infty\}$.

(ii) For every bounded closed subset Δ_0 of Δ and sufficiently small $\epsilon_0 > 0$,

$$\inf\{-i[(G(t+i\epsilon) - G(t-i\epsilon))x, x] : \ t \in \Delta_0, \ 0 < \epsilon \leq \epsilon_0\} > -\infty.$$

If $\infty \in \Delta$, then, in addition, for sufficiently small $\delta_0 > 0$, $\epsilon_0 > 0$,

$$\inf\{-i[(G(-(t+i\epsilon)^{-1}) - G(-(t-i\epsilon)^{-1})x, x] : \ -\delta_0 \leq t \leq \delta_0, \ 0 < \epsilon \leq \epsilon_0\} > -\infty.$$

An open subset $\Delta \subset \Omega \cap \overline{\mathbf{R}}$ is said to be of *negative type with respect to* G if Δ is of positive type with respect to $-G$. Δ is said to be of *definite type with respect to* G if Δ is of positive type or of negative type with respect to G.

Lemma 2.8. *If G is as above and Δ is an open subset of $\Omega \cap \overline{\mathbf{R}}$ the following assertions are equivalent.*

(a) *Δ is of positive type with respect to G.*

(b) (a) *holds, and, for every $x \in \mathcal{H}$, the angular limit $\widehat{\lim}_{\operatorname{Im}\lambda>0,\, \lambda\to t}\,[G(\lambda)x, x]$ exists and belongs to $\mathbf{C}^+ \cup \overline{\mathbf{R}}$ for almost every $t \in \Delta$.*

(c) *If $x \in \mathcal{H}$ and if (λ_n) is a sequence of points of holomorphy of G in $\Omega \cap \mathbf{C}^+$ which converges in $\overline{\mathbf{C}}$ to a point of Δ, then*

$$\liminf_{n\to\infty} \{-i[(G(\lambda_n) - G(\overline{\lambda_n}))x, x]\} \geq 0.$$

(d) *If $x \in \mathcal{H}$, Δ_1 is an open subset of Δ with $\overline{\Delta_1} \subset \Delta$, and if $\alpha > 0$, there exists an open set $\mathcal{O}$ in $\overline{\mathbf{C}}$, $\Delta_1 \subset \mathcal{O}$, such that*

$$\inf\{-i[(G(\lambda) - G(\overline{\lambda}))x, x] : \ \lambda \in \mathcal{O} \setminus \overline{\mathbf{R}}\} > -\alpha.$$

If (a) - (d) *are true, then for every open subset Δ_1 of Δ with $\overline{\Delta_1} \subset \Delta$, there exists an open set $\mathcal{O}$ in $\overline{\mathbf{C}}$, $\Delta_1 \subset \mathcal{O}$, such that*

$$\sup\{\|G(\lambda)\||\operatorname{Im}\lambda|(1+|\lambda|)^{-1} : \ \lambda \in \mathcal{O}, \ \operatorname{Im}\lambda \neq 0\} < \infty.$$

We shall prove this lemma with the help of a similar lemma for $\mathcal{L}(\mathcal{H})$-valued meromorphic functions F in $\psi(\Omega) \setminus \mathbf{T}$ with $F(\bar{z}^{-1}) = -F(z)^+$ such that no point of $\psi(\Omega) \cap \mathbf{T}$ is an accumulation point of poles of F in

$\psi(\Omega)\setminus\mathbf{T}$ and F is holomorphic at 0. For these functions, which are connected by (2.9) with those considered in Definition 2.7, we introduce similar notions (cf. [12, Lemma 1.7]). Observe that if $z\in\psi(\Omega)\cap\mathbf{D}$ is a point of holomorphy of F and $x\in\mathcal{H}$, we have

$$[(F(z)-F(\bar{z}^{-1}))x,x]=2\,\mathrm{Re}\,[F(z)x,x].$$

Definition 2.7′. An open set $\Gamma\subset\psi(\Omega)\cap\mathbf{T}$ is said to be of *positive type with respect to* F if the following conditions are fulfilled for every $x\in\mathcal{H}$.

(i′) $\liminf_{r\uparrow 1}[(F(re^{i\Theta})-F(r^{-1}e^{i\Theta}))x,x]\geq 0$ for almost every $e^{i\Theta}\in\Gamma$.

(ii′) $\inf\{[(F(re^{i\Theta})-F(r^{-1}e^{i\Theta}))x,x]:\,e^{i\Theta}\in\gamma,\,r\in(1-\delta,1)\}>-\infty$ for every closed subarc γ of Γ and sufficiently small $\delta>0$.

An open set $\Gamma\subset\psi(\Omega)\cap\mathbf{T}$ is said to be of *negative type with respect to* F if Γ is of positive type with respect to $-F$. Γ is said to be of *definite type with respect to* F if Γ is of positive type or of negative type with respect to F.

Lemma 2.8′. *Let F be as above and Γ an open subset of $\psi(\Omega)\cap\mathbf{T}$. Then the following assertions are equivalent.*

(a′) *Γ is of positive type with respect to F.*

(b′) (a′) *holds, and, for every $x\in\mathcal{H}$, the angular limit $\widehat{\lim}_{|z|<1,\,z\to s}[F(z)x,x]$ exists and has nonnegative real part for almost every $s\in\psi(\Delta)$.*

(c′) *If $x\in\mathcal{H}$ and if $(z_n)\subset\psi(\Omega)\cap\mathbf{D}$ is a convergent sequence of points of holomorphy of F with $\lim_{n\to\infty}z_n\in\Gamma$, then*

$$\liminf_{n\to\infty}[(F(z_n)-F(\bar{z}_n^{-1}))x,x]\geq 0.$$

(d′) *If $x\in\mathcal{H}$, γ' is a closed subarc of Γ and if $\alpha>0$, there exists a $\delta\in(0,1)$ such that*

$$\inf\{[(F(re^{i\Theta})-F(r^{-1}e^{i\Theta}))x,x]:\,e^{i\Theta}\in\gamma',\,r\in(1-\delta,1)\}>-\alpha.$$

If these assertions hold, then for every closed subarc γ' of Γ and every $r_0\in(0,1)$ such that $1-r_0$ is sufficiently small,

$$\sup\{\|F(re^{i\Theta})\||1-r|:\,e^{i\Theta}\in\gamma',\,r\in[r_0,1)\cup(1,r_0^{-1}]\}<\infty. \tag{2.25}$$

Proof of Lemmas 2.8 and 2.8$'$. 1. We first prove Lemma 2.8$'$. It is easy to see that (c$'$) and (d$'$) are equivalent. Evidently, (c$'$) and (d$'$) imply (a$'$). In order to prove Lemma 2.8$'$ it is sufficient to show that (a$'$) implies (b$'$), (d$'$) and (2.25).

Assume that (a$'$) holds. Let γ' be a compact subarc of Γ, and let γ_0 and γ_1 be open arcs in Γ such that $\gamma' \subset \gamma_0$, $\bar{\gamma}_0 \subset \gamma_1$ and $\bar{\gamma}_1 \subset \Gamma$. We fix some $\epsilon_0 > 0$ such that

$$V_1 := \{re^{i\Theta} : r \in [1-\epsilon_0, 1) \cup (1, (1-\epsilon_0)^{-1}],\, e^{i\Theta} \in \bar{\gamma}_1\}$$

consists of points of holomorphy of F. The function $f(z) := [F(z)x, x]$, $z \in V_1$, satisfies the relation $f(\bar{z}^{-1}) = -\overline{f(z)}$, $z \in V_1$. By (a$'$) there exists an $M \in \mathbf{R}$ such that

$$\begin{aligned} &\operatorname{Re}\{f(z) + M\} = \\ &= \tfrac{1}{2}[(F(z) - F(\bar{z}^{-1}))x, x] + M \geq 0, \qquad z \in V_1 \cap \mathbf{D}. \end{aligned} \tag{2.26}$$

Let now $\mathcal{O}_0$ be a simply connected C^∞ subdomain of $V_1 \cap \mathbf{D}$ such that γ_0 is contained in the boundary of $\mathcal{O}_0$. Let χ be a conformal mapping of $\mathbf{D}$ onto $\mathcal{O}_0$. Then, by (2.26), $f \circ \chi + M$ is a holomorphic function on $\mathbf{D}$ with nonnegative real part. This implies that there is a positive measure μ on $[-\pi, \pi]$, $\mu(\{-\pi\}) = 0$, and $\beta \in \mathbf{R}$ such that

$$(f \circ \chi)(w) + M = i\beta + \int_{-\pi}^{\pi} \tfrac{e^{it}+w}{e^{it}-w}\, d\mu(t),\ w \in \mathbf{D}.$$

As a consequence,

$$(f \circ \chi)(w) = i\beta + \int_{-\pi}^{\pi} \tfrac{e^{it}+w}{e^{it}-w}\, d\nu(t),\ w \in \mathbf{D}, \tag{2.27}$$

where $d\nu(t) = d\mu(t) - M(2\pi)^{-1}dt$. Therefore, the function $f \circ \chi$ has angular limits at almost every point of $\mathbf{T}$. Then, by well-known differentiability properties of χ, $f(z) = [F(z)x, x]$ has angular limits at almost every point of γ_0. In view of (a$'$), these angular limits have nonnegative real part at almost every point of γ_0, which implies (b$'$). For almost every point w_0 of the open arc $\overset{-1}{\chi}(\gamma_0)$ we have

$$\widehat{\lim_{w \to w_0}} \operatorname{Re}(f \circ \chi)(w) \geq 0. \tag{2.28}$$

Let g be a nonnegative continuous function on $\mathbf{T}$ with $\operatorname{supp} g \subset \overset{-1}{\chi}(\gamma_0)$. By

$$\operatorname{Re}(f \circ \chi)(w) = \int_{-\pi}^{\pi} \operatorname{Re}\left\{\tfrac{e^{it}+w}{e^{it}-w}\right\} d\nu(t)$$

and a well-known result (see e.g. [5, Chapter 3]) we have

$$\lim_{r\uparrow 1}(2\pi)^{-1}\int_{-\pi}^{\pi} g(e^{it})\,\mathrm{Re}\,(f\circ\chi)(re^{it})dt = \int_{-\pi}^{\pi} g(e^{it})\,d\nu(t). \tag{2.29}$$

As $\mathrm{Re}\,(f\circ\chi)$ is bounded from below, and by (2.28) the left hand side of (2.29) is nonnegative. Therefore, the measure $d\nu(t)$ is positive on $\{t\in[-\pi,\pi]:e^{it}\in\overset{-1}{\chi}(\gamma_0)\}$.

Define $h(e^{it})$ to be equal to one if $e^{it}\in\overset{-1}{\chi}(\gamma_0)$ and equal to zero if $e^{it}\in\mathbf{T}\setminus\overset{-1}{\chi}(\gamma_0)$. Then the real part of the first term on the right hand side of

$$\begin{aligned}(f\circ\chi)(w) = &\int_{-\pi}^{\pi}\tfrac{e^{it}+w}{e^{it}-w}\,h(e^{it})d\nu(t)+\\ &+\int_{-\pi}^{\pi}\tfrac{e^{it}+w}{e^{it}-w}\,(1-h(e^{it}))d\nu(t),\quad w\in\mathbf{D}.\end{aligned}$$

is nonnegative for all $w\in\mathbf{D}$. The last term is locally holomorphic on $\overset{-1}{\chi}(\gamma_0)$ and has zero real part on $\overset{-1}{\chi}(\gamma_0)$. This implies (d$'$).

From (2.27) we derive, for every $x\in\mathcal{H}$ and γ' and ϵ_0 as above,

$$\sup\{|[F(re^{it})x,x]||1-r|:\ e^{it}\in\gamma',\ r\in[1-\epsilon_0,1)\cup(1,(1-\epsilon_0)^{-1}]\}<\infty.$$

Then the principle of uniform boundedness gives the last assertion of Lemma 2.8$'$.

2. In order to prove Lemma 2.8 we define an $\mathcal{L}(\mathcal{H})$-valued meromorphic function F on $\psi(\Omega)\setminus\mathbf{T}$ by $i\,F\circ\psi=G$ and set $\Gamma:=\psi(\Delta)$. Then, evidently, (b) $\Leftrightarrow$ (b$'$), (c) $\Leftrightarrow$ (c$'$), (d) $\Leftrightarrow$ (d$'$), (b) $\Rightarrow$ (a). It is sufficient to show that (a) implies (b). Assume that (a) holds. Then condition (ii) of Definition 2.7 implies condition (ii$'$) of Definition 2.7$'$, which implies, by part 1 of this proof, that the angular limits of $[F(z)x,x]$, $x\in\mathcal{H}$, exist at almost every point of Γ. It follows that the angular limits of $[G(\lambda)x,x]$ exist for almost every point of Δ. Hence (a) implies (b).

Let, in the rest of Section 2.3 F be the $\mathcal{L}(\mathcal{H})$-valued meromorphic function on $\psi(\Omega)\setminus\mathbf{T}$ defined by $iF\circ\psi=G$. As a consequence of Lemmas 2.8 and 2.8$'$ we obtain the following

Lemma 2.9. *The open set $\Delta\subset\Omega\cap\overline{\mathbf{R}}$ is of positive type with respect to G if and only if $\psi(\Delta)$ is of positive type with respect to F.*

With the help of Definition 2.7′ we can characterize the local positivity of the functional T_F considered in Section 2.1:

Lemma 2.10. *If Γ is an open subset of $\psi(\Omega) \cap \mathbf{T}$, the following conditions are equivalent.*

(α) *Γ is of positive type with respect to F.*

(β) *The order of growth of F near Γ is $\leq m$ for some integer m, and $[(T_F.f)x, x] \geq 0$ for every nonnegative function $f \in C_0^\infty(\Gamma)$ and any $x \in \mathcal{H}$.*

Proof. 1. Assume that (α) holds. Then by the last assertion of Lemma 2.8′ the order of growth of F near Γ is ≤ 1. Let $x \in \mathcal{H}$ and let $f \in C_0^\infty(\Gamma)$ be nonnegative. If $S := \{\Theta \in [0, 2\pi) : \ e^{i\Theta} \in \Gamma\}$, then, by Theorem 2.1,

$$[(T_F.f)x, x] = \lim_{r \uparrow 1} \int_{S_0} f(e^{i\Theta})[\{F(re^{i\Theta}) - F(r^{-1}e^{i\Theta})\}x, x]\, d\Theta,$$

and Lemma 2.8′, (d′) implies $[(T_F.f)x, x] \geq 0$.

2. Assume that (β) holds. We show that F satisfies condition (d′) of Lemma 2.8′. By Lemma 2.4 with $\overline{\Gamma}_1 = \gamma'$ and a nonnegative function χ it is sufficient to prove that for every $x \in \mathcal{H}$, $\alpha > 0$ there exists a $\delta \in (0, 1)$ such that

$$\inf \{[T_F.\chi(h_{re^{i\Theta}} - h_{r^{-1}e^{i\Theta}})x, x] : \ e^{i\Theta} \in \gamma', \ r \in (1 - \delta, 1)\} > -\alpha. \tag{2.30}$$

A simple calculation shows that $h_{re^{i\Theta}}(z) - h_{r^{-1}e^{i\Theta}}(z)$ is nonnegative for $r \in (0, 1)$ and $z \in \mathbf{T}$. Then (β) implies that the left hand side of (2.30) is nonnegative and Lemma 2.10 is proved.

2.4. *Local spectral functions.* Here and in the following Ω, Δ, Γ, A and U are as in Section 2.1. We denote by $\mathcal{B}(\Delta)$ the Boolean ring of all finite unions of connected subsets of Δ whose boundary points (in $\overline{\mathbf{R}}$) belong to Δ.

Definition 2.11. We shall say that A has a *spectral function on* Δ if there exists a strongly σ-additive homomorphism E of $\mathcal{B}(\Delta)$ into a Boolean ring of selfadjoint projections of $\mathcal{H}$ such that the following holds for every $\delta \in \mathcal{B}(\Delta)$.

(i) If $T(A-z)^{-1} = (A-z)^{-1}T$ for a bounded operator T and some $z \in \rho(A)$ (hence, for all $z \in \rho(A)$), then $TE(\delta) = E(\delta)T$.

(ii) $\widetilde{\sigma}(A|E(\delta)\mathcal{H}) \subset \widetilde{\sigma}(A) \cap \overline{\delta}$ and $\widetilde{\sigma}(A|(1-E(\delta))\mathcal{H}) \subset \widetilde{\sigma}(A) \setminus \delta^0$, where δ^0 is the interior of δ with respect to the topology of $\overline{\mathbf{R}}$.

The uniqueness of the spectral function on Δ can be shown in the same way as for a bounded operator A. For the convenience of the reader we give a proof here.

Lemma 2.12. *The operator A cannot have more than one spectral function on Δ.*

Proof. Assume A has the spectral functions E and E' on Δ. Let δ, δ_1, δ_2,..., be closed sets in $\mathcal{B}(\Delta)$ such that the sequence (δ_n) is decreasing, $\bigcap\{\delta_n : n = 1, 2, \ldots\} = \delta$, and $\delta \subset \delta_n^0$ holds for every n. By (i) the spectral functions E and E' commute. By (ii) for any n the spectra of $\psi(A)|E(\delta)\mathcal{H}$ and $\psi(A)|(1 - E'(\delta_n))\mathcal{H}$ are disjoint. It follows that the spectrum of $\psi(A)|E(\delta)(1 - E'(\delta_n))\mathcal{H}$ is empty and, hence, $E(\delta)(1 - E'(\delta_n)) = 0$. By the σ-additivity of E' we obtain $E(\delta)E'(\delta) = E(\delta)$. In the same way we show $E(\delta)E'(\delta) = E'(\delta)$, and it follows that $E(\delta) = E'(\delta)$.

We shall extend this local spectral function, in three consecutive steps, to larger classes of Borel subsets of $\overline{\mathbf{C}}$.

First we extend E to the Boolean ring $\widetilde{\mathcal{B}}(\Delta)$ of all Borel subsets δ of Δ such that $\bar{\delta} \subset \Delta$: For any $x, y \in \mathcal{H}$, $[E(\cdot)x, y]$ has a unique extension to $\widetilde{\mathcal{B}}(\Delta)$ which is σ-additive (see [4, Corollary III.5.9]). Making use of the weak sequential completeness of the Hilbert space we find (with the help of transfinite induction) that even the operator function E has a unique extension to $\widetilde{\mathcal{B}}(\Delta)$. It is well known that the extended spectral function is strongly σ-additive (see e.g. [4, Theorem IV.10.1]), and it is easy to see that all the properties of the spectral function mentioned above are preserved under this extension. For Borel sets δ with $\bar{\delta} \subset \Delta$ we have $E(\delta) = E(\delta \cap \widetilde{\sigma}(A))$.

The second extension is only a formal one: We extend E from a Boolean ring of Borel subsets of Δ to some Boolean algebra of Borel sets in $\overline{\mathbf{C}}$: Consider the Boolean algebra

$$\begin{aligned}&\mathcal{D}_0(\widetilde{\sigma}(A) \setminus \Delta) := \\ &\quad = \{b \,\text{Borel set in}\, \overline{\mathbf{C}} : \; \bar{b} \cap (\widetilde{\sigma}(A) \setminus \Delta) = \emptyset \;\text{or}\; b^0 \supset \widetilde{\sigma}(A) \setminus \Delta\}.\end{aligned}$$

The sets of $\mathcal{D}_0(\widetilde{\sigma}(A) \setminus \Delta)$ contain either the whole set $\widetilde{\sigma}(A) \setminus \Delta$ or no point of $\widetilde{\sigma}(A) \setminus \Delta$. Let $b \in \mathcal{D}_0(\widetilde{\sigma}(A) \setminus \Delta)$. If $\bar{b} \cap (\widetilde{\sigma}(A) \setminus \Delta) = \emptyset$, we set

$$E(b) := E(b \cap \widetilde{\sigma}(A)),$$

and if $b^0 \supset \widetilde{\sigma}(A) \setminus \Delta$, we have $\overline{(\overline{\mathbf{C}} \setminus b)} \cap (\widetilde{\sigma}(A) \setminus \Delta) = \emptyset$ and we set

$$E(b) := 1 - E((\overline{\mathbf{C}} \setminus b) \cap \widetilde{\sigma}(A)).$$

Evidently, E defined on $\mathcal{D}_0(\widetilde{\sigma}(A) \setminus \Delta)$ is a strongly σ-additive homomorphism of $\mathcal{D}_0(\widetilde{\sigma}(A) \setminus \Delta)$ to a Boolean algebra of selfadjoint projections of $\mathcal{H}$ with the properties (i) and (ii).

With the help of the Riesz-Dunford-Taylor projections the spectral function E can be further extended to a Boolean algebra of Borel sets which may contain only a part of the set $\widetilde{\sigma}(A) \setminus \Delta$ of "possible spectral singularities". The following theorem was proved by B. Nagy in [22] for a closed operator A. The proof in [22] is also valid for a closed linear relation, and it is easy to see that condition (i) of Definition 2.11 is preserved under this extension.

Theorem 2.13. *The spectral function E on Δ of A can be extended to a strongly σ-additive homomorphism of the Boolean algebra*

$$\begin{aligned} &\mathcal{A}(\widetilde{\sigma}(A) \setminus \Delta) := \\ &\quad = \{b \text{ Borel set of } \overline{\mathbf{C}} : \ \partial(b \cap \widetilde{\sigma}(A)) \cap (\widetilde{\sigma}(\Delta) \setminus \Delta) = \emptyset^1\} \end{aligned}$$

to a Boolean algebra of projections of $\mathcal{H}$ such that the conditions (i) *and* (ii) *of Definition* 2.11 *are fulfilled. This extension is unique.*

Moreover, $b = b^* \in \mathcal{A}(\widetilde{\sigma}(A) \setminus \Delta)$ implies that $E(b)$ is selfadjoint.

The uniqueness statement can also be verified along the lines of Lemma 2.12. The fact that $b = b^*$ implies the selfadjointness of $E(b)$ is a consequence of the construction in [22].

Remark 2.14. Let Δ' be a connected open subset of $\Omega \cap \overline{\mathbf{R}}$ and e a subset of Δ' which has no point of accumulation in Δ'. Assume that A has a spectral function E on $\Delta := \Delta' \setminus e$. Then all connected subsets δ' of Δ' whose boundary points (in $\overline{\mathbf{R}}$) belong to $\Delta' \setminus e$ are elements of $\mathcal{A}(\widetilde{\sigma}(A) \setminus \Delta)$, i.e. $E(\delta')$ is defined.

In the same way, replacing A, $\overline{\mathbf{R}}$ and Δ by $U = \psi(A)$, $\mathbf{T}$ and $\Gamma = \psi(\Delta)$, the notion of a spectral function of U on Γ can be defined. For the convenience of the reader we repeat the definition. Let $\mathcal{B}(\Gamma)$ be the Boolean ring of all finite unions of connected subsets of Γ whose boundary points (in $\mathbf{T}$) belong to Γ.

Definition 2.11′. We shall say that U has a *spectral function on* Γ if there exists a strongly σ-additive homomorphism F of $\mathcal{B}(\Gamma)$ into a

[1] Here $\partial b'$ denotes the boundary of $b' \subset \widetilde{\sigma}(A)$ with respect to the relative topology of $\widetilde{\sigma}(A)$.

Boolean ring of selfadjoint projections of $\mathcal{H}$ such that the following holds for every $\gamma \in \mathcal{B}(\Gamma)$.

(i′) If $TU = UT$ for a bounded operator T, then $TF(\gamma) = F(\gamma)T$.

(ii′) $\sigma(U|F(\gamma)\mathcal{H}) \subset \sigma(U) \cap \overline{\gamma}$ and $\sigma(U|(1 - F(\gamma))\mathcal{H}) \subset \sigma(U) \setminus \gamma^0$, where γ^0 is the interior of γ with respect to the topology of $\mathbf{T}$.

A spectral function of U on Γ is uniquely determined. This is proved in the same way as in Lemma 2.12.

The selfadjoint relation A has a spectral function E on Δ if and only if U has a spectral function F on Γ, and we have

$$F(\psi(\delta)) = E(\delta)$$

for all $\delta \in \mathcal{B}(\Delta)$. This is an immediate consequence of the spectral mapping theorem for linear relations.

2.5. *Local definiteness.* The following theorem is the main result of this paper.

Theorem 2.15. *Let Ω, A, ψ and U be as at the beginning of Section* 2.1 *and let Δ be an open subset of $\Omega \cap \overline{\mathbf{R}}$. Then the following assertions are equivalent.*

(1) $\Delta \subset \widetilde{\rho}(A) \cup \sigma_{++}(A)$.

(1′) $\psi(\Delta) \subset \rho(U) \cup \sigma_{++}(U)$.

(2) Δ *is of positive type with respect to the function*

$$\lambda \longmapsto \lambda - \operatorname{Re} \lambda_0 + (\lambda - \lambda_0)(\lambda - \overline{\lambda_0})(A - \lambda)^{-1}.$$

(2′) $\psi(\Delta)$ *is of positive type with respect to the function*

$$z \longmapsto (U + z)(U - z)^{-1} = -1 + 2U(U - z)^{-1}.$$

(3) $A \in S^1(\Delta)$ *and* (2) *holds.*

(3′) $U \in S^1(\psi(\Delta))$ *and* (2′) *holds.*

(4) $A \in S^\infty(\Delta)$ *and for every nonnegative $g \in C_0^\infty(\Delta)$ we have $[g(A)x, x] \geq 0$, $x \in \mathcal{H}$.*

($4'$) *$U \in S^\infty(\psi(\Delta))$ and for every nonnegative $f \in C_0^\infty(\psi(\Delta))$ we have $[f(U)x, x] \geq 0$, $x \in \mathcal{H}$.*

(5) *A has a spectral function E on Δ, and for all $\delta \in \mathcal{B}(\Delta)$ we have $[E(\delta)x, x] \geq 0$, $x \in \mathcal{H}$.*

($5'$) *U has a spectral function F on $\psi(\Delta)$, and for all $\gamma \in \mathcal{B}(\psi(\Delta))$ we have $[F(\gamma)x, x] \geq 0$, $x \in \mathcal{H}$.*

(6) *For every open subset Δ_0 of Δ which is a finite union of connected subsets of Δ with $\overline{\Delta}_0 \subset \Delta$, there exists a nonnegative selfadjoint projection E_0 in $\mathcal{H}$ which commutes with every bounded operator that commutes with the resolvent of A, such that the diagonal representation of A,*

$$A = A \cap (E_0\mathcal{H})^2 \dotplus A \cap ((1 - E_0)\mathcal{H})^2$$

has the following properties.

(i) *$\widetilde{\sigma}(A \cap (E_0\mathcal{H})^2) \subset \widetilde{\sigma}(A) \cap \overline{\Delta}_0$, $\quad \widetilde{\sigma}(A \cap ((1 - E_0)\mathcal{H})^2) \subset \widetilde{\sigma}(A) \setminus \Delta_0$.*

(ii) *The boundary points of Δ_0 (in $\overline{\mathbf{R}}$) are no eigenvalues of $A \cap (E_0\mathcal{H})^2$.*

($6'$) *For every open subset Γ_0 of $\psi(\Delta)$ which is a finite union of arcs with $\overline{\Gamma}_0 \subset \psi(\Delta)$, there exists a nonnegative selfadjoint projection F_0 in $\mathcal{H}$ which commutes with every bounded operator that commutes with U, such that the diagonal representation*

$$U = \begin{pmatrix} U_0 & 0 \\ 0 & U_{(0)} \end{pmatrix}$$

with respect to the decomposition $\mathcal{H} = F_0\mathcal{H} \dotplus (1 - F_0)\mathcal{H}$ has the following properties.

(i') *$\sigma(U_0) \subset \sigma(U) \cap \overline{\Gamma}_0$, $\sigma(U_{(0)}) \subset \sigma(U) \setminus \Gamma_0$.*

(ii') *The boundary points of Γ_0 (in $\mathbf{T}$) are no eigenvalues of U_0.*

Similar equivalences hold for all positivity and nonnegativity conditions in (1) - ($6'$) *replaced by the corresponding negativity and nonpositivity conditions.*

Proof. (1) $\Leftrightarrow$ ($1'$): This equivalence follows from the spectral mapping theorem for linear relations (see [3, Section 3]) and Lemma 1.4 with $M = \begin{pmatrix} -\overline{\lambda_0} & 1 \\ \lambda_0 & -1 \end{pmatrix}$.

$(1') \Rightarrow (2')$: Assume that $(2')$ is not true. Then there exist an $x \in \mathcal{H}$, a $\beta < 0$ and two convergent sequences $(e^{i\Theta_n}) \subset \psi(\Delta)$ and $(r_n) \subset (0,1)$ such that $r_n e^{i\Theta_n} \in \rho(U)$, $\lim_{n\to\infty} e^{i\Theta_n} =: e^{i\Theta} \in \psi(\Delta)$, $\lim_{n\to\infty} r_n = 1$ and

$$[U\{(U - r_n e^{i\Theta_n})^{-1} - (U - r_n^{-1} e^{i\Theta_n})^{-1}\}x, x] \le \beta. \tag{2.31}$$

We have

$$\begin{aligned} &[U\{(U - r_n e^{i\Theta_n})^{-1} - (U - r_n^{-1} e^{i\Theta_n})^{-1}\}x, x] = \\ &\quad = (r_n - r_n^{-1}) e^{i\Theta_n} [U(U - r_n e^{i\Theta_n})^{-1} (U - r_n^{-1} e^{i\Theta_n})^{-1} x, x] = \\ &\quad = (r_n - r_n^{-1}) e^{i\Theta_n} [(U - r_n e^{i\Theta_n})^{-1} x, (1 - r_n^{-1} e^{-i\Theta_n} U)^{-1} x, x] = \\ &\quad = (1 - r_n^2)[(U - r_n e^{i\Theta_n})^{-1} x, (U - r_n e^{i\Theta_n})^{-1} x]. \end{aligned} \tag{2.32}$$

Hence (2.31) implies $\alpha_n := \|(U - r_n e^{i\Theta_n})^{-1} x\| \to \infty$ for $n \to \infty$ and, hence, $e^{i\Theta} \in \sigma(U)$.

If $x_n := \alpha_n^{-1}(U - r_n e^{i\Theta_n})^{-1} x$, then $\|x_n\| = 1$ and

$$\|(U - e^{i\Theta}) x_n\| = \alpha_n^{-1} \|x + (r_n e^{i\Theta_n} - e^{i\Theta})(U - r_n e^{i\Theta_n})^{-1} x\| \to 0.$$

By (2.31) and (2.32) we have

$$\liminf_{n\to\infty} [x_n, x_n] \le 0.$$

Therefore, $(1')$ is not true.

$(2) \Leftrightarrow (2')$: We have

$$\begin{aligned} &(U + \psi(\lambda))(U - \psi(\lambda))^{-1} = \\ &\quad = -i(\operatorname{Im}\lambda_0)^{-1}\{\lambda - \operatorname{Re}\lambda_0 + (\lambda - \lambda_0)(\lambda - \overline{\lambda_0})(A - \lambda)^{-1}\} \end{aligned}$$

for every $\lambda \in \rho(A) \cap \Omega$. By Lemma 2.9 this relation implies the equivalence $(2) \Leftrightarrow (2')$.

$(2') \Rightarrow (3')$: Assume that $(2')$ holds. Then the last assertion of Lemma 2.8$'$ applied to

$$F(z) = (U + z)(U - z)^{-1} = -1 + 2U(U - z)^{-1}$$

gives $U \in S^1(\psi(\Delta))$.

$(3) \Leftrightarrow (3')$: That $A \in S^1(\Delta)$ is equivalent to $U \in S^1(\psi(\Delta))$ was shown at the end of Section 2.1.

$(3') \Rightarrow (4')$: By Lemma 2.10 applied to $F(z) = F_U(z)$ condition $(2')$, which is equivalent to $(3')$, implies $(4')$. Or, more directly, if $x \in \mathcal{H}$,

$f \in C_0^\infty(\psi(\Delta))$ and Γ_0 is the union of a finite number of closed subarcs of $\psi(\Delta)$ such that $\operatorname{supp} g \subset \Gamma_0$, then Lemma 2.8′ applied to $F_U(z)$ shows that for every $\alpha > 0$ there exists a $\delta \in (0,1)$ such that

$$\inf \{[U\{(U - re^{i\Theta})^{-1} - (U - r^{-1}e^{i\Theta})^{-1}\}x, x] : e^{i\Theta} \in \Gamma_0, r \in (1-\delta, 1)\} > -\alpha.$$

Then (2.24) gives $[f(U)x, x] \geq 0$.

(4) $\Leftrightarrow$ (4′): It was proved at the end of Section 2.1 that $A \in S^\infty(\Delta)$ and $U \in S^\infty(\psi(\Delta))$ are equivalent. If f runs through all nonnegative functions in $C_0^\infty(\psi(\Delta))$ then $g = f \circ \psi$ runs through all nonnegative functions in $C_0^\infty(\Delta))$, and for these functions we have $f(U) = g(A)$.

(4′) $\Rightarrow$ (5′): Assume that (4′) holds. Let $f_n \in C_0^\infty(\psi(\Delta))$, $n = 1, 2, \ldots$, such that $\operatorname{supp} f_n \subset K$, where K is some compact subset of $\psi(\Delta)$, and assume that the sequence (f_n) converges uniformly to a continuous function f. Let $\chi \in C_0^\infty(\psi(\Delta))$ be a positive function and equal to one on a neighbourhood of K. Then for any $\epsilon > 0$ there exists an N such that for $n, m \geq N$ we have

$$-\epsilon\,[\chi(U)x, x] \leq [(f_n(U) - f_m(U))x, x] \leq \epsilon[\chi(U)x, x]$$

for all $x \in \mathcal{H}$. Hence $f_n(U)$ converges with respect to the operator norm. Therefore the functional calculus for U can be extended to $C_0^0(\psi(\Delta))$.

Let (h_n) be a sequence of uniformly bounded functions belonging to $C_0^0(\psi(\Delta))$ with $\operatorname{supp} h_n \subset K$, $n = 1, 2, \ldots$, for some compact subset K of $\psi(\Delta)$, and assume that the sequence (h_n) is pointwise convergent. Then $(h_n(U))$ converges to a bounded operator in the weak sense. In this way we extend the functional calculus of U to the linear set $B_0(\psi(\Delta))$ of all bounded functions h which are zero outside of a compact subset of $\psi(\Delta)$ and which can be approximated by continuous functions with respect to pointwise convergence. It is easy to see that this extension of the functional calculus remains linear, multiplicative and positive. Moreover, it is continuous with respect to pointwise convergence of uniformly bounded functions and the weak operator topology.

If $\gamma_0 \in \mathcal{B}(\psi(\Delta))$ then the indicator function χ_{γ_0} of γ_0 belongs to $B_0(\psi(\Delta))$, and we define

$$F(\gamma_0) := \chi_{\gamma_0}(U).$$

Then we have $[F(\gamma_0)x, x] \geq 0$ for every $x \in \mathcal{H}$. From the linearity and multiplicativity of the extended functional calculus it follows that F fulfils the homomorphism property of the spectral function. The strong σ-additivity of F is a consequence of the continuity of the extended functional calculus mentioned above and the fact that weakly convergent monotone sequences of bounded selfadjoint operators in a Krein space converge in the strong sense. That F satisfies condition (i′) of Definition 2.11′ follows immediately from the definition of F.

That (ii′) (in Definition 2.11′) holds can be verified in the following way, which is well known: If $z_0 \notin \overline{\gamma_0}$, then the function h defined by $h(z) := (z - z_0)^{-1}\chi_{\gamma_0}(z)$ belongs to $B_0(\psi(\Delta))$ and we have

$$(U - z_0)h(U) = h(U)(U - z_0)F(\gamma_0) = F(\gamma_0).$$

Therefore, $z_0 \notin \sigma(U|F(\gamma_0)\mathcal{H})$.

Let now z_0 be a point of γ_0 but no boundary point of γ_0 (in $\mathbf{T}$), and let k be a function from $C_0^\infty(\psi(\Delta))$ with support contained in the interior of γ_0 such that k is equal to one in some neighbourhood of z_0. Then $h(z) := (1 - k(z))(z - z_0)^{-1}$ belongs to the domain of the functional calculus of U and we have

$$(U - z_0)h(U)(1 - F(\gamma_0)) = h(U)(U - z_0)(1 - F(\gamma_0)) = 1 - F(\gamma_0)$$

and, therefore,

$$z_0 \notin \sigma(U|(1 - F(\gamma_0))\mathcal{H}).$$

(5) $\Leftrightarrow$ (5′): See end of Section 2.4.

(5′) $\Rightarrow$ (6′): If (5′) holds and Γ_0 is as in (6′), then $F_0 := F(\Gamma_0)$ has the required properties.

(6) $\Leftrightarrow$ (6′): Assume that (6) holds. If Γ_0 is as in (6′) we set $F_0 := E_0$ where E_0 is a selfadjoint projection associated with Δ_0, $\psi(\Delta_0) = \Gamma_0$. Then (6′) follows from

$$U = \psi(A) = -1 + (\lambda_0 - \bar{\lambda_0})(A - \bar{\lambda_0})^{-1}$$

and the spectral mapping theorem.

If (6′) holds and if Δ_0 is as in (6), we set $E_0 := F_0$ where F_0 is a selfadjoint projection associated with $\psi(\Delta_0)$. Then (6) follows from

$$A = \left\{ \begin{pmatrix} (U + 1)x \\ (\bar{\lambda}_0 U + \lambda_0)x \end{pmatrix} : \ x \in \mathcal{H} \right\}$$

and the spectral mapping theorem.

(6′) $\Rightarrow$ (1′): Assume that (6′) holds. Let $z_0 \in \psi(\Delta)$ belong to $\sigma(U)$, and choose Γ_0 in (6′) so that $z_0 \in \Gamma_0$. Then there exists a sequence $(x_n) \subset \mathcal{H}$, $n = 1, 2, \ldots$, with $x_n = u_n + v_n$, $u_n \in F_0\mathcal{H}$, $v_n \in (1 - F_0)\mathcal{H}$, $\|u_n\|^2 + \|v_n\|^2 = 1$ and

$$\lim_{n\to\infty} \|(U_0 - z_0)u_n\| = 0, \qquad \lim_{n\to\infty} \|(U_{(0)} - z_0)v_n\| = 0.$$

By the second relation of (i$'$) we have $\lim_{n\to\infty} \|v_n\| = 0$ and, therefore, $\lim_{n\to\infty} \|u_n\| = 1$. $(F_0\mathcal{H}, [\cdot,\cdot])$ is a Hilbert space and there exists an $\alpha > 0$ such that $[u,u] \geq \alpha \|u\|^2$ for all $u \in F_0\mathcal{H}$. This implies

$$\liminf_{n\to\infty}\, [x_n, x_n] = \liminf_{n\to\infty}\, ([u_n, u_n] + [v_n, v_n]) \geq$$
$$\geq \alpha \lim_{n\to\infty} \|u_n\| + \lim_{n\to\infty} [v_n, v_n] \geq \alpha,$$

and Theorem 2.15 is proved.

Remark 2.16. Let assertion (6) of Theorem 2.15 be true, and let $E_0(\cdot)$ be the spectral function of the selfadjoint linear relation $A \cap (E_0\mathcal{H})^2$ in the Hilbert space $(E_0\mathcal{H}, [\cdot,\cdot])$. Define, for any connected subset δ of $\overline{\mathbf{R}}$ with $\bar{\delta} \subset \Delta_0$,

$$E(\delta) := E_0(\delta)E_0.$$

It is easy to see that $E(\cdot)$ possesses all characterizing properties of the local spectral function of A on Δ_0 and, hence (see Lemma 2.12), coincides with the local spectral function of A on Δ_0. The local spectral function of U and the spectral function of U_0 (see condition (6$'$)) are related in a similar way.

In the case when Δ in Theorem 2.15 is finite the function in condition (2) can be replaced by the resolvent of A.

Theorem 2.17. *Let Ω and A be as at the beginning of Section* 2.1 *and Δ be an open subset of $\Omega \cap \overline{\mathbf{R}}$ such that $\infty \notin \Delta$. Then any of the conditions* (1) - (6$'$) *of Theorem* 2.15 *is equivalent to Δ being of positive type with respect to the resolvent of A.*

Proof. We prove that Δ is of positive type with respect to the resolvent of A if and only if condition (2) of Theorem 2.15 holds. Let $x \in \mathcal{H}$. Then for $t \in \Delta$ the angular limit $\widehat{\lim}_{\mathrm{Im}\,>0,\,\lambda\to t}\, [(A-\lambda)^{-1}x, x]$ exists and belongs to $\mathbf{C}^+ \cup \mathbf{R}$ if and only if the angular limit

$$\widehat{\lim}_{\mathrm{Im}\,>0,\,\lambda\to t}\, [\{\lambda - \mathrm{Re}\,\lambda_0 + (\lambda-\lambda_0)(\lambda-\bar{\lambda}_0)(A-\lambda)^{-1}\}x, x]$$

exists and belongs to $\mathbf{C}^+ \cup \mathbf{R}$.

Assume that either Δ is of positive type with respect to the resolvent of A or with respect to

$$\lambda \mapsto \lambda - \mathrm{Re}\,\lambda_0 + (\lambda-\lambda_0)(\lambda-\bar{\lambda}_0)(A-\lambda)^{-1}.$$

We set $g(\lambda) := (\lambda-\lambda_0)(\lambda-\bar{\lambda}_0)$. Then, by the last assertion of Lemma 2.8, for t in a compact subset of Δ and $\epsilon \in (0, \epsilon_0]$ for sufficiently small $\epsilon_0 > 0$, the expressions

$$\|(g(t+i\epsilon) - g(t))(A - (t+i\epsilon))^{-1}\|$$

and

$$\|(g(t-i\epsilon)-g(t))(A-(t-i\epsilon))^{-1}\|$$

are uniformly bounded. Therefore, if $t+i\epsilon$ runs through that points,

$$\inf\ \left\{-i\left[\left(g(t+i\epsilon)(A-(t+i\epsilon))^{-1}-g(t-i\epsilon)(A-(t-i\epsilon))^{-1}\right)x,x\right]\right\}$$

is finite if and only if

$$\inf\ \left\{-ig(t)\left[\left((A-(t+i\epsilon))^{-1}-(A-(t-i\epsilon))^{-1}\right)x,x\right]\right\}$$

is finite. Then applying Lemma 2.8, (a) $\Leftrightarrow$ (b), completes the proof of Theorem 2.17.

Definition 2.18. If A and U are as in Theorem 2.15 and Δ is an open subset of $\Omega\cap\overline{\mathbf{R}}$, then Δ $(\psi(\Delta))$ is said to be of *positive type with respect to* A (resp. U) if one of the equivalent conditions (1) - ($6'$) is satisfied. Open sets of *negative type* are defined in an analogous way. Δ $(\psi(\Delta))$ is said to be of *definite type with respect to* A (resp. U) if Δ (resp. $\psi(\Delta)$) is of positive type or of negative type with respect to A (resp. U).

3. Locally definitizable operators in Krein spaces

3.1. *Definitizable and locally definitizable operators and operator functions.* We recall that a selfadjoint linear relation A (a unitary operator U) in the Krein space $\mathcal{H}$ with $\rho(A)\neq\emptyset$ is called *definitizable* if there exists a rational function $r=r^*$ (resp. $q=\hat{q}$) all poles of which belong to $\widetilde{\rho}(A)$ (resp. $\widetilde{\rho}(U)$), such that

$$[r(A)x,x]\geq 0 \quad (\text{resp. } [q(U)x,x]\geq 0) \quad \text{for all} \quad x\in\mathcal{H}.$$

In the case of a selfadjoint relation this definition is equivalent to that in [3, Section 4].

The functions r and q are called *definitizing* functions for A and U, respectively. If $\lambda_0\in\rho(A)\cap\mathbf{C}^+$, and $U=\psi(A)$, $\psi(\lambda):=-(\lambda-\lambda_0)(\lambda-\bar{\lambda}_0)^{-1}$, then A is definitizable if and only if U is definitizable. The rational function r is definitizing for A if and only if the function q with $q\circ\psi=r$ is definitizing for U.

We also recall the corresponding notions for operator functions. Let G be an $\mathbf{R}$-symmetric function meromorphic in $\mathbf{C}\setminus\mathbf{R}$ and let F be a $\mathbf{T}$-skew-symmetric function meromorphic in $\overline{\mathbf{C}}\setminus\mathbf{T}$ such that 0 and ∞ are points of holomorphy of F. G is called a Nevanlinna function if G is holomorphic in $\mathbf{C} \setminus \mathbf{R}$ and, for every $x \in \mathcal{H}$ and every $\lambda \in \mathbf{C}^+$, Im $[G(\lambda)x, x] \geq 0$ holds. F is called a Caratheodory function if F is holomorphic in $\overline{\mathbf{C}} \setminus \mathbf{T}$ and for every $x \in \mathcal{H}$ and every $z \in \mathbf{D}$, Re $[F(z)x, x] \geq 0$ holds. G (F) is called *definitizable* if there exists a rational function $r = r^*$ (resp. $q = \hat{q}$) such that the poles of r (resp. q) are points of holomorphy of G (resp. F) and rG (resp. qF) is the sum of a Nevanlinna function (resp. a Caratheodory function) and a meromorphic function in $\overline{\mathbf{C}}$. r and q are called *definitizing* functions for G and F, respectively. If $\lambda_0 \in \mathbf{C}^+$ is a point of holomorphy of G and ψ is as above then by the definitions given above G is definitizable if and only if $F := -iG \circ \psi^{-1}$ is definitizable. For more details on definitizable operator functions see [12] and [14].

For operators local versions of definitizability were introduced in [7] and [8]. The definition of local definitizability in Definition 3.3 below will slightly differ from that in [7] and [8]: Here we include a condition on the nonreal or non-unimodular spectrum. Definition 3.3 will be formulated with the help of the resolvent. It is natural, similarly to the considerations of Section 2, to introduce first locally definitizable operator functions with the help of the characteristic "local definitizability properties" of the resolvents of definitzable operators. The concept of locally definitizable operator function (Definitions 3.1 and 3.1′ below) is also a localization of the notion of definitizable operator function (see Proposition 3.2 below). Let Ω, $\lambda_0 \in \Omega \cap \mathbf{C}^+$ and ψ be as in Section 2.1.

Definition 3.1. An $\mathcal{L}(\mathcal{H})$-valued meromorphic function G in $\Omega \setminus \overline{\mathbf{R}}$ symmetric with respect to $\mathbf{R}$ is called *definitizable in* Ω if the following holds.

(α) No point of $\Omega \cap \overline{\mathbf{R}}$ is an accumulation point of nonreal poles of G, and for every finite union Δ_0 of open connected subsets of $\Omega \cap \overline{\mathbf{R}}$ with $\overline{\Delta_0} \subset \Omega \cap \overline{\mathbf{R}}$ there exists a positive integer m such that the order of growth of G near Δ_0 is $\leq m$.

(β) Every point $\lambda \in \Omega \cap \overline{\mathbf{R}}$ has an open connected neighbourhood I_λ in $\overline{\mathbf{R}}$ such that both components of $I_\lambda \setminus \{\lambda\}$ are of definite type with respect to G.

Definition 3.1′. An $\mathcal{L}(\mathcal{H})$-valued meromorphic function F in $\psi(\Omega) \setminus \mathbf{T}$ skew-symmetric with respect to $\mathbf{T}$ is called *definitizable in* $\psi(\Omega)$ if the following holds.

(α') No point of $\psi(\Omega) \cap \mathbf{T}$ is an accumulation point of non-unimodular poles of F, and for every finite union Γ_0 of open arcs of $\psi(\Omega) \cap \mathbf{T}$ with

$\overline{\Gamma}_0 \subset \psi(\Omega) \cap \mathbf{T}$ there exists a positive integer m such that the order of growth of F near Γ_0 is $\leq m$.

(β') Every point $z \in \psi(\Omega) \cap \mathbf{T}$ has an open connected neighbourhood I_z in $\mathbf{T}$ such that both components of $I_z \setminus \{z\}$ are of definite type with respect to F.

In Theorem 3.6 below we shall make use of the following proposition from [12, Proposition 2.2]. For the convenience of the reader we will give a direct proof here.

Proposition 3.2. *An $\mathcal{L}(\mathcal{H})$-valued meromorphic function F in $\overline{\mathbf{C}} \setminus \mathbf{T}$ skew-symmetric with respect to $\mathbf{T}$ which is holomorphic at 0 and ∞, is definitizable in $\overline{\mathbf{C}}$ if and only if it is definitizable.*

Proof. 1. Let F be definitizable, $q = \hat{q}$ a rational function and H a Caratheodory function such that the poles of q are points of holomorphy of F and $qF - H$ can be continued analytically to an $\mathcal{L}(\mathcal{H})$-valued function meromorphic in $\overline{\mathbf{C}}$. Evidently, F has only a finite number of poles in $\overline{\mathbf{C}} \setminus \mathbf{T}$. By the integral representation of Caratheodory functions (see also Lemma 2.8′, (2.25)) the order of growth of H near $\mathbf{T}$ is ≤ 1. Hence there exists an integer m such that the order of growth of F near $\mathbf{T}$ is $\leq m$.

Assume that γ_0 is an open arc of $\mathbf{T}$ such that q has no pole in $\gamma := \overline{\gamma}_0$ and q is positive on γ. Let $x \in \mathcal{H}$. In order to verify that condition (ii′) of Definition 2.7′ is fulfilled it is sufficient to show that

$$\begin{aligned} \inf \{ | [(p(re^{i\Theta})H(re^{i\Theta}) - p(r^{-1}e^{i\Theta})H(r^{-1}e^{i\Theta}))x, x] : \\ e^{i\Theta} \in \gamma,\ r \in (1-\delta, 1)\} > -\infty \end{aligned} \tag{3.1}$$

with $p = q^{-1}$ and for some $\delta > 0$.

We have

$$\begin{aligned} &[(p(re^{i\Theta})H(re^{i\Theta}) - p(r^{-1}e^{i\Theta})H(r^{-1}e^{i\Theta}))x, x] = \\ &\quad = (p(re^{i\Theta}) - p(e^{i\Theta}))[H(re^{i\Theta})x, x] - \\ &\quad - (p(r^{-1}e^{i\Theta}) - p(e^{i\Theta}))[H(r^{-1}e^{i\Theta})x, x] + \\ &\quad + p(e^{i\Theta})[(H(re^{i\Theta}) - H(r^{-1}e^{i\Theta}))x, x]. \end{aligned} \tag{3.2}$$

There is a $\delta > 0$ such that the first two terms on the right hand side of (3.2) are uniformly bounded for $e^{i\Theta} \in \gamma$ and $r \in (1-\delta, 1)$. The third term on the right hand side of (3.2) is nonnegative for $e^{i\Theta} \in \gamma$ and all $r \in (0, 1)$.

Therefore, (3.1) holds. Let μ be a positve measure on $[-\pi, \pi]$ and $\beta \in \mathbf{R}$ such that

$$[H(z)x, x] = i\beta + \int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z}\, d\mu(t), \quad z \in \mathbf{D}.$$

If μ has no mass at $e^{i\Theta} \in \gamma$ then the first two terms on the right hand side of (3.2) converge to zero if $r \uparrow 1$. Therefore condition (i) of Definition 2.7′ is fulfilled, and γ_0 is an arc of positive type with respect to F. If q is negative on γ a similar reasoning applies. Then it follows that F is definitizable in $\overline{\mathbf{C}}$.

2. Let F be definitizable in $\overline{\mathbf{C}}$ and let P be the finite set of all poles of F in $\overline{\mathbf{C}} \setminus \mathbf{T}$, $0, \infty \notin P$. We decompose F as in (2.22): $F = F_0 + F_{(0)}$. Let $g_0 = \hat{g}_0$ be a rational function with the following properties.

(i) The set $P(g_0)$ of all poles of g_0 is contained in $\overline{\mathbf{C}} \setminus (\mathbf{T} \cup P)$.

(ii) g_0 is positive on $\mathbf{T}$.

(iii) $g_0 F_0$ is locally holomorphic on P.

$F_{(0)}$ is locally holomorphic on $\overline{\mathbf{C}} \setminus \mathbf{T}$. Let the order of growth of $F_{(0)}$ near $\mathbf{T}$ be $\leq m$. We denote by e a finite subset of $\mathbf{T}$ such that all connected components of $\mathbf{T} \setminus e$ are of definite type with respect to $F_{(0)}$. Let $g_{(0)} = \hat{g}_{(0)}$ be a rational function with the following properties.

(i′) The set $P(g_{(0)})$ of all poles of $g_{(0)}$ is contained in $\overline{\mathbf{C}} \setminus (\mathbf{T} \cup P)$.

(ii′) All points of e are zeros of $g_{(0)}$ at least of order $m + 2$.

(iii′) For any component γ of $\mathbf{T} \setminus e$ the following holds: If γ is of positive type with respect to $F_{(0)}$, then $g_{(0)}$ is positive on γ, otherwise $g_{(0)}$ is negative on γ.

Let (ϕ_n) be a sequence of functions in $C^\infty(\mathbf{T})$ which converges to $g_{(0)}$ in $C^{m+1}(\mathbf{T})$ such that for every $n = 1, 2, \ldots$, ϕ_n is nonnegative (nonpositive) on those components of $\mathbf{T} \setminus e$ where $g_{(0)}$ is positive (resp. negative) and ϕ_n is zero in a neighbourhood of e. Then, by Lemma 2.10, for every nonnegative function $f \in C^\infty(\mathbf{T})$ we have

$$[(T_{F_{(0)}}.\phi_n g_0 f)x, x] \geq 0, \quad x \in \mathcal{H},$$

and, therefore,

$$[(T_{F_{(0)}}.g_{(0)} g_0 f)x, x] \geq 0, \quad x \in \mathcal{H}.$$

Then the operator function K defined by

$$K(z) := T_{F_{(0)}}.gh_z \text{ with } g = g_0 g_{(0)}, \quad z \in \overline{\mathbf{C}} \setminus \mathbf{T},$$

is a Caratheodory function.

For $\zeta \in \mathbf{T}$ and all points z of holomorphy of g, we have

$$
\begin{aligned}
g(\zeta)h_z(\zeta) &= \frac{1}{4\pi}g(\zeta)\frac{\zeta+z}{\zeta-z} = \\
&= \tilde{g}(\zeta,z) + \frac{1}{4\pi}g(z)\frac{\zeta+z}{\zeta-z},
\end{aligned}
\tag{3.3}
$$

where

$$
\tilde{g}(\zeta,z) := \begin{cases} \frac{1}{4\pi}\frac{g(\zeta)-g(z)}{\zeta-z}(\zeta+z) & \text{if } \ \zeta \neq z \\ \frac{1}{2\pi}g'(\zeta)\zeta & \text{if } \ \zeta = z. \end{cases}
$$

It is easy to see that

$$
z \longmapsto \tilde{g}(\cdot,z) \in C^{m+1}(\mathbf{T})
$$

is complex differentiable in $\overline{\mathbf{C}} \setminus (P(g_0) \cup P(g_{(0)}))$.

Applying $T_{F_{(0)}}$ to both sides of (3.3) gives

$$
K(z) = T_{F_{(0)}}.\tilde{g}(\cdot,z) + g(z)F_{(0)}(z). \tag{3.4}
$$

This shows that $z \longmapsto T_{F_{(0)}}.\tilde{g}(\cdot,z)$ is meromorphic in $\overline{\mathbf{C}}$. Every pole of this function is a pole of g. It follows that

$$
gF = gF_0 + gF_{(0)} = K + g_0 g_{(0)} F_0 - T_{F_{(0)}}.\tilde{g}(\cdot,z).
$$

Since $g_0 g_{(0)} F_0 - T_{F_{(0)}}.\tilde{g}(\cdot,z)$ is a meromorphic function in $\overline{\mathbf{C}}$ the poles of which are contained in $P(g_0) \cup P(g_{(0)}) \subset \overline{\mathbf{C}} \setminus (\mathbf{T} \cup P)$, F is definitizable.

Definition 3.3. The selfadjoint relation A (the unitary operator U) is called *definitizable over* Ω (resp. $\psi(\Omega)$), if $\sigma(A) \cap (\Omega \setminus \overline{\mathbf{R}})$ (resp. $\sigma(U) \cap (\psi(\Omega) \setminus \mathbf{T})$) consists of isolated points which are poles of the resolvent and the function

$$
\lambda \longmapsto \lambda - \operatorname{Re}\lambda_0 + (\lambda-\lambda_0)(\lambda-\overline{\lambda}_0)(A-\lambda)^{-1} =: G_A(\lambda)
$$
$$
(\text{resp. } z \longmapsto (U+z)(U-z)^{-1} =: F_U(z))
$$

is definitizable in Ω (resp. $\psi(\Omega)$).

Remark 3.4. By (2.17) the growth conditions for G_A and F_U contained in Definition 3.3 are equivalent to $A \in S^\infty(\Omega \cap \overline{\mathbf{R}})$ and $U \in S^\infty(\psi(\Omega) \cap \mathbf{T})$, respectively. Conditions equivalent to the "sign" conditions for G_A and F_U are expressed in Theorem 2.15 in terms of A and U. That the isolated spectral points mentioned in Definition 3.3 are poles of the resolvent is also a consequence of the last condition.

Remark 3.5. Let Δ be an open subset of $\overline{\mathbf{R}}$ and let A be a selfadjoint operator such that $\sigma(A) \setminus \overline{\mathbf{R}}$ has no more than a finite number of nonreal accumulation points. Then A is definitizable over Δ in the sense of [8] if for every $\mathbf{R}$-symmetric domain Ω with $\Omega \cap \overline{\mathbf{R}} = \Delta$ such that $\Omega \cap \mathbf{C}^+$ is simply connected and $\sigma(A) \cap (\Omega \setminus \mathbf{R}) = \emptyset$, A is definitizable over Ω. This is a direct consequence of the definitions and Remark 3.4.

We first consider selfadjoint linear relations definitizable over $\overline{\mathbf{C}}$ and show that these relations are just the definitizable ones.

Theorem 3.6. *Let A be a selfadjoint relation in $\mathcal{H}$ with $\rho(A) \neq \emptyset$, let $\lambda_0 \in \rho(A) \cap \mathbf{C}^+$ and let $U = \psi(A) = -1 + (\lambda_0 - \bar{\lambda}_0)(A - \bar{\lambda}_0)^{-1}$. Then the following assertions are equivalent.*

(1) *A is definitizable.*

(1′) *U is definitizable.*

(2) *A is definitizable over $\overline{\mathbf{C}}$.*

(2′) *U is definitizable over $\overline{\mathbf{C}}$.*

(3) *The function G_A (see Definition 3.3) is definitizable.*

(3′) *The function F_U (see Definition 3.3) is definitizable.*

Proof. The assertions (1) and (1′) are equivalent, see the beginning of this section. In view of $F_U \circ \psi = -iG_A$ it follows from Theorem 2.15 and the considerations of Section 2 that (2) and (2′) are equivalent. That (3) and (3′) are equivalent is an immediate consequence of the definitions. It was shown in Proposition 3.2 that (2′) is equivalent to (3′). That (1′) implies (3′) is a consequence of [14, Theorem 1.7]. By [14, Theorem 1.9] (3′) implies (1′).

3.2. Let $\mathcal{H}$ be the orthogonal sum of two Krein spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, $\mathcal{H} = \mathcal{H}_1[+]\mathcal{H}_2$, and let Ω be as above. If A_1 and A_2 are selfadjoint linear relations in $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, such that A_1 is definitizable, $\rho(A_2) \neq \emptyset$ and the resolvent of A_2 is meromorphic in Ω, then by Theorem 3.6 the selfadjoint linear relation $A_1[+]A_2$ is definitizable over Ω. Such an orthogonal sum is not far away from the most general case of a selfadjoint linear relation definitizable over Ω. With the help of Theorem 3.6 it is easy to characterize locally definitizable relations:

Theorem 3.7. *Let Ω, λ_0, ψ, A and U be as at the beginning of Section* 2.1. *Then the following assertions are equivalent.*

(1) *A is definitizable over Ω.*

(1′) *U is definitizable over $\psi(\Omega)$.*

(2) *For every closed set $K \subset \Omega \cap \overline{\mathbf{R}}$ there exist an open subset Δ_0 of $\Omega \cap \overline{\mathbf{R}}$ which is a finite union of open connected sets, such that $K \subset \Delta_0$, $\overline{\Delta_0} \subset \Omega \cap \overline{\mathbf{R}}$, and a selfadjoint projection E_0 in $\mathcal{H}$ which commutes with every bounded operator that commutes with the resolvent of A such that the corresponding diagonal representation of A,*

$$A = A \cap (E_0\mathcal{H})^2 \dotplus A \cap ((1 - E_0)\mathcal{H})^2,$$

has the following properties.

(o) *$A \cap (E_0\mathcal{H})^2$ is definitizable.*

(i) $\widetilde{\sigma}(A \cap (E_0\mathcal{H})^2) \subset \widetilde{\sigma}(A) \cap \overline{\Delta_0}$, $\quad \widetilde{\sigma}(A \cap ((1 - E_0)\mathcal{H})^2) \subset \widetilde{\sigma}(A) \setminus \Delta_0$.

(ii) *The boundary points of Δ_0 (in $\overline{\mathbf{R}}$) are no eigenvalues of $A \cap (E_0\mathcal{H})^2$.*

(2′) *For every closed set $K' \subset \psi(\Omega) \cap \mathbf{T}$ there exist an open subset Γ_0 of $\psi(\Omega) \cap \mathbf{T}$ which is a finite union of open arcs, such that $K' \subset \Gamma_0$, $\overline{\Gamma_0} \subset \psi(\Omega) \cap \mathbf{T}$, and a selfadjoint projection F_0 in $\mathcal{H}$ which commutes with every bounded operator that commutes with U such that the corresponding diagonal representation of U,*

$$U = \begin{pmatrix} U_0 & 0 \\ 0 & U_{(0)} \end{pmatrix},$$

with respect to the decomposition $\mathcal{H} = F_0\mathcal{H} \dotplus (1 - F_0)\mathcal{H}$ has the following properties.

(o′) *U_0 is definitizable.*

(i′) $\sigma(U_0) \subset \sigma(U) \cap \overline{\Gamma_0}$, $\sigma(U_{(0)}) \subset \sigma(U) \setminus \Gamma_0$.

(ii′) *The boundary points of Γ_0 are no eigenvalues of U_0.*

Proof. That the assertions (1), (1′) are equivalent follows from what was proved in Section 2. Evidently, (2) is equivalent to (2′) with $E_0 = F_0$. It remains to prove that (1′) and (2′) are equivalent.

Assume that $(1')$ holds. In the following, subarcs of the unit circle will be denoted similar to real intervals: If $\alpha = e^{i\theta}$, $\beta = e^{i\Theta}$, $0 < \Theta - \theta < 2\pi$, we define $(\alpha, \beta) := \{e^{it} : \ t \in (\theta, \Theta)\}$.

Let Γ_0 be the union of open arcs $(\alpha_j, \beta_j) \subset \psi(\Omega) \cap \mathbf{T}$, $j = 1, \ldots, k$, with pairwise positive distance from each other such that $K' \subset \Gamma_0$, $\overline{\Gamma}_0 \subset \psi(\Omega) \cap \mathbf{T}$ and there are open arcs $\gamma(\alpha_j)$, $\gamma(\beta_j)$, $j = 1, \ldots, k$, of definite type with respect to U such that $\alpha_j \in \gamma(\alpha_j)$, $\beta_j \in \gamma(\beta_j)$. Assume, in addition, that the arcs $\gamma(\alpha_j)$, $\gamma(\beta_j)$, $j = 1, \ldots, k$, are pairwise disjoint. In view of Theorem 2.15 it is easy to see that U has a spectral function F on $\bigcup_{j=1}^{k}(\gamma(\alpha_j) \cup \gamma(\beta_j))$.

We choose $\epsilon > 0$ so small that

$$\alpha_j e^{i\epsilon} \in \gamma(\alpha_j), \quad \beta_j e^{-i\epsilon} \in \gamma(\beta_j), \quad j = 1, \ldots k,$$

and set

$$\widetilde{\gamma} = \bigcup_{j=1}^{k} (\alpha_j, \alpha_j e^{i\epsilon}) \cup (\beta_j e^{-i\epsilon}, \beta_j).$$

Then

$$\sigma(U|F(\widetilde{\gamma})\mathcal{H}) \subset \overline{\widetilde{\gamma}}, \quad \sigma(U|(1 - F(\widetilde{\gamma}))\mathcal{H}) \subset \sigma(U) \setminus \widetilde{\gamma}.$$

The sets $(\alpha_j, \beta_j) \setminus \widetilde{\gamma}$, $j = 1, \ldots, k$, are Dunford spectral sets of $U|(1 - F(\widetilde{\gamma}))\mathcal{H}$. Let P_j, $j = 1, \ldots, k$, be the corresponding Riesz-Dunford projections in $(1 - F(\widetilde{\gamma}))\mathcal{H}$, which are selfadjoint. The selfadjoint projection

$$F_0 := F(\widetilde{\gamma}) + \sum_{j=1}^{k} P_j (1 - F(\widetilde{\gamma}))$$

commutes with every bounded operator that commutes with U. This follows from an analogous property of F. Then U can be written as a diagonal matrix as in $(3')$. The properties (i') and (ii') are consequences of the definition of F_0.

The local finite order growth of the resolvent of U near $\psi(\Omega) \cap \mathbf{T}$ implies growth of finite order of the resolvent of U_0 near $\mathbf{T}$. Every subarc of $\psi(\Omega) \cap \mathbf{T}$ of definite type with respect to U has the same definite type with respect to U_0 (Theorem 2.15, $(1')$). Then $\sigma(U_0)$ is contained up to a finite subset in a union of a finite number of open arcs of definite type with respect to U_0. Hence by Theorem 3.6 U_0 is definitizable.

Assume that $(2')$ holds. Let s_0 be an arbitrary point of $\psi(\Omega) \cap \mathbf{T}$. We choose Γ_0 as in $(2')$ so that $s_0 \in \Gamma_0$. If U_0 is as in $(2')$, then the resolvent of U_0 and, hence, the resolvent of U has finite order of growth near Γ_0.

Assume that a one-sided neighbourhood $\mathcal{U}_0$ of s_0 in Γ_0 belongs to $\sigma_{++}(U_0) \cup \rho(U_0)$. Let $s \in \mathcal{U}_0 \cap \sigma(U_0) = \mathcal{U}_0 \cap \sigma(U)$ and let $(x_n)_{n=1}^{\infty}$ be a sequence in $\mathcal{H}$ with $\|x_n\| = 1$ and $\lim_{n\to\infty} \|(U - s)x_n\| = 0$. Then $\lim_{n\to\infty} \|(U - s)F_0 x_n\| = 0$ and $\lim_{n\to\infty} \|(U-s)(1-F_0)x_n\| = 0$, which implies $\lim_{n\to\infty} \|(1-F_0)x_n\| = 0$. Therefore, by $s \in \sigma_{++}(U_0)$,

$$\liminf_{n\to\infty} [x_n, x_n] = \liminf_{n\to\infty} [F_0 x_n, F_0 x_n] \geq 0$$

and $\mathcal{U}_0 \subset \sigma_{++}(U) \cup \rho(U)$. A similar reasoning applies for $\sigma_{++}(U_0) \cup \rho(U_0)$ replaced by $\sigma_{--}(U_0) \cup \rho(U_0)$, and $(1')$ is proved.

Remark 3.8. Let A be a selfadjoint linear relation definitizable over Ω and let δ be a closed subset of $\Omega \cap \overline{\mathbf{R}}$ which is a finite union of connected sets such that all boundary points of δ in $\overline{\mathbf{R}}$ belong to $\sigma_{++}(A) \cup \sigma_{--}(A) \cup \rho(A)$. If in assertion (2) we set $K = \delta$, and Δ_0 and E_0 are as in (2), then for the extended local spectral function $E(\cdot)$ (see Theorem 2.13) we have $E(\delta) = E_0(\delta)E_0$, where $E_0(\cdot)$ is the spectral function of $A \cap (E_0\mathcal{H})^2$. This is a consequence of Remark 2.16 and the uniqueness of the extension of the local spectral function.

This work was supported by the Hochschulsonderprogramm III / 1.6 Land Brandenburg.

References

[1] Colojoară, I., C. Foiaş: Theory of generalized spectral operators, New York, 1968.

[2] Dijksma, A., H. S. V. de Snoo: Symmetric and selfadjoint relations in Krein spaces I, Operator Theory: Advances and Applications Vol. 24 (1987), Birkhäuser Verlag Basel, 145-166.

[3] Dijksma, A., H. S. V. de Snoo: Symmetric and selfadjoint relations in Krein spaces II, Ann. Acad. Sci. Fenn., Ser. A. I. Mathematica, **12** (1987), 199-216.

[4] Dunford, N., J. T. Schwartz: Linear operators. Part I, Wiley-Interscience, New-York 1958.

[5] Hoffman, K.: Banach spaces of analytic functions, Prentice-Hall, Englewood Cliffs, N.J., 1962.

[6] Jonas, P.: On the functional calculus and the spectral function for definitizable operators in Krein space, Beiträge zur Analysis **16** (1981), 121-135.

[7] Jonas, P.: On a class of unitary operators in Krein space.- Advances in Invariant Subspaces and Other Results of Operator Theory. Operator Theory: Advances and Applications **17**. Birkhäuser Verlag Basel-Boston-Stuttgart, 1986, 151-172.

[8] Jonas, P.: On a class of selfadjoint operators in Krein space and their compact perturbations, Integral Equations Operator Theory **11** (1988), 351-384.

[9] Jonas, P.: On a problem of the perturbation theory of selfadjoint operators in Krein space, J. Operator Theory, **25**(1991), 183–211.

[10] Jonas, P.: A note on perturbations of selfadjoint operators in Krein spaces. Operator Theory: Advances and Applications **43**. Birkhäuser Verlag Basel-Boston-Stuttgart, 1990, 229-235.

[11] Jonas, P.: On the trace formula of perturbation theory. I, Math.Nachr. **137** (1988), 257-281.

[12] Jonas, P.: A class of operator-valued meromorphic functions on the unit disc, Ann. Acad. Sci. Fenn., Ser. A. I. Mathematica, **17** (1992), 257-284.

[13] Jonas, P.: On the spectral theory of operators associated with perturbed Klein-Gordon and wave type equations, J. Operator Theory **29** (1993), 207-224.

[14] Jonas, P.: Operator representations of definitizable functions, Ann. Acad. Sci. Fenn., Ser. A. I. Mathematica, **25** (2000), 41-72.

[15] Köthe, G.: Die Randeigenschaften analytischer Funktionen, Math. Z. **57** (1952/53), 13 - 33.

[16] Köthe, G.: Topologische lineare Räume. I, Berlin 1960.

[17] Lancaster, P., A. Markus, V. Matsaev: Definitizable operators and quasihyperbolic operator polynomials, J. Funct. Anal. **131** (1995), 1-28.

[18] Langer, H.: Spektraltheorie linearer Operatoren in J-Räumen und einige Anwendungen auf die Schar $L(\lambda) = \lambda^2 I + \lambda B + C$, Habilitationsschrift, Dresden, 1965.

[19] Langer, H.: Spektralfunktionen einer Klasse J-selbstadjungierter Operatoren, Math. Nachr. **33** (1967), 107-120.

[20] Langer, H.: Spectral functions of definitizable operators in Krein spaces, in: Functional Analysis, Proceedings of a Conference held at Dubrovnik, Lecture Notes in Mathematics **948**, Berlin, 1982.

[21] Langer, H., A. Markus, V. Matsaev: Locally definite operators in indefinite inner product spaces, Math. Annalen **308** (1997), 405-424.

[22] Nagy, B.: Operators with spectral singularities. J. Operator Theory **15** (1986), 307-325.

[23] Shmulyan, Yu. L.: Transformers of linear relations in J-spaces. Funktsional. Anal. i Prilozh. **14** (1980), 39-44 [Russian. English translation: Functional Anal. Appl. **14** (1980), 110-113.]

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