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SUFFICIENT SECOND-ORDER OPTIMALITY CONDITIONS FOR A PARABOLIC OPTIMAL CONTROL PROBLEM WITH POINTWISE STATE CONSTRAINTS

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Abstract. An optimal control problem for a semilinear parabolic equation is investigated, where pointwise constraints are given on the control and the state. The state constraints are of mixed (bottleneck) type, where associated Lagrange multipliers can assumed to be bounded and measurable functions. Based on this property, a second-order sufficient optimality condition is established that considers strongly active constraints.

Keywords: Optimal control, heat equation, parabolic differential equation, sufficient second-order optimality condition, pointwise mixed control-state constraints, bottleneck constraints

AMS subject classification: 49K20, 90C48

1. Introduction. In this paper we consider the optimal control problem to minimize

$$F(y,u) = \int_\Omega \omega(x, y(T,x))\, dx + \int_\Sigma \sigma(t,x,y(t,x),u(t,x))\, dS_x dt + \int_Q q(t,x,y(t,x))\, dxdt \tag{1.1}$$

subject to the state equations

$$\begin{aligned} y_t + Ay &= 0 && \text{in } Q = (0,T)\times\Omega \\ \partial_n y &= b(t,x,y,u) && \text{in } \Sigma = (0,T)\times\Gamma \\ y(0,x) &= y_o(x) && \text{in } \Omega, \end{aligned} \tag{1.2}$$

and subject to the mixed control-state constraints

$$0 \le u(t,x) \le c(t,x) + \gamma(t,x)y(t,x) \quad \text{for } (t,x)\in\Sigma. \tag{1.3}$$

The main task of our paper is to establish second-order sufficient optimality conditions that are close to the associated necessary ones. For control-constrained problems, this issue was discussed quite completely in literature for semilinear elliptic and parabolic equations. We mention Bonnans [3], Casas/Tröltzsch/Unger [5], Goldberg/Tröltzsch [8], Heinkenschloss/Tröltzsch [9].

The main difficulty in our problem is the presence of the pointwise control-state constraint $u(t,x) \le c(t,x) + \gamma(t,x)y(t,x)$ in (1.3). If pointwise state contraints are given, then the theory of sufficient second-order conditions is faced with specific difficulties that are still far from being solved. In particular these problems arise for pointwise state constraints of the type $y(t,x) \le c$. Here, the difficulties are caused by the low regularity of Lagrange multipliers associated with the pointwise state constraints. The multipliers are Borel measures. We refer to Casas/Tröltzsch/Unger [6] and Raymond/Tröltzsch [12].

In our problem (1.1–1.3), the situation is slightly simpler, since the constraint (1.3) is a *mixed* control-state constraint of bottleneck type. In this case, the Lagrange multipliers are more regular, they can assumed to be functions of $L^\infty(\Sigma)$, see Bergounioux/Tröltzsch [2] or Arada/Raymond [1].

Higher regularity of the multipliers is the main advantage enabling us to establish second-order conditions. Moreover, the second-order conditions should require minimum assumptions, i.e. they should be as close as possible to associated necessary conditions. Often, this task is accomplished by considering strongly active sets (see [7] for control-constrained optimal control of ordinary differential equations). Here, we extend this technique to our case of mixed constraints. Our analysis will show that this is by far not an easy problem. It indicates again that pointwise state constraints of more general type will rise even more difficult techniques. Our paper extends the results of [15], where second-order conditions were derived for a weakly singular integral state equation. This problem covered the one-dimensional parabolic case.

The paper is organized as follows: In section 2 we formulate first and second-order optimality conditions and state the main result. Section 3 contains several auxiliary results. The proof that our second-order conditions are sufficient for local optimality is presented in section 4.

In the paper we use the following notations: By $b'(t,x,y,u)$ and $b''(t,x,y,u)$ we denote the gradient and the Hessian matrix of b with respect to (y,u):

$$b'(t,x,y,u) = \begin{pmatrix} b_y(t,x,y,u) \\ b_u(t,x,y,u) \end{pmatrix}, \quad b''(t,x,y,u) = \begin{pmatrix} b_{yy}(t,x,y,u) & b_{yu}(t,x,y,u) \\ b_{yy}(t,x,y,u) & b_{uu}(t,x,y,u) \end{pmatrix}.$$

Here we used the notation $b_y(t,x,y,u) = D_y b(t,x,y,u)$ and $b_{yy}(t,x,y,u) = D_{yy}b(t,x,y,u)$. The norms $|b'|$, $|b''|$ are defined by adding the absolute values of all entries of b' and b'', respectively. By ∂_n we denote the outward normal derivative at Γ.

The following assumptions are required:

(A1) The function $b : \Sigma \times \mathbb{R}^2 \to \mathbb{R}$, $b = b(t,x,y,u)$, satisfies the following Carathéodory condition: b is of class C^2 with respect to (y,u). Moreover, for all $(y,u) \in \mathbb{R}^2$, it is measurable with respect to t and x.

For all $M > 0$, there are a constant $C_M > 0$ and a continuous, monotone increasing function $\eta \in C(\mathbb{R}^+ \cup \{0\})$ with $\eta(0) = 0$ such that:
$|b(t,x,0,0)| + |b'(t,x,0,0)| + |b''(t,x,0,0)| \le C_M$,
$|b''(t,x,y_1,u_1) - b''(t,x,y_2,u_2)| \le \eta(|y_1 - y_2| + |u_1 - u_2|)$
for almost all (t,x) and all $|y|, |u|, |y_1|, |y_2|, |u_1|, |u_2| \le M$. The same conditions are imposed for $\sigma = \sigma(t,x,y,u)$.

In addition, we suppose on $\Sigma \times \mathbb{R}^2$ that

$$b_u(t,x,y,u) \ge 0, \quad b_y(t,x,y,u) \le 0, \quad |b(t,x,y_1,u) - b(t,x,y_2,u)| \le L|y_1 - y_2|$$

holds for all $|y|, |u| \le M$ and $y_1, y_2 \in \mathbb{R}$.

(A2) The function $\omega : \Omega \times \mathbb{R} \to \mathbb{R}$, $\omega = \omega(x,y)$, is of class C^2 with respect to y. Furthermore, ω is measurable with respect to x for all $y \in \mathbb{R}$. We assume $\omega(\cdot,0) \in L^\infty(\Omega)$, $\omega_y(\cdot,0) \in L^\infty(\Omega)$, $\omega_{yy}(\cdot,0) \in L^\infty(\Omega)$, and

$$|\omega_{yy}(x,y_1) - \omega_{yy}(x,y_2)| \le \eta(|y_1 - y_2|)$$

for almost all $x \in \Omega$ and all $|y_1| \le M$, $|y_2| \le M$.

The function $q = q(t,x,y)$ is assumed to satisfy the assumptions on ω, where Q is substituted for Ω and (t,x) is substituted for x.

(A3) We assume that $c, \gamma \in C(\bar{\Sigma})$, and $c(t,x) > 0$, $\gamma(t,x) \ge 0 \quad \forall (t,x) \in \bar{\Sigma}$. In addition, we require $y_0 \in C(\bar{\Omega})$.

(A4) The domain $\Omega \subset \mathbb{R}^m$ has a boundary Γ of class C^2. The elliptic operator A is defined by

$$Ay(x) = \sum_{i,j=1}^{m} D_i(a_{ij}(x) D_j y(x)),$$

where $a_{ij} \in C^{1,\nu}$ satisfy, for some positive m_0, the condition of ellipticity

$$\sum_{i,j=1}^{m} a_{ij}(x)\xi_i\xi_j \ge m_0|\xi|^2.$$

Other estimates of b, ω, σ, q and their first derivatives can be derived from (A1), (A2) by the mean value theorem. For convenience, in (A1) we assume a global Lipschitz continuity for b with respect to y. This is not really a strong assumption. The maximum principle of the parabolic equation ensures a-priori-bounds on the solution of the parabolic equation. Therefore, the Lipschitz continuity with respect to y is only needed on a bounded set that is predetermined by the given data.

2. First- and second-order optimality conditions. First, we introduce the spaces $V = H^1(\Omega)$ and $W(0,T) = \{v \in L^2(0,T;V) : v_t \in L^2(0,T;V^*)\}$. Since $W(0,T)$ is not embedded in the space $C(\bar{Q})$, which is needed to differentiate the superposition operators associated with the nonlinear functions ω, σ, q, and b, we fix $\alpha > \frac{m}{2} + 1$, $s > m + 1$ and introduce the state space

$$Y = \{y \in W(0,T) | y_t + Ay \in L^\alpha(Q), \partial_n y \in L^s(\Sigma), y(0) \in C(\bar{\Omega})\}$$

endowed with the norm

$$\|y\|_Y = \|y\|_{W(0,T)} + \|y_t + Ay\|_{L^\alpha(Q)} + \|\partial_n y\|_{L^s(\Sigma)} + \|y(0)\|_{C(\bar{\Omega})}.$$

Due to the choice of α and s, the embedding of Y into $C(\bar{Q})$ is continuous, [4], [13].

A function $y \in Y$ is said to be a (weak) solution of (1.2), if y satisfies the initial value problem

$$\begin{aligned} \frac{d}{dt}(y(t), v)_{L^2(\Omega)} + (\nabla y(t), \nabla v)_{L^2(\Omega)} &= (b(t, \cdot, y(t), u(t)), v)_{L^2(\Gamma)} \\ y(0, \cdot) &= y_0 \end{aligned} \tag{2.1}$$

for almost all t and all $v \in V$.

LEMMA 2.1. *For each $u \in L^\infty(\Sigma)$, equation (1.2) admits a unique solution $y \in Y$.*

For the proof we refer to [11] and [13]. In these papers, the authors use a weak solution approach. It is also possible to get a similar result by semigroup techniques, see for instance [14].

By Lemma 2.1, a solution mapping $G : L^\infty(\Sigma) \to C(\bar{Q})$ is defined that assigns to $u \in L^\infty(\Sigma)$ the solution y of (1.2). The boundary values of y are of particular importance for us. Thus we define the mapping $S : L^\infty(\Sigma) \to C(\bar{\Sigma})$ with $S = \tau G$ that assigns to u the boundary values of y. It is known from literature that G and S are twice continuously Fréchet-differentiable. Nevertheless, for our further results it is useful to briefly sketch the proof. Let $(\bar{y}, \bar{u}) \in Y \times L^\infty(\Sigma)$ be a fixed reference pair. Later, this couple will stand for a local minimum of (1.1)–(1.3). Below we use the abbreviations $\bar{b}_u = b_u(t, x, \bar{u}(t, x), \bar{y}(t, x))$, $\bar{b}_y = b_y(t, x, \bar{u}(t, x), \bar{y}(t, x))$ with $\bar{y} = S(\bar{u})$. In the same way $\bar{b}_{yy}$, $\bar{b}_{uu}$, $\bar{\sigma}_y$ etc. are defined.

LEMMA 2.2. *The nonlinear mapping $S : L^\infty(\Sigma) \to C(\bar{Q})$ is of class C^1. Its Fréchet derivative $S'(\bar{u})$ at $\bar{u}$ in direction u is given by $S'(\bar{u})u = w|_\Sigma$, where w is the solution of the initial-boundary value problem*

$$\begin{aligned} w_t + Aw &= 0 && \text{in } Q \\ \partial_n w - \bar{b}_y\, w &= \bar{b}_u\, u && \text{in } \Sigma \\ w(0, x) &= 0 && \text{in } \Omega. \end{aligned} \tag{2.2}$$

Proof. Let w be the solution of (2.2) with $u = \tilde{u} - \bar{u}$ and set

$$z := \tilde{y} - \bar{y} - w = S(\tilde{u}) - S(\bar{u}) - w. \tag{2.3}$$

Next, we perform a Taylor expansion for $b(t, x, \tilde{u}(t, x), \tilde{y}(t, x))$,

$$\begin{aligned} b(t, x, \tilde{u}(t, x), \tilde{y}(t, x)) &= b(t, x, \bar{u}(t, x), \bar{y}(t, x)) + \bar{b}_u(\tilde{u}(t, x) - \bar{u}(t, x)) \\ &\quad + \bar{b}_y(\tilde{y}(t, x) - \bar{y}(t, x)) + r(t, x). \end{aligned} \tag{2.4}$$

The remainder term $r = r(t, x)$ depends on the point $\bar{u}$ and on the direction h. It is known that

$$\frac{\|r(\bar{u}, h)\|_{L^\infty(\Sigma)}}{\|h\|_{L^\infty(\Sigma)}} \to 0 \quad \text{as } \|h\|_{L^\infty(\Sigma)} \to 0. \tag{2.5}$$

One can easily verify that z solves the initial-boundary value problem

$$\begin{aligned} z_t + Az &= 0 && \text{in } Q \\ \partial_n z - \bar{b}_y\, z &= r && \text{in } \Sigma \\ z(0, x) &= 0 && \text{in } \Omega. \end{aligned} \tag{2.6}$$

The estimate (2.5) of the remainder r implies a similar property for z,

$$\frac{\|z(\bar{u},h)\|_{C(\bar{Q})}}{\|h\|_{L^\infty(\Sigma)}} \to 0 \quad \text{as } \|h\|_{L^\infty(\Sigma)} \to 0, \tag{2.7}$$

and the differentiability of S is readily seen from $S(\tilde{u}) = S(\bar{u}) + w + z(\bar{u},h)$. □

It is possible to extend the operator $S'(\bar{u})$ to a linear continuous operator in $\mathcal{L}(L^2(\Sigma))$. From now on, we consider $S'(\bar{u})$ in this way. The known property

$$\frac{\|r(\bar{u},h)\|_{L^2(\Sigma)}}{\|h\|_{L^2(\Sigma)}} \to 0 \quad \text{as } \|h\|_{L^\infty(\Sigma)} \to 0$$

(see Maurer [10]) implies a similar property for $z|_\Sigma$

$$\frac{\|z|_\Sigma(\bar{u},h)\|_{L^2(\Sigma)}}{\|h\|_{L^2(\Sigma)}} \to 0 \quad \text{as } \|h\|_{L^\infty(\Sigma)} \to 0.$$

Next, we introduce the L^2-adjoint operator $S'(\bar{u})^* \in \mathcal{L}(L^2(\Sigma))$. This operator is given by $S'(\bar{u})^*\mu = \varphi|_\Sigma$, where φ is the solution of the well-posed parabolic backward problem

$$\begin{aligned} -\varphi_t + A^*\varphi &= 0 && \text{in } Q \\ \partial_n\varphi - \bar{b}_y\ \varphi &= \bar{b}_u\ \mu && \text{in } \Sigma \\ \varphi(T,x) &= 0 && \text{in } \Omega, \end{aligned} \tag{2.8}$$

where A^* is the formal adjoint operator to A. In all what follows let $(\bar{y},\bar{u})$ be a locally optimal reference solution of (1.1–1.3). Let us set up the associated first-order necessary optimality conditions in form of a Karush-Kuhn-Tucker type theorem.

To this aim, we introduce the Lagrange functional $L : Y \times L^\infty(\Sigma) \times Y \times L^\infty(\Sigma)^2 \to \mathbb{R}$,

$$\begin{aligned} L(y,u,p,\mu_1,\mu_2) = F(y,u) &- \int_Q (y_t + Ay)p\ dxdt - \int_\Sigma (\partial_n y - b)p\ dS_x dt \\ &- \int_\Sigma \mu_1 u\ dS_x dt + \int_\Sigma (u - c - \gamma y)\mu_2\ dS_x dt, \end{aligned}$$

where dS_x denotes the Lebesgue surface measure induced on Γ with respect to x.

Let us now comment on this choice for L. The heat quation (1.2) is considered in Y, while the inequality constraints (1.3) are posed in $L^\infty(\Sigma)$. Knowing the general Karush-Kuhn-Tucker theory in Banach spaces, one expects associated Lagrange multipliers $p \in Y^*$ and $\mu_i \in (L^\infty(\Sigma))^*$, together with a related quite complicated Lagrange functional. In contrast to this, special techniques for optimal control problems of bottleneck type have shown that, under natural assumptions, the Lagrange multipliers can be expressed by regular functions, i.e. $p \in W(0,T) \cap C(\bar{Q})$ and $\mu_i \in L^\infty(\Sigma)$, see Bergounioux/Tröltzsch [2] and Arada/Raymond [1]. This well known advantage of bottleneck type problems is our key idea to establish special second-order sufficient optimality conditions, which are hardly to be expected for $\mu_i \in (L^\infty(\Sigma))^*$. The existence of such regular multipliers can be shown under a Slater type condition and the assumption $\gamma(t,x) \geq 0$. Here, the nonnegativity of γ plays a crucial role.

Therefore we are justified to *assume* that an adjoint state $\bar{p} \in W(0,T) \cap C(\bar{Q})$ and Lagrange multipliers $\bar{\mu}_i \in L^\infty(\Sigma)$ exist such that $(\bar{y},\bar{u},\bar{p},\bar{\mu}_1,\bar{\mu}_2)$ satisfies the following first-order necessary optimality system (FON),

$$
(FON) \begin{cases}
D_y L(\bar y, \bar u, \bar p, \bar\mu_1, \bar\mu_2) & = \; 0 \\
D_u L(\bar y, \bar u, \bar p, \bar\mu_1, \bar\mu_2) & = \; 0 \\
\text{and for almost all } (t,x) \in \Sigma & \\
\bar\mu_1(t,x) & \geq \; 0 \\
\bar\mu_2(t,x) & \geq \; 0 \\
\bar u(t,x)\bar\mu_1(t,x) & = \; 0 \\
(\bar u(t,x) - c(t,x) - \gamma(t,x)\bar y(t,x))\bar\mu_2(t,x) & = \; 0.
\end{cases}
$$

Let us express these optimality conditions also in terms of partial differential equations. As it is well known, the first equation of (FON) is represented by the adjoint equation

$$
(2.9) \qquad \begin{aligned}
-\bar p_t + A^* \bar p &= q_y(t,x,\bar y) && \text{in } Q \\
\partial_n \bar p - b_y(t,x,\bar y,\bar u)\bar p &= \sigma_y(t,x,\bar y,\bar u) - \gamma \bar\mu_2 && \text{in } \Sigma \\
\bar p(T,x) &= \omega_y(x, \bar y(T,x)) && \text{in } \Omega.
\end{aligned}
$$

The second equation of (FON) is equivalent to

$$
(2.10) \qquad \sigma_u(t,x,\bar y,\bar u) + b_u(t,x,\bar y,\bar u)\bar p - \bar\mu_1 + \bar\mu_2 = 0.
$$

Next, we discuss a sufficient second-order optimality condition (SSC). For this purpose, we define *strongly active sets* and the associated *critical subspace*. Assume that $(\bar y, \bar u, \bar p, \bar\mu_1, \bar\mu_2)$ fulfils (FON).

DEFINITION 2.3. *Let $\delta_1, \delta_2 > 0$ be real numbers and $\bar\mu_1, \bar\mu_2 \in L^\infty(\Sigma)$ be the Lagrange multipliers introduced in (FON). The sets*

$$
(2.11) \qquad A_1(\delta_1) := \{(t,x) \in \Sigma : \bar\mu_1(t,x) \geq \delta_1\},
$$
$$
(2.12) \qquad A_2(\delta_2) := \{(t,x) \in \Sigma : \bar\mu_2(t,x) - (S'(\bar u)^* \gamma \bar\mu_2)(t,x) \geq \delta_2\}
$$

are called strongly active sets. We say that $(y,u) \in C(\bar Q) \times L^\infty(\Sigma)$ belongs to the critical subspace, if

$$
(2.13) \qquad u = 0 \qquad \textit{on } A_1,
$$
$$
(2.14) \qquad u = \gamma y|_\Sigma \qquad \textit{on } A_2,
$$

and

$$
(2.15) \qquad \begin{aligned}
y_t + Ay &= 0 && \textit{in } Q \\
\partial_n y - \bar b_y \; y &= \bar b_u \; u && \textit{in } \Sigma \\
y(0,x) &= 0 && \textit{in } \Omega.
\end{aligned}
$$

Notice that (2.15) implies $y|_\Sigma = S'(\bar u)u$. In (2.12), the expression $S'(\bar u)^*\gamma\bar\mu_2$ can be evaluated by solving the backward problem

$$
(2.16) \qquad \begin{aligned}
-\kappa_t + A^*\kappa &= 0 && \text{in } Q \\
\partial_n \kappa - \bar b_y \; \kappa &= \bar b_u \; \gamma \, \bar\mu_2 && \text{in } \Sigma \\
\kappa(T,x) &= 0 && \text{in } \Omega.
\end{aligned}
$$

The boundary values of κ deliver the desired expression, $\kappa|_\Sigma = S'(\bar u)^*\gamma\bar\mu_2$. Knowing κ, it is easy to determine the strongly active set A_2.

Before we formulate the second-order sufficient optimality condition, we mention for convenience the explicit expression of $L''_{(u,y)}(\bar y, \bar u, \bar p, \bar\mu_1, \bar\mu_2)[h_y, h_u]^2$:

$$
\begin{aligned}
L''_{(u,y)}(\bar y, \bar u, \bar p, \bar\mu_1, \bar\mu_2)[h_y, h_u]^2 &= \int_\Omega \bar\omega_{yy} h_y^2 \, dx + \int_Q \bar q_{yy} h_y^2 \, dxdt \\
&\quad + \int_\Sigma (\bar\sigma_{yy} h_y^2 + 2\bar\sigma_{yu} h_y h_u + \bar\sigma_{uu} h_u^2) \, dS_x dt \\
&\quad + \int_\Sigma (\bar b_{yy} h_y^2 + 2\bar b_{yu} h_y h_u + \bar b_{uu} h_u^2) \bar p \, dS_x dt. \qquad (2.17)
\end{aligned}
$$

Here, h_y, h_u denote arbitrary increments of y and u, respectively. Now we state the main result of our paper, the second-order sufficient condition.

(SSC): The coercivity condition

$$
L''_{(u,y)}(\bar y, \bar u, \bar p, \bar\mu_1, \bar\mu_2)[h_y, h_u]^2 \ge \delta \|h_u\|^2_{L^2(\Sigma)} \qquad (2.18)
$$

holds true for some $\delta > 0$ and all (h_y, h_u) belonging to the critical subspace.

THEOREM 2.4. *(Second-order sufficiency) Assume that $(\bar y, \bar u, \bar p, \bar\mu_1, \bar\mu_2)$ fulfils the first-order optimality system (FON). If the second-order condition (SSC) is satisfied, then there exist $\delta_s > 0$ and $\varepsilon > 0$ such that the quadratic growth condition*

$$
F(y, u) - F(\bar y, \bar u) \ge \delta_s \|u - \bar u\|^2_{L^2(\Sigma)} \qquad (2.19)
$$

holds for all admissible pairs (y, u) with $\|u - \bar u\|_{L^\infty(\Sigma)} < \varepsilon$. Therefore, $\bar u$ is a locally optimal control in the norm of $L^\infty(\Sigma)$.

The proof is contained in Section 4.

3. Auxiliary results. LEMMA 3.1. *Let $\beta \in L^\infty(\Sigma)$ and $f \in L^2(\Sigma)$ be given and let v be the solution of the initial boundary value problem*

$$
\begin{aligned}
v_t + Av &= 0 && \text{in } Q \\
\partial_n v + \beta v &= f && \text{in } \Sigma \\
v(0, x) &= 0 && \text{in } \Omega.
\end{aligned}
$$

If $f \ge 0$ holds a.e. on Σ, then also $v \ge 0$ holds true a.e. on Σ.

For a proof of this *comparison principle* we refer to Raymond/Zidani [13].

DEFINITION 3.2. *A continuous linear operator A in $\mathcal{L}(L^2(\Sigma))$ is said to be nonnegative, if $u \ge 0$ a.e. on Σ implies $Au \ge 0$ a.e. on Σ. In this case, we write $A \ge 0$.*

LEMMA 3.3. *(Comparison principle) Under the assumptions (A1)-(A4), the nonnegativity properties*

$$
S'(\bar u) \ge 0, \qquad (3.1)
$$

$$
(I - \gamma S'(\bar u))^{-1} \ge 0 \qquad (3.2)
$$

hold true.

Proof. The operator $S'(\bar u) : u \mapsto w_\Sigma$, is defined upon equation (2.2). In (A1) we have assumed $b_u \ge 0$. Hence Lemma 3.1, applied with $\beta = -\bar b_y$, $f = \bar b_u u$, yields that $u \ge 0$ implies $w|_\Sigma \ge 0$ and (3.1) is proved.

To prove (3.2), we apply Lemma 3.1 to the system

$$
\begin{aligned}
v_t + Av &= 0 && \text{in } Q \\
\partial_n v - (\bar b_y + \gamma \bar b_u) v &= \bar b_u u && \text{in } \Sigma \qquad (3.3) \\
v(0, x) &= 0 && \text{in } \Omega.
\end{aligned}
$$

Invoking Lemma 3.1 again, the implication $u \geq 0 \Rightarrow v|_\Sigma \geq 0$ holds. A comparison of (3.3) with (2.2) shows that

$$S'(\bar{u})(\gamma v + u) = v$$

holds. Setting $z = \gamma v + u$, we get $z = \gamma S'(\bar{u})(\gamma v + u) + u = \gamma S'(\bar{u}) z + u$, hence $z = (I - \gamma S'(\bar{u}))^{-1} u$. Thanks to the implication $u \geq 0 \Rightarrow v|_\Sigma \geq 0$, (A3), and $z = \gamma v + u$, we obtain $u \geq 0 \Rightarrow z \geq 0$. This proves (3.2). ∎

COROLLARY 3.4. *The property (3.1) extends to the adjoint operator $S'(\bar{u})^*$,*

$$S'(\bar{u})^* \geq 0. \tag{3.4}$$

In what follows, we repeatedly need controls u defined as follows: Let M_1, M_2 be disjoint measurable subsets of Σ such that $M_1 \cup M_2 = \Sigma$, and let $f \in L^\infty(\Sigma)$ be given. Then u satisfies

$$u(t,x) = \begin{cases} f(t,x) & \text{on } M_1, \\ f(t,x) + \gamma(t,x)(S'(\bar{u})u)(t,x) & \text{on } M_2. \end{cases} \tag{3.5}$$

We shall see that this setting is correct: Suppose that $u \in L^\infty(\Sigma)$ satisfies (3.5). Put $v := S'(\bar{u})u$. Then v satisfies the heat equation with homogeneous initial data and the boundary condition

$$\partial_n v - \bar{b}_y v = \begin{cases} \bar{b}_u f & \text{on } M_1, \\ \bar{b}_u (f + \gamma v) & \text{on } M_2, \end{cases} \tag{3.6}$$

that is

$$\partial_n v - (\bar{b}_y v + \chi_{M_2} \bar{b}_u \gamma) v = \bar{b}_u f \quad \text{on } \Sigma. \tag{3.7}$$

This solution v is unique. Therefore, if u satisfies (3.5), then $v = S'(\bar{u})u$ is unique, hence u is unique, because of

$$u = \begin{cases} f & \text{on } M_1, \\ f + \gamma v & \text{on } M_2. \end{cases} \tag{3.8}$$

On the other hand, starting from M_1, M_2, and f, the solution v of the heat equation with homogeneous initial data and boundary condition (3.7) is defined, and the function u given by (3.8) satisfies (3.5), since, by definition of v, $u = S'(\bar{u})v$.

Finally, by Lemma 3.1 applied to (3.7) with $f := \bar{b}_u f$, the relation $f \geq 0$ implies $v \geq 0$, hence also $u \geq 0$. In this way, we have proved

LEMMA 3.5. *For all disjoint measurable subsets M_1, M_2 of Σ with $M_1 \cup M_2 = \Sigma$ and all $f \in L^\infty(\Sigma)$, there is exactly one function $u \in L^\infty(\Sigma)$ that satisfies condition (3.4). In addition, the implication*

$$f \geq 0 \quad \Rightarrow \quad u \geq 0 \tag{3.9}$$

holds true. Moreover, the estimates

$$\|u\|_{L^\infty(\Sigma)} \leq c_1 \|\tilde{f}\|_{L^\infty(\Sigma)}, \tag{3.10}$$

$$\|u\|_{L^2(\Sigma)} \leq c_2 \|\tilde{f}\|_{L^2(\Sigma)} \tag{3.11}$$

hold with certain constants c_1, c_2 that do not depend on M_1, M_2, and f.

The estimates (3.10) and (3.11) follow immediately from those for v and (3.8).

To prove the main result, we later have to compare the reference pair $(\bar{y}, \bar{u})$ with another admissible pair (y,u), where $y = S(u)$. Then we have to estimate the difference

$$y - \bar{y} = S(u) - S(\bar{u}) = S'(\bar{u})(u - \bar{u}) + r_1(\bar{u}, u - \bar{u}),$$

where r_1 stands for the associated first-order remainder term. In the following, we denote for short the remainder $r_1(\bar{u}, u - \bar{u})$ and the derivative $S'(\bar{u})$ by r_1 and S', respectively, if there is no risk of notational confusion.

Before continuing our analysis of second-order suffiency, we briefly discuss the main difficulties and our main ideas to resolve them. We start with the case of pure control constraints, which is covered by our setting for $\gamma(t,x) \equiv 0$. Then the constraints are simple box constraints,

$$0 \le u(t,x) \le c(t,x).$$

On A_1, we have $\bar{u}(t,x) \equiv 0$, hence $u - \bar{u} \ge 0$ on A_1, while $\bar{u}(t,x) = c(t,x)$ holds on A_2, thus $u - \bar{u} \le 0$ on A_2. The associated terms in the Lagrange functional can be estimated as

$$\begin{aligned}\int_{A_1} \bar{\mu}_1(u-\bar{u})\, dS_x dt - \int_{A_2} \bar{\mu}_2(u-\bar{u})\, dS_x dt &\ge \int_{A_1} \delta_1(u-\bar{u})\, dS_x dt + \int_{A_2} \delta_2(u-\bar{u})\, dS_x dt \\ &= \delta_1 \|u-\bar{u}\|_{L^1(A_1)} + \delta_2 \|u-\bar{u}\|_{L^1(A_2)}.\end{aligned}$$

In the proof of the suffiency theorem the L^1-norms on the right-hand side will compensate for the lack of coercivity, since (2.18) does not help on $A_1 \cup A_2$.

Now we return to the given mixed control-state constraints

$$0 \le u(t,x) \le c(t,x) + \gamma(t,x)\, y(t,x).$$

To simplify our explanation, assume for a while that the control-state mapping is linear. This holds for $y_0 = 0$ functions b being linear with respect to y and u. Then $S' = S$, hence

$$0 \le u \le c + \gamma S' u \tag{3.12}$$

holds for any admissible control u. On A_1, we have again $0 = \bar{u} \le u$, hence $u - \bar{u} \ge 0$ on A_1. However, in contrast to the case of pure control constraints, the relation $u \le \bar{u}$ cannot be expected on A_2 now. If $u > \bar{u}$ holds somewhere on $\Sigma \setminus A_2$, then $S'u > S'\bar{u}$ can hold on A_2. Then the right-hand side of (3.12) is greater than $c + \gamma S'\bar{u}$ and $u > \bar{u}$ can happen.

To overcome this difficulty, we represent u in the form $u = u_1 + u_2$, such that $u_1 \le \bar{u}$ can be shown on A_2 and u_2 stands for the additional margin of freedom that is caused by $u > \bar{u}$ outside of A_2. Hence we split u in two parts, $u = u_1 + u_2$ on Σ, where

$$\begin{array}{lll} u_1 = \bar{u}, & u_2 = u - \bar{u} & \text{on } \Sigma \setminus A_2, \\ u_2 = \gamma(S'u_2 + r_1), & u_1 = u - u_2 & \text{on } A_2. \end{array} \tag{3.13}$$

The functions u_1 and u_2 are well defined. To see this, we apply Lemma 3.5, where $M_1 = \Sigma \setminus A_2$ and $M_2 = A_2$. On M_1, u_2 is given by $u - \bar{u}$. On M_2, $u_2 = \gamma r_1 + \gamma S' u_2$, hence

$$u_2 = \begin{cases} f & \text{on } M_1, \\ f + \gamma S' u_2 & \text{on } M_2. \end{cases}$$

where $f = u - \bar{u}$ on M_1, $f = \gamma r_1$ on M_2. Then u_2 is well defined by Lemma 3.5. Note that $S'u_2 = S'(\bar{u})(\chi_{M_1}(u - \bar{u}) + \chi_{M_2} u_2)$. From (3.10) and the properties of the remainder r_1 we get easily

$$\|u_2\|_{L^\infty(\Sigma)} \le c_3 \|u - \bar{u}\|_{L^\infty(\Sigma)}.$$

Therefore, we find

$$\begin{aligned}\|u_1 - \bar{u}\|_{L^\infty(A_2)} &\le \|u - \bar{u}\|_{L^\infty(A_2)} + \|u_2\|_{L^\infty(A_2)} \\ &\le c_4 \|u - \bar{u}\|_{L^\infty(\Sigma)}.\end{aligned} \tag{3.14}$$

Lemma 3.6. *It holds*

$$\bar{u} - u_1 \ge 0 \quad \textit{a.e. on } \Sigma. \tag{3.15}$$

Proof. On A_2, the inequality, $\bar{\mu}_2 \geq \bar{\mu}_2 - \gamma S'(\bar{u})^*\bar{\mu}_2 \geq \delta_2 > 0$ holds. Therefore, (FON) implies $\bar{u} = c + \gamma\bar{y}$ there. In addition, we know on A_2 that $u - \gamma y \leq c = \bar{u} - \gamma\bar{y}$, hence

$$u - \gamma S(u) \leq \bar{u} - \gamma S(\bar{u})$$

holds there. In view of this, we find on A_2

$$\begin{aligned} u - \gamma(S(u) - S(\bar{u})) &\leq \bar{u} \\ u - \gamma(S'(\bar{u})(u - \bar{u}) + r_1) &\leq \bar{u} \\ u_1 - \gamma S' u_1 + (u_2 - \gamma(S' u_2 + r_1)) &\leq \bar{u} - \gamma S'\bar{u} \\ u_1 - \gamma S' u_1 &\leq \bar{u} - \gamma S'\bar{u} \\ (I - \gamma S')(u_1 - \bar{u}) &\leq 0, \end{aligned} \tag{3.16}$$

where we have inserted the definition of u_2. Outside of A_2, it holds by definition $u_1 = \bar{u}$. We are now again in the situation that was described in Lemma 3.5. Indeed, taking $M_1 := \Sigma \setminus A_2$, $M_2 = A_2$, $f = 0$ on M_1 and $f = (I - \gamma S')(\bar{u} - u_1)$ on M_2, we have $f \geq 0$. Applying (3.9), we obtain

$$\bar{u} - u_1 \geq 0 \quad \text{a.e. on } \Sigma,$$

which is just inequality (3.15). ∎

LEMMA 3.7. *Assume that $(\bar{y}, \bar{u}, \bar{p}, \bar{\mu}_1, \bar{\mu}_2)$ fulfil the first-order optimality system (FON). Then the estimates*

$$\int_\Sigma (u - \bar{u})\bar{\mu}_1 \, dS_x dt \geq \frac{\delta_1}{\varepsilon} \|u - \bar{u}\|^2_{L^2(A_1)}, \tag{3.17}$$

$$-\int_\Sigma (u - \bar{u} - \gamma(y - \bar{y}))\bar{\mu}_2 \, dS_x dt \geq \frac{\delta_2}{c_4 \varepsilon} \|u_1 - \bar{u}\|^2_{L^2(A_2)} \tag{3.18}$$

are valid for all $\varepsilon > 0$ and all admissible pairs (u, y) satisfying $\|u - \bar{u}\|_{L^\infty(\Sigma)} < \varepsilon$.

Proof. **(i)** Because of (FON), $\bar{\mu}_1 > 0$ can only hold if $\bar{u} = 0$. If $\bar{u} > 0$, then $\bar{\mu}_1 = 0$. Moreover, u is admissible, hence $u \geq 0$ and we have almost everywhere

$$(u - \bar{u})\bar{\mu}_1 \geq 0.$$

Therefore we get by (2.11)

$$\int_\Sigma (u - \bar{u})\bar{\mu}_1 \, dS_x dt \geq \int_{A_1} (u - \bar{u})\bar{\mu}_1 \, dS_x dt \geq \delta_1 \|u - \bar{u}\|_{L^1(A_1)}.$$

By our assumption, we have $\|u - \bar{u}\|_{L^\infty(\Sigma)} < \varepsilon$. In particular this inequality includes $\|u - \bar{u}\|_{L^\infty(A_1)} < \varepsilon$. Consequently,

$$\int_\Sigma (u - \bar{u})\bar{\mu}_1 \, dS_x dt \geq \delta_1 \|u - \bar{u}\|_{L^1(A_1)} \frac{\|u - \bar{u}\|_{L^\infty(A_1)}}{\varepsilon} \geq \frac{\delta_1}{\varepsilon} \|u - \bar{u}\|^2_{L^2(A_1)},$$

and (3.17) is proved.

(ii) Next, we discuss the integral in (3.18). Because of (FON), $\bar{\mu}_2 > 0$ can only hold for $\bar{u} - c - \gamma\bar{y} = 0$. In addition, (y, u) is admissible, hence in particular $u \leq c + \gamma y$. Therefore, we obtain almost everywhere

$$-(u - \bar{u} - \gamma(y - \bar{y}))\bar{\mu}_2 \geq 0$$

and

$$-\int_\Sigma (u - \bar{u} - \gamma(y - \bar{y}))\bar{\mu}_2 \, dS_x dt \geq -\int_{A_2} (u - \bar{u} - \gamma(y - \bar{y}))\bar{\mu}_2 \, dS_x dt.$$

Let us discuss this integral more detailed. Expressing $y-\bar{y}$ in terms of the controls,

$$-\int_{A_2}(u-\bar{u}-\gamma(y-\bar{y}))\bar{\mu}_2\,dS_xdt=-\int_{A_2}(u-\bar{u}-\gamma(S'(\bar{u})(u-\bar{u})+r_1))\bar{\mu}_2\,dS_xdt \tag{3.19}$$

is found. The definition of u_1 and u_2 yields on A_2

$$u-\gamma(S'u+r_1)=u_1+u_2-\gamma S'u_1-\gamma S'u_2-\gamma r_1=u_1-\gamma S'u_1.$$

Inserting the last equation in (3.19), we continue by

$$\begin{aligned}\int_{A_2}(u-\bar{u}-\gamma(y-\bar{y}))\bar{\mu}_2\,dS_xdt&=\int_{A_2}(u_1-\bar{u}-\gamma(S'(\bar{u})(u_1-\bar{u})))\bar{\mu}_2\,dS_xdt\\&=\int_{\Sigma}(u_1-\bar{u}-\gamma(S'(\bar{u})(u_1-\bar{u})))\chi_{A_2}\bar{\mu}_2\,dS_xdt\\&=\int_{\Sigma}(u_1-\bar{u})(I-(\gamma S')^*)(\chi_{A_2}\bar{\mu}_2)\,dS_xdt\\&=\int_{A_2}(u_1-\bar{u})(I-(\gamma S')^*)(\chi_{A_2}\bar{\mu}_2)\,dS_xdt.\end{aligned} \tag{3.20}$$

To deduce the last equation, we used $\bar{u}-u_1=0$ outside of A_2. Now we discuss the expression $(I-(\gamma S')^*)(\chi_{A_2}\bar{\mu}_2)$ in (3.20). On A_2 we have

$$(I-(\gamma S')^*)(\chi_{A_2}\bar{\mu}_2)=\chi_{A_2}\bar{\mu}_2-(\gamma S')^*(\chi_{A_2}\bar{\mu}_2)=\bar{\mu}_2-(\gamma S')^*(\chi_{A_2}\bar{\mu}_2).$$

Using the non-negativity of S'^* following from (3.4), together with $\chi_{A_2}\bar{\mu}_2\le\bar{\mu}_2$, we obtain

$$(\gamma S')^*(\chi_{A_2}\bar{\mu}_2)=(S')^*(\gamma\chi_{A_2}\bar{\mu}_2)\le(S')^*(\gamma\bar{\mu}_2)=(\gamma S')^*\bar{\mu}_2.$$

Combining these results, we continue by

$$(I-(\gamma S')^*)(\chi_{A_2}\bar{\mu}_2)=\bar{\mu}_2-(\gamma S')^*(\chi_{A_2}\bar{\mu}_2)\ge(I-(\gamma S')^*)\bar{\mu}_2\ge\delta_2, \tag{3.21}$$

where the last inequality follows from the definition (2.12) of A_2. Inserting (3.15) and (3.21) in (3.20), we infer

$$\begin{aligned}-\int_{A_2}(u-\bar{u}-\gamma(y-\bar{y}))\bar{\mu}_2\,dS_xdt&=-\int_{A_2}(u_1-\bar{u})(I-(\gamma S')^*)(\chi_{A_2}\bar{\mu}_2)\,dS_xdt\\&\ge\delta_2\|\bar{u}-u_1\|_{L^1(A_2)}.\end{aligned}$$

Invoking again $\|u-\bar{u}\|_{L^\infty(\Sigma)}<\varepsilon$ and (3.14), we obtain

$$\begin{aligned}-\int_{A_2}(u-\bar{u}-\gamma(y-\bar{y}))\bar{\mu}_2\,dS_xdt&\ge\delta_2\|u_1-\bar{u}\|_{L^1(A_2)}\cdot\frac{\|u-\bar{u}\|_{L^\infty(A_2)}}{\varepsilon}\\&\ge\frac{\delta_2}{c_4\varepsilon}\|u_1-\bar{u}\|^2_{L^2(A_2)},\end{aligned}$$

implying inequality (3.18). □

If $A_1\cup A_2=\Sigma$, then the critical subspace contains only the zero-function. Then the assumptions of Theorem 2.4 are trivially fulfilled. In this case, (3.17) and (3.18) express the so-called *first-order sufficient optimality conditions*.

4. Second-order sufficient optimality condition. This section includes the proof of the sufficiency Theorem 2.4. We start from an admissible control u in a sufficiently small L^∞-neighborhood of $\bar{u}$ and have to show that $F(y,u)\ge F(\bar{y},\bar{u})$. Let us introduce the increments $\delta u:=u-\bar{u}$ and $\delta y:=S'(\bar{u})\delta u$. We split $\delta u=u_0+u_+$, where

$$\begin{array}{lll}u_0=0, & u_+=\delta u & \text{on } A_1,\\ u_0=\delta u, & u_+=0 & \text{on } \Sigma\setminus(A_1\cup A_2),\\ u_0=\gamma S'(\bar{u})u_0, & u_+=\delta u-u_0 & \text{on } A_2.\end{array}$$

Thanks to Lemma 3.5, the definition of u_0 and hence u_+ is correct. We take $M_1 = \Sigma \backslash A_2$, $M_2 = A_2$, $f := 0$ on $A_1 \cup A_2$, $f := \delta u$ on $\Sigma \setminus (A_1 \cup A_2)$. The part u_0 belongs to the critical subspace, while u_+ is the part of δu that accounts for the effects of first-order sufficiency. Furthermore, we define $y_0 := S' u_0$ and $y_+ := S' u_+$. By the linearity of S', we have $\delta y = y_0 + y_+$.

In the Lemma below, we estimate the difference $L(y, u, \bar{p}, \bar{\mu}_1, \bar{\mu}_2) - L(\bar{y}, \bar{u}, \bar{p}, \bar{\mu}_1, \bar{\mu}_2)$. Let us write for short $L(y, u) - L(\bar{y}, \bar{u})$, since $(\bar{p}, \bar{\mu}_1, \bar{\mu}_2)$ remains fixed in all the next considerations. We also do not explicitely indicate the point $(\bar{y}, \bar{u}, \bar{p}, \bar{\mu}_1, \bar{\mu}_2)$ where all derivatives are taken, i.e. we write $L_u u$ instead of $(D_u L)(\bar{y}, \bar{u}, \bar{p}, \bar{\mu}_1, \bar{\mu}_2) u$.

LEMMA 4.1. *Under the assumptions of Theorem 2.4,*

$$L(y, u) - L(\bar{y}, \bar{u}) \geq \frac{\delta}{4} \|u_0\|^2_{L^2(\Sigma)} - \frac{c_s}{2} \|u_+\|^2_{L^2(\Sigma)} + r_2 + \tilde{r}_2 \tag{4.1}$$

holds, where r_2, $\tilde{r}_2$ are second-order remainder terms with

$$\frac{|r_i|}{\|u - \bar{u}\|^2_{L^2(\Sigma)}} \to 0 \quad \textit{if } \|u - \bar{u}\|_{L^\infty(\Sigma)} \to 0.$$

Proof. Using a Taylor expansion, in view of (FON) we get

$$\begin{aligned} L(y, u) - L(\bar{y}, \bar{u}) &= L_u[u - \bar{u}] + L_y[y - \bar{y}] + \frac{1}{2}(L_{uu}[u - \bar{u}]^2 \\ &\quad + 2L_{uy}[u - \bar{u}, y - \bar{y}] + L_{yy}[y - \bar{y}]^2) + r_2 \\ &= \frac{1}{2}(L_{uu}[u - \bar{u}]^2 + 2L_{uy}[u - \bar{u}, y - \bar{y}] + L_{yy}[y - \bar{y}]^2) + r_2. \end{aligned} \tag{4.2}$$

The following property of the remainder is known

$$\frac{|r_2(\bar{u}, h)|}{\|h\|^2_{L^2(\Sigma)}} \to 0 \quad \text{as } \|h\|_{L^\infty(\Sigma)} \to 0.$$

For the proof we refer to [16]. According to the notation of Lemma 3.5, we get $y - \bar{y} = \delta y + r_1$. Replacing $y - \bar{y}$ by δy in (4.2), we cause a small error of second order

$$\begin{aligned} \tilde{r}_2 := &\frac{1}{2}(L_{uu}[u - \bar{u}]^2 + 2L_{uy}[u - \bar{u}, y - \bar{y}] + L_{yy}[y - \bar{y}]^2) \\ &- \frac{1}{2}(L_{uu}[\delta u]^2 + 2L_{uy}[\delta u, \delta y] + L_{yy}[\delta y]^2). \end{aligned}$$

It is easy to show that

$$\frac{|\tilde{r}_2|}{\|u - \bar{u}\|^2_{L^2(\Sigma)}} \to 0 \quad \text{as } \|u - \bar{u}\|_{L^\infty(\Sigma)} \to 0.$$

With these notations, we can express (4.2) in the form

$$L(y, u) - L(\bar{y}, \bar{u}) = \frac{1}{2}(L_{uu}[\delta u]^2 + 2L_{uy}[\delta u, \delta y] + L_{yy}[\delta y]^2) + r_2 + \tilde{r}_2. \tag{4.3}$$

We continue by splitting the Lagrange functional,

$$\begin{aligned} L_{uu}[\delta u]^2 + 2L_{uy}[\delta u, \delta y] + L_{yy}[\delta y]^2 = &L_{uu}[u_0]^2 + 2L_{uy}[u_0, y_0] + L_{yy}[y_0]^2 \\ &+ L_{uu}[u_+]^2 + 2L_{uy}[u_+, y_+] + L_{yy}[y_+]^2 \\ &+ 2L_{uu}[u_0, u_+] + 2L_{uy}[u_0, y_+] \\ &+ 2L_{uy}[u_+, y_0] + 2L_{yy}[y_0, y_+]. \end{aligned}$$

As u_0 belongs to the critical subspace, the second-order condition (SSC) yields

$$L''[u_0, y_0]^2 = L_{uu}[u_0]^2 + 2L_{uy}[u_0, y_0] + L_{yy}[y_0]^2 \geq \delta \|u_0\|^2_{L^2(\Sigma)}.$$

The other terms are easily estimated by $\|y_0\|^2_{L^2(\Sigma)} \le \|S'\|^2\|u_0\|^2_{L^2(\Sigma)}$, $\|y_+\|^2_{L^2(\Sigma)} \le \|S'\|^2\|u_+\|^2_{L^2(\Sigma)}$, and by means of Young's inequality,

$$\begin{aligned}|L_{uu}[u_+]^2 &+ 2L_{uy}[u_+, y_+] + L_{yy}[y_+]^2 \\ &+2L_{uu}[u_0, u_+] + 2L_{uy}[u_0, y_+] \\ &+2L_{uy}[u_+, y_0] + 2L_{yy}[y_0, y_+]| \le \frac{\delta}{2}\|u_0\|^2_{L^2(\Sigma)} + c_s\|u_+\|^2_{L^2(\Sigma)}.\end{aligned}$$

In this setting, c_s is a certain (large) constant. Combining the last two results, we arrive at

$$L_{uu}[\delta u]^2 + 2L_{uy}[\delta u, \delta y] + L_{yy}[\delta y]^2 \ge \frac{\delta}{2}\|u_0\|^2_{L^2(\Sigma)} - c_s\|u_+\|^2_{L^2(\Sigma)}.$$

Returning to (4.3), we end up with

$$L(y, u) - L(\bar{y}, \bar{u}) \ge \frac{\delta}{4}\|u_0\|^2_{L^2(\Sigma)} - \frac{c_s}{2}\|u_+\|^2_{L^2(\Sigma)} + r_2 + \tilde{r}_2,$$

which is exactly the assertion. ☐

In the next Lemma, we estimate the term $\|u_+\|^2_{L^2(\Sigma)}$ in (4.1).

LEMMA 4.2. *Under the assumptions of Theorem 2.4,*

$$\left(\frac{c_s}{2} + \frac{\delta}{4}\right)\|u_+\|^2_{L^2(\Sigma)} \le c_5\|u_1 - \bar{u}\|^2_{L^2(A_2)} + c_6\|r_1\|^2_{L^2(\Sigma)} + c_7\|u - \bar{u}\|^2_{L^2(A_1)} \tag{4.4}$$

holds with certain positive constants c_5, c_6, and c_7.

Proof. First, we get on A_1

$$\|u_+\|_{L^2(A_1)} = \|\delta u\|_{L^2(A_1)} = \|u - \bar{u}\|_{L^2(A_1)}. \tag{4.5}$$

On the whole set Σ we have

$$u_+ + u_0 = \delta u = u - \bar{u}.$$

We apply the operator $I - \gamma S'$ to this equation and consider the image only on the set A_2. Using $u_0 = \gamma S' u_0$ on A_2, we find

$$u_+ - \gamma S' u_+ = u - \gamma S' u - (\bar{u} - \gamma S' \bar{u}) \text{ on } A_2.$$

Now, u is again replaced by $u_1 + u_2$, see (3.13), to obtain on A_2

$$u_+ - \gamma S' u_+ = u_1 - \gamma S' u_1 + u_2 - \gamma S' u_2 - (\bar{u} - \gamma S' \bar{u}).$$

On A_2, by definition, the equation $u_2 - \gamma S' u_2 = r_1$ is satisfied. Therefore, here we are able to continue by

$$u_+ - \gamma S' u_+ = u_1 - \bar{u} - (\gamma S'(\bar{u})(u_1 - \bar{u})) + r_1 \text{ on } A_2.$$

Due to our definitions, $u_+ = \delta u = u - \bar{u}$ holds on A_1. In addition, u_+ vanishes on $\Sigma \setminus (A_1 \cup A_2)$. Therefore, we find

$$u_+ = \begin{cases} u_1 - \bar{u} + \gamma S'(\bar{u})(u_+ - u_1 + \bar{u}) + r_1 & \text{on } A_2 \\ u - \bar{u} & \text{on } A_1 \\ 0 & \text{on } \Sigma \setminus (A_1 \cup A_2). \end{cases}$$

Again we have a construction that was investigated in Lemma 3.5. Setting $M_2 = A_2$, $M_1 = \Sigma \setminus A_2$ and applying (3.11), we get the inequality

$$\|u_+\|_{L^2(\Sigma)} \le c_2\|f\|_{L^2(\Sigma)},$$

where f is defined by

$$f = \begin{cases} r_1 + (u_1 - \bar{u}) - \gamma S'(\bar{u})(u_1 - \bar{u}) & \text{on } A_2 \\ u - \bar{u} & \text{on } A_1 \\ 0 & \text{on } \Sigma \setminus (A_1 \cup A_2). \end{cases}$$

Therefore, we obtain

$$\|u_+\|_{L^2(\Sigma)} \le c_2(\|u - \bar{u}\|_{L^2(A_1)} + c_8\|u_1 - \bar{u}\|_{L^2(\Sigma)} + \|r_1\|_{L^2(A_2)}),$$

where the positive constant c_8 is related to $\|S'\|$. Using $\|u_1 - \bar{u}\|_{L^2(\Sigma)} = \|u_1 - \bar{u}\|_{L^2(A_2)}$,

$$\|u_+\|_{L^2(\Sigma)} \le c_9\|u_1 - \bar{u}\|_{L^2(A_2)} + c_2\|r_1\|_{L^2(A_2)} + c_2\|u - \bar{u}\|_{L^2(A_1)}$$

is found. Young's inequality yields

$$\|u_+\|^2_{L^2(\Sigma)} \le 3c_9\|u_1 - \bar{u}\|^2_{L^2(A_2)} + 3c_2\|r_1\|^2_{L^2(\Sigma)} + 3c_2\|u - \bar{u}\|^2_{L^2(A_1)}.$$

A multiplication by $(\frac{c_s}{2} + \frac{\delta}{4})$

$$\left(\frac{c_s}{2} + \frac{\delta}{4}\right) \|u_+\|^2_{L^2(\Sigma)} \le c_5\|u_1 - \bar{u}\|^2_{L^2(A_2)} + c_6\|r_1\|^2_{L^2(\Sigma)} + c_7\|u - \bar{u}\|^2_{L^2(A_1)}$$

concludes the proof of the Lemma. ▯

Now we are able to prove our main result Theorem 2.4.

Proof. (Theorem 2.4) Inserting (4.4) in (4.1),

$$\begin{aligned} L(y,u) - L(\bar{y},\bar{u}) \ge{}& \frac{\delta}{4}(\|u_0\|^2_{L^2(\Sigma)} + \|u_+\|^2_{L^2(\Sigma)}) + r_2 + \tilde{r}_2 \\ & -c_7\|u - \bar{u}\|^2_{L^2(A_1)} - c_5\|u_1 - \bar{u}\|^2_{L^2(A_2)} - c_6\|r_1\|^2_{L^2(\Sigma)} \end{aligned}$$

is obtained. Next, we return to the objective F,

$$\begin{aligned} L(y,u) - L(\bar{y},\bar{u}) &= F(y,u) - F(\bar{y},\bar{u}) \\ &\quad - \int_Q (y_t + Ay)\bar{p}\, dxdt - \int_\Sigma (\partial_n y - b)\bar{p}\, dS_x dt \\ &\quad + \int_Q (\bar{y}_t + A\bar{y})\bar{p}\, dxdt + \int_\Sigma (\partial_n \bar{y} - \bar{b})\bar{p}\, dS_x dt \\ &\quad - \int_\Sigma \bar{\mu}_1 u\, dS_x dt + \int_0^T \bar{\mu}_1 \bar{u}\, dS_x dt \\ &\quad + \int_\Sigma (u - c - \gamma y)\bar{\mu}_2\, dS_x dt \\ &\quad - \int_\Sigma (\bar{u} - c - \gamma \bar{y})\bar{\mu}_2\, dS_x dt \\ &= F(y,u) - F(\bar{y},\bar{u}) \\ &\quad - \int_\Sigma \bar{\mu}_1(u - \bar{u})\, dS_x dt \\ &\quad + \int_\Sigma (u - \bar{u} - \gamma(y - \bar{y})\bar{\mu}_2\, dS_x dt. \end{aligned}$$

Using Lemma 3.7 we find

$$\begin{aligned} F(y,u) - F(\bar{y},\bar{u}) \ge{}& \frac{\delta}{4}(\|u_0\|^2_{L^2(\Sigma)} + \|u_+\|^2_{L^2(\Sigma)}) + r_2 + \tilde{r}_2 \\ & +(\frac{\delta_1}{\varepsilon} - c_7)\|u - \bar{u}\|^2_{L^2(A_1)} + (\frac{\delta_2}{c_4\varepsilon} - c_5)\|u_1 - \bar{u}\|^2_{L^2(A_2)} \\ & -c_6\|r_1\|^2_{L^2(\Sigma)}. \end{aligned} \tag{4.6}$$

Next, $\|\delta u\|_{L^2(\Sigma)} = \|u_0 + u_+\|^2_{L^2(\Sigma)} \leq 2\|u_0\|^2_{L^2(\Sigma)} + 2\|u_+\|^2_{L^2(\Sigma)}$ is applied to continue by

$$
\begin{aligned}
F(y,u) - F(\bar{y},\bar{u}) \geq{}& \frac{\delta}{8}\|\delta u\|^2_{L^2(\Sigma)} + r_2 + \tilde{r}_2 \\
&+(\frac{\delta_1}{\varepsilon} - c_7)\|u-\bar{u}\|^2_{L^2(A_1)} + (\frac{\delta_2}{c_4\varepsilon} - c_5)\|u_1 - \bar{u}\|^2_{L^2(A_2)} \\
&-c_6\|r_1\|^2_{L^2(\Sigma)}.
\end{aligned} \tag{4.7}
$$

Take now ε sufficiently small, such that

$$\frac{\delta_1}{\varepsilon} - c_7 \geq 0 \quad \text{and} \quad \frac{\delta_2}{c_4\varepsilon} - c_5 \geq 0.$$

Then we can omit the associated terms in (4.7),

$$F(y,u) - F(\bar{y},\bar{u}) \geq \frac{\delta}{8}\|\delta u\|^2_{L^2(\Sigma)} + r_2 + \tilde{r}_2 - c_6\|r_1\|^2_{L^2(\Sigma)}. \tag{4.8}$$

Due to the discussions during the proof, all terms of the right-hand side (except the first one) are small with respect to $\|u-\bar{u}\|^2_{L^2(\Sigma)}$. Therefore

$$F(y,u) - F(\bar{y},\bar{u}) \geq \frac{\delta}{16}\|u-\bar{u}\|^2_{L^2(\Sigma)} \tag{4.9}$$

holds if $\|u-\bar{u}\|_{L^\infty(\Sigma)} < \varepsilon$ and ε is sufficiently small. The quadratic growth condition is proved. We can choose $\delta_s = \delta/16$. □

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