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Abstract. The regularity and stability of optimal controls of instationary Navier-Stokes equations is investigated. Under suitable assumptions every control satisfying first-order necessary conditions is shown to be a continuous function in both space and time. Moreover, the behaviour of a locally optimal control under certain perturbations of the cost functional and the state equation is investigated. Lipschitz stability is proven provided a second-order sufficient optimality condition holds.

Key Words. Optimal control, Navier-Stokes equations, control constraints, Lipschitz stability

AMS Subject Classifications. Primary 49N60, Secondary 49K40

1. Introduction. We are considering optimal control of the instationary Navier-Stokes equations. As model problem serves the minimization of the quadratic objective functional

$$\min J(y,u) = \frac{\alpha_T}{2}\int_\Omega |y(x,T)-y_T(x)|^2dx + \frac{\alpha_Q}{2}\int_Q |y(x,t)-y_Q(x,t)|^2dxdt + \frac{\alpha_R}{2}\int_Q |\operatorname{rot} y(x,t)|^2dxdt + \frac{\gamma}{2}\int_Q |u(x,t)|^2dxdt \quad (1.1)$$

subject to the instationary Navier-Stokes equations

$$\begin{aligned} y_t - \nu\Delta y + (y\cdot\nabla)y + \nabla p &= u \quad \text{in } Q,\\ \operatorname{div} y &= 0 \quad \text{in } Q,\\ y(0) &= y_0 \quad \text{in } \Omega, \end{aligned} \quad (1.2)$$

and the control constraints $u \in U_{ad}$ with control set defined by

$$U_{ad} = \{u \in L^2(Q)^n :\ u_{a,i}(x,t) \le u_i(x,t) \le u_{b,i}(x,t) \text{ a.e. on } Q,\ i = 1..2\}.$$

Here, Ω is an open bounded subset of $\mathbb{R}^2$ with C^3-boundary Γ, such that Ω is locally on one side of Γ, and Q is defined by $Q = (0,T)\times\Omega$. Further, functions $y_T \in L^2(\Omega)^2$, $y_Q \in L^2(Q)^2$, and $y_0 \in H \subset L^2(\Omega)^2$ are given. The parameters γ and ν are positive real numbers. The constraints u_a, u_b are required to be in $L^2(Q)^2$ with $u_{a,i}(x,t) \le u_{b,i}(x,t)$ a.e. on Q, $i = 1,2$.

In this article, we deal with two questions arising in the optimal control of partial differential equations:

1. How smooth is a locally optimal control?
2. Does a locally optimal control enjoy stability under perturbations of the data?

Both are related in the following sense: if one has that the optimal control enjoy some regularity then one wants that under reasonable perturbations this regularity is preserved.

Actually, regularity results for optimal controls can be derived from the first order necessary optimality system. It introduces some coupling between the control and

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the adjoint state. The adjoint state itself is solution of a partial differential equation and therefore has some regularity, which is inherited by an optimal control. We want to show that any control satisfying the first-order necessary optimality conditions of problem (1.1) is a continuous function in both - space and time.

Since the 1980s the investigation of stability of optimal controls has attracted much interest. Once a stability result holds true, one easily can prove convergence of numerical methods such as the SQP-method for instance, [5]. The first stability results for optimal control of partial differential equations are due to Tröltzsch [26], where a linar-quadratic control problem is studied. For the treatment of general state equations including instationary ones we refer to [7, 13, 17, 27] and the references cited therein. For the control of the stationary Navier-Stokes system, we refer to [20]. The stability of optimal controls of the instationary Navier-Stokes equations was presented in the recent research paper [11].

The control of instationary Navier-Stokes flow has been studied very intensively since the pioneering work [1], see for instance [3, 6, 8, 9, 10, 11, 12, 13, 22, 28]. Stability problems were addressed in [11] to prove convergence of the SQP-method.

Since we will show that optimal controls of (1.1) has to be continuous, we will give stability results in the associated L^∞-norm. This extends the results obtained in [11], where stability of optimal controls in $L^s(Q)^2$, $s < 7/2$, was achieved. But, it requires a change in the methods too. Using Hilbert-space theory of the instationary Navier-Stokes equations one can prove stability of optimal controls in the space $L^s(0,T;L^\infty(\Omega)^2)\cap L^\infty(0,T;L^s(\Omega)^2)$ only for $s<\infty$. We will close the gap to $s=\infty$ employing a L^p-solution theory due to v.Wahl [29, 30].

The outline of the article is as follows. In Section 2, we will introduce some notation and state common results concerning solvability of the instationary Navier-Stokes system (1.2). Section 3 contains a brief summary of known facts about optimality conditions. The continuity of optimal controls is proven in Section 4. Finally, Section 5 is devoted to the study of stability of optimal controls.

2. Notations and preliminary results. Here, we will restrict ourselve to the two-dimensional case, $n = 2$, since a satisfactory theory of the instationary Navier-Stokes equations is only available for this space dimension. First, we introduce some notations and provide some results that we need later on.

To begin with, we define the solenoidal spaces

$$H_p := \{v \in L^p(\Omega)^2 : \operatorname{div} v = 0\}, \qquad V_p := \{v \in W_0^{1,p}(\Omega)^2 : \operatorname{div} v = 0\}.$$

Here, p denotes an arbitrary exponent $p \geq 2$. These spaces are Banach spaces with their norms denoted by $|\cdot|_p$ respectively $|\cdot|_{1,p}$. For $p = 2$, we get the frequently used solenoidal spaces $H := H_2$ and $V := V_2$, which are Hilbert spaces with scalar products $(\cdot,\cdot)_H$ respectively $(\cdot,\cdot)_V$. The dual of V with respect to the scalar product of H we denote by V' with the duality pairing $\langle\cdot,\cdot\rangle_{V',V}$.

We shall work in the standard space of abstract functions from $[0,T]$ to a real Banach space X, $L^p(0,T;X)$, endowed with its natural norm,

$$\|y\|_{L^p(X)} := \|y\|_{L^p(0,T;X)} = \left(\int_0^T |y(t)|_X^p \,dt\right)^{1/p} \qquad 1 \leq p < \infty,$$

$$\|y\|_{L^\infty(X)} := \operatorname*{vrai\,max}_{t\in(0,T)} |y(t)|_X.$$

In the sequel, we will identify the spaces $L^p(0,T;L^p(\Omega)^2)$ and $L^p(Q)^2$ for $1<p<\infty$, and denote their norm by $\|u\|_p := |u|_{L^p(Q)^2}$. The usual $L^2(Q)^2$-scalar product we denote by $(\cdot,\cdot)_Q$ to avoid ambiguity.

In all what follows, $\|\cdot\|$ stands for norms of abstract functions, while $|\cdot|$ denotes norms of "stationary" spaces like H and V.

To deal with the time derivative in (1.2), we introduce the common spaces of functions y whose time derivatives y_t exist as abstract functions,

$$W^\alpha(0,T;V) := \{y \in L^2(0,T;V) :\ y_t \in L^\alpha(0,T;V')\}, \quad W(0,T) := W^2(0,T;V),$$

where $1 \le \alpha < \infty$. Endowed with the norm

$$\|y\|_{W^\alpha} := \|y\|_{W^\alpha(0,T;V)} = \|y\|_{L^2(V)} + \|y_t\|_{L^\alpha(V')},$$

these spaces are Banach spaces, respectively Hilbert spaces in the case of $W(0,T)$. Every function of $W(0,T)$ is, up to changes on sets of zero measure, equivalent to a function of $C([0,T],H)$, and the imbedding $W(0,T) \hookrightarrow C([0,T],H)$ is continuous, cf. [2, 16].

Furthermore, we introduce the following space of abstract functions in the L^p-context:

$$W_p^{2,1} := \{y \in L^p(0,T;W^{2,p}(\Omega)^2 \cap V_p) :\ y_t \in L^p(0,T;L^p(\Omega)^2)\},$$

which is continuously imbedded in $C([0,T],W_0^{2-2/p,\,p}(\Omega)^2)$, [15]. Here, $W_0^{2-2/p,\,p}(\Omega)^2$ denotes the space of solenoidal $W^{2-2/p,\,p}$-functions where zero boundary values are prescribed if $p \ge 4/3$. We abbreviate $H^{2,1} = W_2^{2,1}$ for $p=2$. Note, that in this case we have $W_0^{2-2/2,\,2}(\Omega)^2 = V$. In this article, we will use exponents $p \ge 2$.

We define the trilinear form $b : V \times V \times V \mapsto \mathbb{R}$ by

$$b(u,v,w) = ((u\cdot\nabla)v,w)_2 = \int_\Omega \sum_{i,j=1}^{2} u_i \frac{\partial v_j}{\partial x_i}\ w_j\ dx.$$

To specify the problem setting, we introduce a linear operator $A : L^2(0,T;V) \mapsto L^2(0,T;V')$ by

$$\int_0^T \langle (Ay)(t),\ v(t)\rangle_{V',V}\, dt := \int_0^T (y(t),v(t))_V\, dt,$$

and a nonlinear operator B by

$$\int_0^T \big\langle \big(B(y)\big)(t),\ v(t)\big\rangle_{V',V}\, dt := \int_0^T b(y(t),y(t),v(t))dt.$$

B is continuous for instance as operator from $W(0,T)$ to $L^2(0,T;V')$. Further properties can be found in [4, 23, 31]. For convenience, we will use in the sequel the notation

$$b_Q(y,v,w) = \int_0^T b(y(t),v(t),w(t))dt.$$

2.1. The state equation. We begin with the notation of weak solutions for the instationary Navier-Stokes equations (1.2) in the Hilbert space setting.

DEFINITION 2.1 (Weak solution). *Let $f \in L^2(0,T;V')$ and $y_0 \in H$ be given. A function $y \in L^2(0,T;V)$ with $y_t \in L^2(0,T;V')$ is called weak solution of (1.2) if*

$$\begin{aligned} y_t + \nu Ay + B(y) &= f, \\ y(0) &= y_0. \end{aligned} \tag{2.1}$$

Results concerning the solvability of (2.1) are standard, cf. [4, 23] for proofs and further details.

THEOREM 2.2 (Existence and uniqueness of solutions). *For every $f \in L^2(0,T;V')$ and $y_0 \in H$, the equation (2.1) has a unique solution $y \in W(0,T)$. Moreover, the mapping $(y_0, u) \mapsto y$ is locally Lipschitz continuous form $H \times L^2(0,T;V')$ to $W(0,T)$.*

For more regular data, one expects more regular solutions. The next theorem states some well-known facts, see for instance [23] for the details and [24] for more regularity results.

THEOREM 2.3 (Regularity). *For the higher regularity of the weak solutions of (2.1) the following holds.*

(i) Let $y_0 \in V$ and $f \in L^2(Q)^2$ be given. Then the weak solution of (2.1) fulfills

$$\begin{aligned} y &\in L^2(0,T;H^2(\Omega)^2) \cap L^\infty(0,T;V), \\ y_t &\in L^2(0,T;H). \end{aligned}$$

(ii) Let additionally, $y_0 \in H^2(\Omega)^2 \cap V$ and $f_t \in L^2(0,T;V')$ and $f(0) \in L^2(\Omega)^2$ be given. Then the weak solution y of (2.1) satisfies

$$y_t \in L^2(0,T;V) \cap L^\infty(0,T;H).$$

(iii) If moreover $f \in L^\infty(0,T;L^2(\Omega)^2)$ then it holds

$$y \in L^\infty(0,T;H^2(\Omega)^2).$$

The solution mapping $(f, y_0) \mapsto (y, y_t)$ is locally Lipschitz continuous between the mentioned spaces.

For the proofs of the three statements we refer to Temam [23, Theorems III.3.10, 3.5, 3.6].

REMARK 2.4 (Linearized state equation). *We consider the linearized equation*

$$\begin{aligned} y_t + \nu Ay + B'(\bar{y})y &= f, \\ y(0) &= y_0, \end{aligned} \tag{2.2}$$

for a given state $\bar{y}$, which is usually the solution of the nonlinear system (2.1). Following the lines of Temam, one can proof existence and uniqueness of a weak solution $y \in W(0,T)$. Regularity results similar to (i)–(iii) hold, provided the state $\bar{y}$ has the same regularity as one wants to get for the solution of the linearized equation, see also [13].

Notice, that result *(ii)* implies in particular $y \in C(\bar{Q})^2$. But the pre-requisites are quite restrictive with respect to f. We need that its time-derivative has some regularity. Also, any other result in the Hilbert theory which leads to continuous states of class $C(\bar{Q})^2$ needs regularity of space or time derivatives of the right hand side, cf. [4, 21, 23, 31]. In the context of optimal control this is quite problematic. We will comment on it later on, see remarks at the end of Section 5.3.

Can we gain some 'intermediate' regularity of the solution if the right-hand side is in $L^p(Q)^2$ with $p > 2$? If then the weak solution would satisfy $y \in L^p(0,T;W^{2,p}(\Omega)^2)$ and $y_t \in L^p(0,T;L^p(\Omega)^2)$, we would get immediately $y \in C([0,T];W^{1,p}(\Omega)^2) \hookrightarrow C(\bar{Q})^2$. Actually, such a result is available. At first, we have to specify the notation of a strong solution in the L^p-setting.

DEFINITION 2.5 (Strong solution in L^p). *Let $f \in L^p(Q)^2$ and $y_0 \in W_0^{2-2/p,\,p}(\Omega)^2$ be given. A function $y \in L^p(0,T;V_p)$ with $y_t \in L^p(0,T;L^p(\Omega)^2)$ is called strong solution to the exponent $p > 2$ of* (1.2) *if*

$$-\int_0^T (y,\phi')dt + \nu\int_0^T (\nabla y,\nabla\phi)dt + \int_0^T b(y,y,\phi) = \int_0^T (f,\phi)dt + (y_0,\phi(0)) \quad (2.3)$$

for all test functions $\phi \in L^q(0,T;V_q)$ with $\phi_t \in L^q(0,T;L^q(\Omega)^2)$ and $\phi(T) = 0$, where q is the dual exponent to p, $1/q + 1/p = 1$.

Here the space $W_0^{2-2/p,\,p}(\Omega)^2$ is the natural trace space. Every abstract function of $L^p(0,T;W^{2,p}(\Omega)^2)$ with time derivative in $L^p(0,T;L^p(\Omega)^2)$ is - after changes on a zero measure set - continuous with values in this space, [15]. Obviously, every strong L^p-solution is a weak solution. For existence of L^p-solutions we have the following theorem.

THEOREM 2.6 (L^p-solutions). *Let $f \in L^p(Q)^2$ and $y_0 \in W_0^{2-2/p,\,p}(\Omega)^2$ be given with $p \geq 2$. Then the weak solution y of* (2.1) *in the sense of Definition 2.1 is a strong solution and satisfies*

$$y \in L^p(0,T;W^{2,p}(\Omega)^2 \cap V_p),$$
$$y_t \in L^p(0,T;L^p(\Omega)^2).$$

There exists a constant $C > 0$ such that

$$\|y\|_{L^p(W^{2,p})} + \|y_t\|_p + \|y\|_{L^\infty(W^{2-2/p,\,p})} \leq c\,\{|y_0|_{W^{2,p}} + \|f\|_p\}$$

Moreover, the mapping $(f,y_0) \mapsto y$ is locally Lipschitz continuous, hence the strong solution y is unique.

If $p = 2$ this result reduces to Theorem 2.3(i). For the non-Hilbert space case $p > 2$, it is proven in [29, 30].

3. The optimal control problem.

3.1. First order necessary optimality conditions. Now we return to our optimal control problem. We briefly recall the necessary conditions for local optimality. For the proofs and further discussion see [1, 3, 9, 12, 28] and the references cited therein.

DEFINITION 3.1 (Locally optimal control). *A control $\bar{u} \in U_{ad}$ is said to be locally optimal in $L^2(Q)^2$, if there exists a constant $\rho > 0$ such that*

$$J(\bar{y},\bar{u}) \leq J(y_\rho,u_\rho)$$

holds for all $u_\rho \in U_{ad}$ with $\|\bar{u} - u_\rho\|_2 \le \rho$. Here, $\bar{y}$ and y_ρ denote the states associated with $\bar{u}$ and u_ρ, respectively.

In the following, we denote by $B'(\bar{y})^*$ the formal adjoint of $B'(\bar{y})$. For $\bar{y} \in W(0,T)$, it is a continuous linear operator from $L^2(0,T;V)$ to $L^{4/3}(0,T;V')$.

THEOREM 3.2 (Necessary condition). *Let $\bar{u}$ be a locally optimal control with associated state $\bar{y} = y(\bar{u})$. Then there exists a unique solution $\bar{\lambda} \in W^{4/3}(0,T;V)$ of the adjoint equation*

$$\begin{aligned} -\bar{\lambda}_t + \nu A\bar{\lambda} + B'(\bar{y})^*\bar{\lambda} &= \alpha_Q(\bar{y} - y_Q) + \alpha_R \vec{\mathrm{rot}}\,\mathrm{rot}\,\bar{y} \\ \bar{\lambda}(T) &= \alpha_T(\bar{y}(T) - y_T). \end{aligned} \tag{3.1}$$

Moreover, the variational inequality

$$(\gamma\bar{u} + \bar{\lambda}, u - \bar{u})_{L^2(Q)^2} \ge 0 \quad \forall u \in U_{ad} \tag{3.2}$$

is satisfied.

Proofs can be found in [9, 10, 28]. The regularity of $\bar{\lambda}$ is proven in [13].

In the sequel we need the notation of the normal cone $N_{U_{ad}}(\bar{u})$ of the set of admissible controls given by

$$N_{U_{ad}}(\bar{u}) = \begin{cases} \{z \in L^2(Q)^2 : (z, u - \bar{u})_2 \le 0 \ \forall u \in U_{ad}\} & \text{if } \bar{u} \in U_{ad} \\ \emptyset & \text{otherwise.} \end{cases} \tag{3.3}$$

Then the variational inequality (3.2) can be written equivalently as the inclusion

$$\nu\bar{u} + \bar{\lambda} + N_{U_{ad}}(\bar{u}) \ni 0. \tag{3.4}$$

The regularity of the adjoint state depending on the regularity of the data is stated more precisely in the next lemma. It can be proven following the lines of Temam [23], see also [13, 19]. For convenience, we denote by f the right-hand side of (3.1), and by λ_T the initial value $\alpha_T(\bar{y}(T) - y_T)$.

THEOREM 3.3 (Regularity of the adjoint state).

(i) *Let $\lambda_T \in H$, $f \in L^2(0,T;V')$, and $\bar{y} \in L^2(0,T;V) \cap L^\infty(0,T;H)$ be given. Then there exists a unique weak solution λ of (3.1) satisfying*

$$\begin{aligned} \lambda &\in L^2(0,T;V), \\ \lambda_t &\in L^{4/3}(0,T;V'). \end{aligned} \tag{3.5}$$

(ii) *Let $\lambda_T \in V$, $f \in L^2(Q)^2$, and $\bar{y} \in L^2(0,T;H^2(\Omega)^2) \cap L^\infty(0,T;V)$ be given. Then the unique weak solution λ of (3.1) satisfies*

$$\begin{aligned} \lambda &\in L^2(0,T;H^2(\Omega)^2) \cap L^\infty(0,T;V), \\ \lambda_t &\in L^2(0,T;H). \end{aligned} \tag{3.6}$$

(iii) *Additionally, let $\lambda_T \in H^2(\Omega)^2 \cap V$, $f \in L^\infty(0,T;L^2(\Omega)^2)$, $f_t \in L^2(0,T;V')$, $\bar{y}_t \in L^2(0,T;V) \cap L^\infty(0,T;H)$, and $\bar{y}(0) \in H^2(\Omega)^2 \cap V$ be given. Then the weak solution λ of (3.1) satisfies*

$$\begin{aligned} \lambda &\in L^\infty(0,T;H^2(\Omega)^2) \cap L^\infty(0,T;V), \\ \lambda_t &\in L^\infty(0,T;H) \cap L^2(0,T;V). \end{aligned}$$

The mapping $(f, \lambda_T) \mapsto \lambda$ is continuous in the mentioned spaces.

The existence of L^p-solutions of the adjoint equation is topic of the next Theorem.

Theorem 3.4. *Let $f \in L^p(Q)^2$ and $\lambda_T \in W_0^{2-2/p,\,p}(\Omega)^2$ be given with $p \geq 2$. If $\bar{y} \in L^p(0,T; W^{2,p}(\Omega)^2 \cap V_p)$, then the weak solution λ of (3.1) is a strong solution and satisfies*

$$\begin{aligned} \lambda &\in L^p(0,T; W^{2,p}(\Omega)^2 \cap V_p), \\ \lambda_t &\in L^p(0,T; L^p(\Omega)^2). \end{aligned}$$

Moreover, the mapping $(f, \lambda_T) \mapsto \lambda$ is continuous, hence the weak solution λ is unique.

The result in the case $p = 2$ is equivalent to Theorem 3.3(ii). Following the lines of [29, 30] one can proof the claim also for $p > 2$.

Let us introduce the Lagrange function $\mathcal{L} : W(0,T) \times L^2(Q)^2 \times W^{4/3}(0,T) \mapsto \mathbb{R}$ for the optimal control problem as follows:

$$\mathcal{L}(y,u,\lambda) = J(u,y) - \nu\,(y,\lambda)_V - b(y,y,\lambda) + (u,\lambda)_2.$$

This function is twice Fréchet-differentiable with respect to $(y,u) \in W(0,T) \times L^2(Q)^2$, cf. [28]. The reader can readily verify that the necessary conditions can be expressed equivalently by

$$\mathcal{L}_y(\bar{y},\bar{u},\bar{\lambda})\,h = 0 \quad \forall h \in W(0,T) \text{ with } h(0) = 0,$$

and

$$\mathcal{L}_u(\bar{y},\bar{u},\bar{\lambda})(u - \bar{u}) \geq 0 \quad \forall u \in U_{ad}.$$

Here, $\mathcal{L}_y$, $\mathcal{L}_u$ denote the partial Fréchet-derivative of $\mathcal{L}$ with respect to y and u.

In the sequel we denote the pair of state and control (y,u) by v for convenience. The second derivative of the Lagrangian $\mathcal{L}$ at $y \in W(0,T)$ with associated adjoint state λ in the directions $v_1 = (z_1,h_1)$, $v_2 = (z_2,h_2) \in W(0,T) \times L^2(Q)^2$ is given by

$$\mathcal{L}_{vv}(y,u,\lambda)[v_1,v_2] = \mathcal{L}_{yy}(y,u,\lambda)[z_1,z_2] + \mathcal{L}_{uu}(y,u,\lambda)[h_1,h_2] \tag{3.7}$$

with

$$\begin{aligned} \mathcal{L}_{yy}(y,u,\lambda)[z_1,z_2] = \alpha_T(z_1(T),z_2(T))_H + \alpha_Q(z_1,z_2)_Q + \alpha_R(\operatorname{rot} z_1, \operatorname{rot} z_2)_Q \\ - b_Q(z_1,z_2,\lambda) - b_Q(z_2,z_1,\lambda) \end{aligned}$$

and

$$\mathcal{L}_{uu}(y,u,\lambda)[h_1,h_2] = \gamma(h_1,h_2)_2.$$

It satisfies the estimate

$$|\mathcal{L}_{yy}(y,u,\lambda)[z_1,z_2]| \leq c\,\left(\|z_1\|_{L^\infty(H)} + \|z_1\|_{L^2(V)}\right)\left(\|z_2\|_{L^\infty(H)} + \|z_2\|_{L^2(V)}\right)\|\lambda\|_{L^2(V)} \tag{3.8}$$

for all $z_1, z_2 \in W(0,T)$, confer [28].

3.2. Second-order sufficient optimality conditions. Let $\bar v := (\bar y, \bar u)$ be an admissible reference pair satisfying the first-order necessary optimality conditions. We assume further that the reference pair $\bar v = (\bar y, \bar u)$ satisfies the following coercivity assumption on $\mathcal{L}''(\bar v, \bar\lambda)$, in the sequel called second-order sufficient condition:

(SSC) *There exists $\delta > 0$ such that*

$$\mathcal{L}_{vv}(\bar v, \bar\lambda)[(z,h)]^2 \geq \delta \, \|h\|_2^2$$

holds for all pairs $(z,h) \in W(0,T) \times L^2(Q)^2$ with $z \in W(0,T)$ being the weak solution of the linearized equation

$$\begin{aligned} z_t + Az + B'(\bar y)z &= h \\ z(0) &= 0. \end{aligned}$$

THEOREM 3.5. *Let $\bar v = (\bar y, \bar u)$ be admissible for the optimal control problem and suppose that $\bar v$ fulfills the first-order necessary optimality condition with associated adjoint state $\bar\lambda$. Assume further that (SSC) is satisfied at $\bar v$. Then there exist $\alpha > 0$ and $\rho > 0$ such that*

$$J(v) \geq J(\bar v) + \alpha \, \|u - \bar u\|_2^2$$

holds for all admissible pairs $v = (y,u)$ with $\|u - \bar u\|_\infty \leq \rho$.

For a proof we refer to [28]. There, Theorem 3.5 was proven in a slightly weaker form: The space of directions in which $\mathcal{L}_{vv}$ has to be positive definite was shrinked using the concept of strong active control constraints. Sufficiency was achieved in a L^s-neighborhood of the reference control, whereas the quadratic growth takes place in the L^q-norm with $q \leq 2 \leq s \leq \infty$, $s = q/(2-q)$. The usage of the L^s-norm with $s < \infty$ was motivated as follows: if one utilize a L^∞-neighborhood of the reference control then jumps of the optimal control has to be known a-priorily. For general objective functionals such jumps can not be excluded. It is one goal of the present article to show that the quadratic functional given by (1.1) results in continuous optimal controls without jumps.

4. Regularity of extremal controls. In this section, we are going to prove continuity in space and time of extremal controls, i.e. controls satisfying the first-order necessary optimality conditions. The key tool in our analysis is the well-known projection formula

$$u(x,t) = \mathrm{Proj}_{[u_a(x,t), u_b(x,t)]}\left(-\frac{1}{\gamma}\bar\lambda(x,t)\right) \qquad \text{a.e. on } Q, \tag{4.1}$$

which is equivalent to the variational inequality (3.2).

To begin with, we state the assumption imposed on the various ingredients of the optimal control problem (1.1).

(A1) *The bounds u_a, u_b are of class $C(\bar Q)^2$. Their time derivatives $u_{a,t}$, $u_{b,t}$ exist as functions in $L^2(Q)^2$.*

(A2) *$y_0 \in H^2(\Omega)^2 \cap V$.*
Either $\alpha_T = 0$ or $y_T \in H^2(\Omega)^2 \cap V$.
Either $\alpha_Q = 0$ or $y_Q \in L^\infty(0,T; L^2(\Omega)^2)$ and $y_{Q,t} \in L^2(0,T;V')$.

Assuming this allows us to prove

THEOREM 4.1. *Let $\bar{u} \in U_{ad}$ satisfy the first-order necessary conditions of the optimal control problem (P). Then, $\bar{u}$ is continuous in $\bar{Q}$, i.e. $\bar{u} \in C(\bar{Q})^2$.*

Proof. For convenience, we denote the right-hand side of the adjoint equation (3.1) by f, i.e. $f := \alpha_Q(\bar{y} - y_Q) + \alpha_R \vec{\mathrm{rot}}\,\mathrm{rot}\,\bar{y}$.

Since $\bar{u} \in U_{ad}$ it follows $u \in L^2(Q)^2$. Then Theorem 2.3(i) yields the regularity of the associated state $y \in L^2(0,T;H^2(\Omega)^2) \cap L^\infty(0,T;V)$, $y(T) \in V$. The right-hand side f of the adjoint equation is therefore at least of class $L^2(\Omega)^2$. Additionally, the initial value $\lambda(T)$ is in V. By Theorem 3.3(ii) we conclude $\bar{\lambda} \in L^2(0,T;H^2(\Omega)^2) \cap L^\infty(0,T;V)$, $\bar{\lambda}_t \in L^2(0,T;H)$.

The projection formula (4.1) gives $u \in L^\infty(0,T;L^2(\Omega)^2)$. With a well-known result of Stampacchia [14, Thm. II.3.1] we conclude $u_t \in L^2(Q)^2$. Now, we can apply Theorem 2.3(ii) and 2.3(iii) to obtain $y \in L^\infty(0,T;H^2(\Omega)^2)$, $y_t \in L^2(0,T;V) \cap L^\infty(0,T;H)$. Then the right-hand side of the adjoint equation satisfies $f \in L^\infty(0,T;L^2(\Omega)^2)$, $f_t \in L^2(0,T;V')$. The initial value $\lambda(T)$ is now of class H^2. Thus, Theorem 3.3(iii) implies $\lambda \in L^\infty(0,T;H^2(\Omega)^2)$ and $\lambda_t \in L^2(0,T;V)$.

Finally, we want to prove $\lambda \in C(\bar{Q})^2$. To this aim, observe that

$$\lambda \in Y = \left\{ w |\, w \in L^2(0,T;H^2(\Omega)^2),\ \frac{dw}{dt} \in L^2(0,T;V) \right\}.$$

Every function in Y is - up to changes on a zero-measure set - a continuous function with values in $[H^2(\Omega)^2, V]_{1/2}$. And the imbedding of Y in $C([0,T],[H^2(\Omega)^2,V]_{1/2})$ is linear and continuous, [16, Theorem 1.3.1] . Here, $[\cdot,\cdot]_\theta$ denotes the complex interpolation functor, cf. [25]. The interpolation identity $[H^2(\Omega)^2, V]_{1/2} = H^{3/2}(\Omega)^2$ is proven for instance in [16, 25] . The space $H^{3/2}(\Omega)$ is continuously imbedded in $C(\bar{\Omega})$, cf. [2]. Thus, we got $\lambda \in C([0,T], C(\bar{\Omega})^2) = C(\bar{Q})^2$.

Now, the projection formula (4.1) together with the assumptions on the box constraints in (A1) gives $u \in C(\bar{Q})^2$. ◻

As the proof shows, one can even prove H^1-regularity of the extremal controls, if the bounds are smooth enough.

(A3) *The functions u_a, u_b are of class $H^1(Q)^2$.*

Using again, Stampacchia's Theorem, we have the following

COROLLARY 4.2. *Let (A1),(A2), and (A3) be satisfied. Then every extremal solution $\bar{u} \in U_{ad}$ is in $H^1(Q)^2$.*

Note, that this result for bounded optimal control is maximal in the L^2-theory. In the L^p-context it is possible to get $\bar{u} \in W^{1,p}(Q)^2$ for $p < \infty$. The projection mapping is only bounded in spaces with differentiation order less or equal than 1. Hence without further assumptions one cannot prove higher regularity than $W^{1,p}$ of optimal controls.

Since the proof is analogous to the previous one, we only state the result $\bar{u} \in C(\bar{Q})^2$ using the L^p-solution theory. For $2 < p < \infty$ let the following pre-requisites be fulfilled.

($\mathbf{A1}_p$) *The bounds u_a, u_b are of class $C(\bar{Q})^2$.*

$$
(\mathbf{A2}_p)\begin{cases} y_0 \in W_0^{2-2/p,\,p}(\Omega)^2. \\ \textit{Either } \alpha_T = 0 \textit{ or } y_T \in W_0^{2-2/p,\,p}(\Omega)^2. \\ \textit{Either } \alpha_Q = 0 \textit{ or } y_Q \in L^p(Q)^2. \end{cases}
$$

COROLLARY 4.3. *Let (A1$_p$) and (A2$_p$) be satisfied. Then every extremal solution $\bar u \in U_{ad}$ is in $C(\bar Q)^2$. The associated state $\bar y$ and adjoint $\bar\lambda$ are at least in $W_p^{2,1}$.*

5. Local stability analysis. Finally, we are dealing with stability of a locally optimal reference tripel $(\bar y, \bar u, \bar\lambda)$ of the original problem (1.1). To be more specific, consider the perturbed optimal control problem with perturbation vector $z = (z_y, z_0, z_Q, z_T, z_u)$ in some function space Z

$$
\begin{aligned} \min J(y,u,z) = \frac{\alpha_T}{2}|y(\cdot,T) - y_T|_2^2 + (z_T, y(T))_\Omega + \frac{\alpha_Q}{2}\|y - y_Q\|_2^2 + (z_Q, y)_Q \\ + \frac{\alpha_R}{2}\|\operatorname{rot} y\|_2^2 + \frac{\gamma}{2}\|u\|_2^2 - (z_u, u)_Q \end{aligned} \tag{5.1}
$$

subject to the perturbed Navier-Stokes equations

$$
\begin{aligned} y_t - \nu\Delta y + (y\cdot\nabla)y + \nabla p = u + z_y \quad & \text{in } Q, \\ \operatorname{div} y = 0 \quad & \text{in } Q, \\ y(0) = y_0 + z_0 \quad & \text{in } \Omega, \end{aligned} \tag{5.2}
$$

and the constraint

$$
u \in U_{ad}.
$$

Here arises the natural question: How depends the optimal tripel (y, u, λ) on the perturbation z? This question is answered in the rest of this article.

The plan of this section is as follows: At first, we will introduce the concept of generalized equations, where we emphasize on an abstract stability result due to Robinson. Secondly, the optimality system is written as a generalized equation in function spaces. Finally, we prove stability of optimal controls provided a second-order sufficient optimality condition holds. Under suitable assumptions, we get even stability of optimal controls with respect to the L^∞-norm.

5.1. Generalized equations. In the sequel, we will apply a result on generalized equations due to Robinson [18]. First, we recall some basic notations. We consider the generalized equation

$$
0 \in F(x) + N(x), \tag{5.3}
$$

where F is a C^1-mapping between to Banach spaces X and Z, while $N : X \mapsto 2^Z$ is a set-valued mapping with closed graph.

Let $\bar x$ be a solution of (5.3). The generalized equation is said to be *strongly regular* at the point $\bar x$, if there are open balls $B_X(\bar x, \rho_x)$ and $B_Z(0, \rho_z)$ such that for all $z \in B_Z(0, \rho_z)$ the linearized and perturbed equation

$$
z \in F(\bar x) + F'(\bar x)(x - \bar x) + N(x)
$$

admits a unique solution $x = x(z)$ in $B_X(\bar x, \rho_x)$, and the mapping $z \mapsto x$ is Lipschitz continuous $B_Z(0, \rho_z)$ from to $B_X(\bar x, \rho_x)$. The following theorem allows to get from stability results for the perturbed linearized equation to similar results for the perturbed nonlinear problem.

THEOREM 5.1. *Let $\bar{x}$ be a solution of* (5.3) *and assume that* (5.3) *is strongly regular at $\bar{x}$. Then there exist open balls $B_X(\bar{x}, \rho'_x)$ and $B_Z(0, \rho'_z)$ such that for all $z \in B_Z(0, \rho'_z)$ the perturbed equation*

$$z \in F(x) + N(x)$$

has a unique solution in $x = x(z) \in B_X(\bar{x}, \rho'_x)$, and the solution mapping $z \mapsto x(z)$ is Lipschitz continuous from $B_Z(0, \rho'_z)$ to $B_X(\bar{x}, \rho'_x)$.

5.2. Perturbed optimal control problem. Let $(\bar{y}, \bar{u}, \bar{\lambda})$ satisfy the first-order necessary optimality conditions, see Theorem 3.2, together with the second-order sufficient optimality conditions (SSC). The optimality system consisting of state equation (1.2), adjoint equation (3.1) and variational inequality (3.2), can be written in the condensed form

$$F(\bar{y}, \bar{u}, \bar{\lambda}) + \big(0, 0, 0, 0, N_{U_{ad}}(\bar{u})\big)^T \ni 0, \tag{5.4}$$

where the function F,

$$F : H^{2,1} \times L^2(Q)^2 \times H^{2,1} \to L^2(Q)^2 \times V \times L^2(Q)^2 \times V \times L^2(Q)^2 \tag{5.5}$$

is given by

$$F(y, u, \lambda) = \begin{pmatrix} y_t + \nu A y + B(y) \\ y(0) \\ -\lambda_t + \nu A\lambda + B'(y)^*\lambda \\ \lambda(T) \\ \gamma u + \lambda \end{pmatrix} - \begin{pmatrix} u \\ y_0 \\ \alpha_Q(y - y_Q) + \alpha_R \vec{\mathrm{rot}}\, \mathrm{rot}\, y \\ \alpha_T(y(T) - y_T) \\ 0 \end{pmatrix}. \tag{5.6}$$

We will apply Theorem 5.1 to the generalized equation (5.4). To do so, we have to show strong regularity of this equation at the reference tripel $(\bar{y}, \bar{u}, \bar{\lambda})$. At first, we investigate the mapping F.

COROLLARY 5.2. *The function F defined by* (5.6) *is continuously differentiable in the setting* (5.5).

Proof. The components of F are affine linear functions except F_1, which contains the nonlinear part $B(y)$. We derive for $y, h \in H^{2,1}$, $v \in L^2(Q)^2$

$$\begin{aligned} B(y + h)v - B(y)v &= \int_0^T b(y + h, y + h, v) - b(y, y, v)dt \\ &= \int_0^T b(y, h, v) + b(h, y, v) + b(h, h, v)dt. \end{aligned}$$

This gives immediately the directional derivative of B in direction h as $B'(y)h = \int_0^T b(y, h, v) + b(h, y, v)dt$. We proceed with

$$\begin{aligned} \|B(y + h) - B(y) - B'(y)h\|_2 &= \sup_{v \in L^2(Q)^2 \setminus \{0\}} \|v\|_2^{-1} \int_0^T b(h, h, v)dt \\ &\le \sup_{v \in L^2(Q)^2 \setminus \{0\}} \|v\|_2^{-1} c\, \|h\|_{L^4(W^{1,4})} \|h\|_4 \|v\|_2 \le c\, \|h\|^2_{H^{2,1}}, \end{aligned}$$

which proves Frechét-differentiability of $B(y)$. To prove continuous differentiability we take $y_1, y_2 \in H^{2,1}$. Then for any direction $h \in H^{2,1}$ and element $v \in L^2(Q)^2$ we obtain

$$\begin{aligned}\left|(B'(y_1)h - B'(y_2)h)v\right| &= \left|\int_0^T b(y_1 - y_2, h, v) + b(h, y_1 - y_2, v)dt\right| \\ &\leq c\|y_1 - y_2\|_{H^{2,1}}\|h\|_{H^{2,1}}\|v\|_2,\end{aligned}$$

which shows that the mapping $y \mapsto B'(y)$ is even Lipschitz continuous from $H^{2,1}$ in the space $\mathcal{L}(H^{2,1}, L^2(Q)^2)$. ∎

For convenience, we introduce the space of perturbation vectors Z as

$$Z := L^2(Q)^2 \times V \times L^2(Q)^2 \times V \times L^2(Q)^2 \tag{5.7}$$

equipped with the norm $\|z\|_Z = \|z_y\|_2 + |z_0|_V + \|z_Q\|_2 + |z_T|_V + \|z_u\|_2$.

The optimality system of the perturbed problem (5.1) is equivalent to the generalized equation

$$F(y, u, \lambda) + \big(0, 0, 0, 0, N_{U_{ad}}(u)\big)^T \ni z, \tag{5.8}$$

where $z = (z_y, z_0, z_Q, z_T, z_u) \in Z$. The components one to four of this inclusion are in fact equations.

The next step in proving strong regularity of (5.4) is the investigation of the linearized version of the inclusion (5.8)

$$F(\bar{y}, \bar{u}, \bar{\lambda}) + F'(\bar{y}, \bar{u}, \bar{\lambda})(y - \bar{y}, u - \bar{u}, \lambda - \bar{\lambda}) + \big(0, 0, 0, 0, N_{U_{ad}}(u)\big) \ni z.$$

This generalized equation corresponds to the following system. It consists of the state equations

$$\begin{aligned}y_t + \nu A y + B'(\bar{y})y &= u + B(\bar{y}) + z_y \\ y(0) &= y_0 + z_0,\end{aligned}$$

the adjoint equations

$$\begin{aligned}-\lambda_t + \nu A\lambda + B'(\bar{y})^*\lambda &= -B'(y - \bar{y})^*\bar{\lambda} + \alpha_Q(y - y_Q) + \alpha_R \vec{\operatorname{rot}}\operatorname{rot} y + z_Q \\ \lambda(T) &= \alpha_T(y(T) - y_T) + z_T,\end{aligned}$$

and the variational inequality

$$\gamma u + \lambda + N_{U_{ad}}(u) \ni z_u.$$

This altogether builds up the optimality system of the perturbed linear-quadratic optimization problem given by

$$\begin{aligned}\min J^{(z)}(y, u) = \frac{\alpha_T}{2}|y(T) - y_d|_H^2 + \frac{\alpha_Q}{2}\|y - y_Q\|_2^2 + \frac{\alpha_R}{2}\|\operatorname{rot} y\|_2^2 + \frac{\gamma}{2}\|u\|_2^2 \\ + (z_Q, y)_Q + (z_T, y(T))_\Omega - (z_u, u)_Q - b_Q(y - \bar{y}, y - \bar{y}, \bar{\lambda})\end{aligned} \tag{5.9}$$

subject to the linearized state equation

$$\begin{aligned}y_t + \nu A y + B'(\bar{y})y &= u + B(\bar{y}) + z_y \\ y(0) &= y_0 + z_0\end{aligned}$$

and the control constraint

$$u \in U_{ad}.$$

The existence of a unique optimal control of the problem (5.9) is an easy consequence of the coercivity assumption (SSC). Let us denote the Lagrangian associated to (5.9) by $\mathcal{L}^{(z)}$. Then it holds for all y, u, λ

$$\mathcal{L}^{(z)}_{vv}(y, u, \lambda) = \mathcal{L}_{vv}(\bar{y}, \bar{u}, \bar{\lambda}).$$

Hence, the second-order sufficient condition yields unique solvability of (5.9) as a linear-quadratic optimization problem with strong convex objective functional. We denote its unique solution of (5.9) by $u_z = u(z)$ with associated state y_z and adjoint state λ_z. For a more detailed discussion of those aspects we refer to [20], where the stability analysis is made for the stationary Navier-Stokes system.

5.3. Stability of optimal controls in $L^2(Q)^2$. Now, we are ready to prove stability of optimal controls in the setting given in the last section. To verify strong regularity we have to prove Lipschitz continuity of the solution mapping $z \mapsto (y_z, u_z, \lambda_z)$ of the perturbed linearized problem (5.9).

THEOREM 5.3. *Let (SSC) be satisfied for the reference solution $\bar{v}$ with adjoint state $\bar{\lambda}$. Let additionally y_0, $y_T \in V$, $y_Q \in L^2(Q)^2$ be given. Then the mapping $z \to (y_z, u_z, \lambda_z)$ is Lipschitz continuous from Z to $H^{2,1} \times L^2(Q)^2 \times H^{2,1}$.*

Proof. Let $z_1, z_2 \in Z$ be given. Denote the optimal controls of the perturbed problem by $u_i := u_{z_i}$ with associated states y_i and adjoints λ_i, $i = 1, 2$. Denote the differences by $z = z_1 - z_2$, $u := u_1 - u_2$, $y = y_1 - y_2$, and $\lambda = \lambda_1 - \lambda_2$.

Throughout the proof we abbreviate $(\cdot, \cdot) := (\cdot, \cdot)_Q$ and $\langle \cdot, \cdot \rangle := \langle \cdot, \cdot \rangle_{L^2(V'), L^2(V)}$.

At first, we consider the variational inequality connected with the constraint $u_i \in U_{ad}$,

$$(\gamma u_i + \lambda_i - z_{u,i},\, u - u_i) \geq 0 \qquad \forall u \in U_{ad}.$$

Testing the inequality for u_i, $i = 1, 2$ with u_j, $j = 2 - i$, and adding them, we find

$$-(\lambda, u) + (u, z_u) \geq \gamma \|u\|_2^2. \tag{5.10}$$

Secondly, we consider the state equation. The difference y is the weak solution of

$$\begin{aligned} y_t + \nu A y + B'(\bar{y}) y &= u + z_y \\ y(0) &= z_0. \end{aligned} \tag{5.11}$$

We test this equation by $\lambda = \lambda_1 - \lambda_2$ to obtain

$$\langle y_t,\, \lambda \rangle + \nu (y, \lambda)_{L^2(V)} + b_Q(\bar{y}, y, \lambda) + b_Q(y, \bar{y}, \lambda) = (u, \lambda) + (z_y, \lambda). \tag{5.12}$$

And third, we investigate the adjoint equations. The difference λ of the adjoint states satisfies

$$\begin{aligned} -\lambda_t + \nu A \lambda + B'(\bar{y})^* \lambda &= -B'(y)^* \bar{\lambda} + \alpha_Q y + \alpha_R \vec{\operatorname{rot}} \operatorname{rot} y + z_Q \\ \lambda(T) &= \alpha_T y(T) + z_T. \end{aligned} \tag{5.13}$$

Testing this equation by $y = y_1 - y_2$ yields

$$
\begin{aligned}
&-\langle \lambda_t,\, y\rangle + \nu(\lambda, y)_{L^2(V)} + b_Q(\bar y, y, \lambda) + b_Q(y, \bar y, \lambda) = \\
&\qquad - 2b_Q(y, y, \bar\lambda) + \alpha_Q\|y\|_2^2 + \alpha_R\|\operatorname{rot} y\|_2^2 + (z_Q, y).
\end{aligned} \tag{5.14}
$$

By integration by parts we find

$$
\begin{aligned}
-\langle \lambda_t,\, y\rangle &= \langle y_t,\, \lambda\rangle - (\lambda(T), y(T))_H + (\lambda(0), y(0))_H \\
&= \langle y_t,\, \lambda\rangle - \alpha_T\|y(T)|_H^2 - (z_T, y(T))_H + (\lambda(0), z_0)_H
\end{aligned} \tag{5.15}
$$

Combining (5.12), (5.14), and (5.15), the equation

$$
\begin{aligned}
(u, \lambda) + (z_y, \lambda) &= \alpha_T\|y(T)|_H^2 + (z_T, y(T))_H - (\lambda(0), z_0)_H \\
&\qquad - 2b_Q(y, y, \bar\lambda) + \alpha_Q\|y\|_2^2 + \alpha_R\|\operatorname{rot} y\|_2^2 + (z_Q, y)
\end{aligned} \tag{5.16}
$$

is found.

We introduce the auxiliary function $\tilde y$ as the weak solution of (5.11) with $u = 0$. Now, the coercivity assumption of $\mathcal{L}_{vv}$ comes into play. The tupel $(y - \tilde y, u)$ fits in the assumptions of (SSC). With $\mathcal{L}_{vv}$ given by (3.7), we derive

$$
\begin{aligned}
\delta\|u\|_2^2 &\le \mathcal{L}_{vv}(\bar v, \bar\lambda)[(y - \tilde y, u)]^2 \\
&= \mathcal{L}_{uu}(\bar v, \bar\lambda)[u]^2 + \mathcal{L}_{yy}(\bar v, \bar\lambda)[y]^2 + 2\mathcal{L}_{yy}(\bar v, \bar\lambda)[y, \tilde y]^2 + \mathcal{L}_{yy}(\bar v, \bar\lambda)[\tilde y]^2
\end{aligned} \tag{5.17}
$$

The first and second addend we write according to (3.7) as

$$
\mathcal{L}_{uu}(\bar v, \bar\lambda)[u]^2 + \mathcal{L}_{yy}(\bar v, \bar\lambda)[y]^2 = \alpha_T|y(T)|_H^2 + \alpha_Q\|y\|_2^2 + \alpha_R\|\operatorname{rot} y\|_2^2 - 2b_Q(y, y, \bar\lambda) + \gamma\|u\|^2.
$$

Using (5.16) and inequality (5.10), we continue

$$
\begin{aligned}
&\mathcal{L}_{uu}(\bar v, \bar\lambda)[u]^2 + \mathcal{L}_{yy}(\bar v, \bar\lambda)[y]^2 \\
&\qquad = (u, \lambda) + (z_y, \lambda) - (z_T, y(T))_H + (\lambda(0), z_0)_H - (z_Q, y) + \gamma\|u\|^2 \\
&\qquad = (z_u, u) + (z_y, \lambda) - (z_T, y(T))_H + (\lambda(0), z_0)_H - (z_Q, y) \\
&\qquad \le c\|z\|_Z \left\{\|y\|_{H^{2,1}} + \|\lambda\|_{H^{2,1}}\right\} + (z_u, u).
\end{aligned} \tag{5.18}
$$

Since $\tilde y$ is the weak solution of a linearized equation, we can conclude

$$
\|\tilde y\|_{W(0,T)} \le c\|\tilde y\|_{H^{2,1}} \le c\left\{\|z_y\|_2 + |z_0|_V\right\} \le c\|z\|_Z.
$$

Applying (3.8), we can estimate the third and fourth addend in (5.17) by

$$
\begin{aligned}
2\mathcal{L}_{yy}(\bar v, \bar\lambda)[y, \tilde y]^2 + \mathcal{L}_{yy}(\bar v, \bar\lambda)[\tilde y]^2 &\le c\big\{\left(\|y\|_{L^\infty(H)} + \|y\|_{L^2(V)}\right)\|\tilde y\|_{W(0,T)} + \|\tilde y\|_{W(0,T)}^2\big\} \\
&\le c\big\{\|y\|_{W(0,T)}\|z\|_Z + \|z\|_Z^2\big\}.
\end{aligned} \tag{5.19}
$$

Collecting (5.17)–(5.19), we find

$$
\delta\|u\|_2^2 \le c\left(\|z\|_Z^2 + \|z\|_Z\left\{\|y\|_{H^{2,1}} + \|\lambda\|_{H^{2,1}}\right\}\right) + (z_u, u). \tag{5.20}
$$

By Theorems 2.3(i) and Lemma 3.3(ii), we estimate the differences of the states and adjoints as weak solutions of (5.11) and (5.13)

$$
\begin{aligned}
\|y\|_{H^{2,1}} &\le c\left\{\|u\|_2 + \|z_y\|_2 + |z_0|_V\right\} \\
\|\lambda\|_{H^{2,1}} &\le c\left\{\|y\|_{L^\infty(H)} + \|y\|_{L^2(H^2)} + \|z_Q\|_2 + |z_T|_V\right\},
\end{aligned}
$$

which gives immediately

$$\|y\|_{H^{2,1}} + \|\lambda\|_{H^{2,1}} \le c\left\{\|u\|_2 + \|z\|_Z\right\}. \tag{5.21}$$

Combining (5.20) and (5.21) we get

$$\begin{aligned}\delta\|u\|_2^2 &\le c\|z\|_Z\left\{\|y\|_{H^{2,1}} + \|\lambda\|_{H^{2,1}} + \|z\|_Z\right\} + (z_u, u)\\ &\le c\|z\|_Z\left\{\|u\|_2 + \|z\|_Z\right\}\\ &\le c\|z\|_Z^2 + \frac{\delta}{2}\|u\|_2^2,\end{aligned}$$

and the claim is proven. ◻

So far, we provided all pre-requisites to prove the L^2-stability theorem.

THEOREM 5.4. *Let (SSC) be satisfied for the reference solution $\bar{v}$ with adjoint state $\bar{\lambda}$. Let additionally y_0, $y_T \in V$, $y_Q \in L^2(Q)^2$ be given. Then there exists $\rho > 0$, such that for all $z \in Z$ with $\|z\|_Z \le \rho$, the perturbed optimal control problem* (5.1) *admits a unique solution (y_z, u_z, λ_z). Moreover, the mapping $z \mapsto (y_z, u_z, \lambda_z)$ is Lipschitz continuous from Z to $H^{2,1} \times L^2(Q)^2 \times H^{2,1}$.*

Proof. Theorem 5.3 yields strong regularity of the equation (5.4) at the point $(\bar{y}, \bar{u}, \bar{\lambda})$. So we can apply Theorem 5.1 which finishes the proof. ◻

If the vector of perturbations z is slightly more regular than stated in (5.7), say

$$z \in \tilde{Z} = L^2(Q)^2 \times V \times L^2(Q)^2 \times V \times L^s(Q)^2$$

for some $2 < s < \infty$ equipped with norm $\|z\|_{\tilde{Z}} = \|z_y\|_2 + |z_0|_V + \|z_Q\|_2 + |z_T|_V + \|z_u\|_s$, then one can show the following

THEOREM 5.5. *Let (SSC) be satisfied for the reference solution $\bar{v}$ with adjoint state $\bar{\lambda}$. Let additionally $y_0, y_T \in V$, $y_Q \in L^2(Q)^2$ be given. Then there exists $\rho > 0$, such that for all $z \in \tilde{Z}$ with $\|z\|_{\tilde{Z}} \le \rho$, the perturbed optimal control problem* (5.1) *admits a unique solution (y_z, u_z, λ_z). Moreover, the mapping $z \mapsto (y_z, u_z, \lambda_z)$ is Lipschitz continuous from $\tilde{Z}$ to $H^{2,1} \times L^s(Q)^2 \times H^{2,1}$.*

Proof. The proof is very similar to the proof of Theorem 5.8, see below. It uses the stability result of the previous Theorem 5.4, the projection formula (4.1), and the imbedding $H^{2,1} \hookrightarrow L^s(Q)^2$ for $s < \infty$. ◻

However, this result is maximal in the following sense. Stability of optimal controls in $C(\bar{Q})^2$ can not be achieved using Hilbert space results, since it is not possible to derive a Lipschitz estimate for the time derivatives of the controls which would be necessary to employ Theorem 2.3(ii). To this end consider the following example.

EXAMPLE 5.6. *Let $\lambda_1, \lambda_2 \in C^1[0,T]$ be given by $\lambda_1(t) = \sin(nt) + 2$ and $\lambda_2(t) = \sin(nt) - 2$. Then it holds $\lambda_1(t) - \lambda_2(t) = 4$ and $\frac{d}{dt}(\lambda_1(t) - \lambda_2(t)) = 0$. With $u_a(t) = 0$, $u_b(t) = +\infty$, and $\gamma = 1$ we get*

$$\mathrm{Proj}_{[0,+\infty)}(-\lambda_1(t)) - \mathrm{Proj}_{[0,+\infty)}(-\lambda_2(t)) = 0 - (2 - \sin(nt)) = \sin(nt) - 2,$$

hence,

$$\frac{d}{dt}\Big(\mathrm{Proj}_{[0,+\infty)}(-\lambda_1(t)) - \mathrm{Proj}_{[0,+\infty)}(-\lambda_2(t))\Big) = n\cos(nt) \ne 0,$$

which proves that we cannot show Lipschitz dependancy of the time derivatives of the projected adjoint states λ_i.

At this point, we have to use L^p-methods to derive a stability result in the $C(\bar{Q})^2$-norm.

REMARK 5.7. *Obviously, this difficulties do not appear for the unconstrained problem* $U_{ad} = L^2(Q)^2$, *where the variational inequality is equivalent to* $u = -\frac{1}{\gamma}\lambda$. *Then, any extremal control* $\bar{u}$ *is as smooth as the associated adjoint* $\bar{\lambda}$ *and admits almost the same stability properties, i.e.* $z \mapsto u_z$ *is Lipschitz from* Z *to* $H^{2,1}$.

5.4. L^∞-stability of optimal controls. Here, we give the stability result of optimal controls in norms adequate to the regularity achieved in Section 4. Again, we are considering the inclusion (5.4) and the linearized and perturbed problem. Now, we regard F to be a function in the setting

$$F : W_p^{2,1} \times L^\infty(Q)^2 \times W_p^{2,1} \to \\ L^p(Q)^2 \times W_0^{2-2/p,\,p}(\Omega)^2 \times L^p(Q)^2 \times W_0^{2-2/p,\,p}(\Omega)^2 \times L^\infty(Q)^2, \quad (5.22)$$

Thus, F is continuously differentiable with respect to the spaces given by (5.22).

Accordingly, the perturbation vector z has to be in the smaller space of perturbations Z_p,

$$Z_p := L^p(Q)^2 \times W_0^{2-2/p,\,p}(\Omega)^2 \times L^p(Q)^2 \times W_0^{2-2/p,\,p}(\Omega)^2 \times L^\infty(Q)^2$$

which we endow with the norm

$$\|z\|_{Z_p} = \|(z_y, z_0, z_Q, z_T, z_u)\|_{Z_p} := \|z_y\|_p + |z_0|_{W^{2-2/p,\,p}} + \|z_Q\|_p + |z_T|_{W^{2-2/p,\,p}} + \|z_u\|_\infty.$$

Finally, we have to modify the definition of the normal cone $N_{U_{ad}}$ in (3.3). Here, this set has to be a subset of $L^\infty(Q)^2$,

$$\tilde{N}_{U_{ad}}(\bar{u}) := \begin{cases} \left\{z \in L^\infty(Q)^2 :\ (z, u - \bar{u})_2 \le 0 \ \forall u \in U_{ad}\right\} & \text{if } \bar{u} \in U_{ad} \\ \emptyset & \text{otherwise.} \end{cases} \quad (5.23)$$

Observe that $\tilde{N}_{U_{ad}}(u)$ is a non-empty, closed and convex subset of $L^\infty(Q)^2$.

THEOREM 5.8. *Let (SSC) be satisfied for the reference solution* $\bar{v} = (\bar{y}, \bar{u})$ *with adjoint state* $\bar{\lambda}$. *Moreover, assume that* $y_0, y_T \in W_0^{2-2/p,p}(\Omega)^2$, $y_Q \in L^p(Q)^2$ *for some* p *satisfying* $2 < p < \infty$, *and* $u_a, u_b \in L^\infty(Q)^2$.

Then the mapping solution mapping $z \to (y_z, u_z, \lambda_z)$ *associated to* (5.9) *is Lipschitz continuous from* Z_p *to* $W_p^{2,1} \times L^\infty(Q)^2 \times W_p^{2,1}$.

Proof. To begin with, notice that the assumptions imply $\bar{u} \in L^p(Q)^2$. Thus $\bar{y}$ as well as $\bar{\lambda}$ are strong solutions of the respective equations, i.e. $\bar{y}$, $\bar{\lambda} \in W_p^{2,1}$, see Theorems 2.6 and 3.4.

Let $z_1, z_2 \in Z_p$ be given. Denote the optimal controls of the perturbed problem by $u_i := u_{z_i}$ with associated states y_i and adjoints λ_i, $i = 1, 2$.

At first, Theorem 5.3 yields stability of control, state, and adjoint in $L^2(Q)^2 \times H^{2,1} \times H^{2,1}$,

$$\|u_1 - u_2\|_2 + \|y_1 - y_2\|_{H^{2,1}} + \|\lambda_1 - \lambda_2\|_{H^{2,1}} \le c\,\|z_1 - z_2\|_{Z_2}.$$

By imbedding arguments, we have

$$\|\lambda_1 - \lambda_2\|_p \le c\|\lambda_1 - \lambda_2\|_{L^\infty(V)} \le c\|z_1 - z_2\|_{Z_2}.$$

The projection formula (4.1) yields,

$$\|u_1 - u_2\|_p \le c\,\{\|\lambda_1 - \lambda_2\|_p + \|z_{u,1} - z_{u,2}\|_p\} \le c\,\|z_1 - z_2\|_{Z_p}.$$

By Theorem 2.6 the weak solution $y_1 - y_2$ of (5.11) is also a strong solution and satisfies

$$\begin{aligned}\|y_1 - y_2\|_{L^p(W^{2,p})} &+ \|y_1 - y_2\|_{L^\infty(W^{2-2/p,\,p})} + \|y_{1,t} - y_{2,t}\|_p \\ &\le c\,\{|z_{0,1} - z_{0,2}|_{W^{2-2/p,\,p}} + \|z_{y,1} - z_{y,2}\|_p + \|u_1 - u_2\|_p\} \le c\,\|z_1 - z_2\|_{Z_p}.\end{aligned}$$

A similar estimate is valid also for the adjoint states, cf. Theorem 3.4,

$$\begin{aligned}\|\lambda_1 - \lambda_2\|_{L^p(W^{2,p})} &+ \|\lambda_1 - \lambda_2\|_{L^\infty(W^{2-2/p,\,p})} + \|\lambda_{1,t} - \lambda_{2,t}\|_p \\ &\le c\,\left\{\|z_1 - z_2\|_{Z_p} + \|y_1 - y_2\|_{L^p(W^{2,p})} + \|y_1 - y_2\|_{L^\infty(W^{2-2/p,\,p})}\right\} \\ &\le c\,\|z_1 - z_2\|_{Z_p}.\end{aligned}$$

This actually means that the mapping $z \mapsto \lambda$ is Lipschitz form Z_p to $W_p^{2,1}$. The space $W_p^{2,1}$ is continuously imbedded in $L^\infty(Q)^2$. Hence, it follows using the projection formula a last time

$$\|u_1 - u_2\|_\infty \le c\,\left\{\|\lambda_1 - \lambda_2\|_{L^\infty(Q)^2} + \|z_{u,1} - z_{u,2}\|_\infty\right\} \le c\,\|z_1 - z_2\|_{Z_p}.$$

□

Thus, we proved strong regularity of the equation (5.4) in the stronger setting (5.22), and Theorem 5.1 is applicable.

Theorem 5.9. *Let (SSC) be satisfied for the reference solution $\bar{v}$ with adjoint state $\bar{\lambda}$. Additionally, assume that $y_0, y_T \in W_0^{2-2/p,\,p}(\Omega)^2$, $y_Q \in L^p(Q)^2$ for some p satisfying $2 < p < \infty$, and $u_a, u_b \in L^\infty(Q)^2$. Then there exists $\rho > 0$ such that for all $z \in Z_p$ with $\|z\|_{Z_p} \le \rho$, the perturbed optimal control problem* (5.1) *admits a unique solution (y_z, u_z, λ_z). Moreover, the mapping $z \mapsto (y_z, u_z, \lambda_z)$ is Lipschitz continuous from Z_p to $W_p^{2,1} \times L^\infty(Q)^2 \times W_p^{2,1}$.*

Proof. Theorem 5.8 yields strong regularity of the equation (5.4) at the point $(\bar{y}, \bar{u}, \bar{\lambda})$. So we can apply Theorem 5.1 to conclude the claim. □

As already mentioned in Remark 5.7, it is not possible to derive stability results for *bounded* optimal controls in W_p^1-norms, $1 \le p \le \infty$. So the result of Theorem 5.9 cannot be improved in this direction.

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REFERENCES

[1] F. Abergel and R. Temam. On some control problems in fluid mechanics. *Theoret. Comput. Fluid Dynam.*, 1:303–325, 1990.
[2] R. A. Adams. *Sobolev spaces.* Academic Press, San Diego, 1978.

[3] E. Casas. An optimal control problem governed by the evolution Navier-Stokes equations. In S. S. Sritharan, editor, *Optimal control of viscous flows*, Frontiers in applied mathematics. SIAM, Philadelphia, 1993.

[4] P. Constantin and C. Foias. *Navier-Stokes equations*. The University of Chicago Press, Chicago, 1988.

[5] A. L. Dontchev, W. W. Hager, A. B. Poore, and B. Yang. Optimality, stability, and convergence in optimal control. *Appl. Math. Optim.*, 31:297–326, 1995.

[6] H. O. Fattorini and S. Sritharan. Necessary and sufficient for optimal controls in viscous flow problems. *Proc. Royal Soc. of Edinburgh*, 124:211–251, 1994.

[7] H. Goldberg and F. Tröltzsch. On a Lagrange-Newton method for a nonlinear parabolic boundary control problem. *Optimization Methods and Software*, 8:225–247, 1998.

[8] M. D. Gunzburger, editor. *Flow control*. Springer, New York, 1995.

[9] M. D. Gunzburger and S. Manservisi. The velocity tracking problem for Navier-Stokes flows with bounded distributed controls. *SIAM J. Contr. Optim.*, 37:1913–1945, 1999.

[10] M. D. Gunzburger and S. Manservisi. Analysis and approximation of the velocity tracking problem for Navier-Stokes flows with distributed control. *SIAM J. Numer. Anal.*, 37:1481–1512, 2000.

[11] M. Hintermüller and M. Hinze. A SQP-semi-smooth Newton-type algorithm applied to control of the instationary Navier-Stokes system subject to control constraints. Technical Report TR03-11, Department of Computational and Applied Mathematics, Rice University, 2003.

[12] M. Hinze. *Optimal and instantaneous control of the instationary Navier-Stokes equations*. Habilitation, TU Berlin, 1999.

[13] M. Hinze and K. Kunisch. Second order methods for optimal control of time-dependent fluid flow. *SIAM J. Contr. Optim.*, 40:925–946, 2001.

[14] D. Kinderlehrer and G. Stampacchia. *An introduction to variational inequalities and their applications*. Academic Press, New York, 1980.

[15] O. A. Ladyzhenskaya, V. A. Solonnikov, and N. N. Uralceva. *Linear and quasilinear equations of parabolic type*. AMS Publications, Providence, 1968.

[16] J. L. Lions and E. Magenes. *Non-homogeneous boundary value problems and applications*, volume I. Springer, Berlin, 1972.

[17] K. Malanowksi and F. Tröltzsch. Lipschitz stability of solutions to parametric optimal control for elliptic equations. *Control and Cybernetics*, 29:237–256, 2000.

[18] S. M. Robinson. Strongly regular generalized equations. *Mathematics of Operation Research*, 5:43–62, 1980.

[19] A. Rösch and D. Wachsmuth. Imbeddings of abstract functions with application to an optimal control problem. 2004. In preparation.

[20] T. Roubíček and F. Tröltzsch. Lipschitz stability of optimal controls for the steady-state Navier-Stokes equations. *Control and cybernetics. To appear*, 2002.

[21] H. Sohr. *The Navier-Stokes equations*. Birkhäuser, Basel, 2001.

[22] S. Sritharan. Dynamic programming of the Navier-Stokes equations. *Systems & Control Letters*, 16:299–307, 1991.

[23] R. Temam. *Navier-Stokes equations*. North Holland, Amsterdam, 1979.

[24] R. Temam. *Navier-Stokes equations and nonlinear functional analysis*. SIAM, Philadelphia, 2nd edition, 1995.

[25] H. Triebel. Function spaces in Lipschitz domains and on Lipschitz manifolds. Characteristic functions as pointwise multipliers. *Revista Matemática Complutense*, 15:475–524, 2002.

[26] F. Tröltzsch. A stability theorem for linear-quadratic parabolic control problems. In E. Casas, editor, *Control of partial differential equations and applications*, volume 174 of *Lecture Notes in Pure and Appl. Math.*, pages 287–296. Dekker, New York, 1996.

[27] F. Tröltzsch. Lipschitz stability of solutions of linear-quadratic parabolic control problems with respect to perturbations. *Dyn. Contin. Discrete Impulsive Syst.*, 7:289–306, 2000.

[28] F. Tröltzsch and D. Wachsmuth. Second-order sufficient optimality conditions for the optimal control of Navier-Stokes equations. Preprint 30-2003, Institut für Mathematik, TU Berlin, submitted, 2003.

[29] W. von Wahl. Instationary Navier-Stokes equations and parabolic systems. *Pacific J. Math.*, 72(2):557–569, 1977.

[30] W. von Wahl. Regularitätsfragen für die instationären Navier-Stokesschen Gleichungen in höheren Dimensionen. *J. Math. Soc. Japan*, 32(2):263–283, 1980.

[31] W. von Wahl. *The equations of Navier-Stokes and abstract parabolic equations*. Vieweg, Braunschweig, 1985.