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Distributed Control for a Class of Non-Newtonian Fluids

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Abstract

We consider control problems with a general cost functional where the state equations are the stationary, incompressible Navier-Stokes equations with shear-dependent viscosity. The equations are quasi-linear. The control function is given as the inhomogeneity of the momentum equation. In this paper we study a general class of viscosity functions which correspond to shear-thinning or shear-thickening behavior. The basic results concerning existence, uniqueness, boundedness, and regularity of the solutions of the state equations are reviewed. The main topic of the paper is the proof of Gâteaux differentiability, which extends known results. It is shown that the derivative is the unique solution to a linearized equation. Moreover necessary first order optimality conditions are stated, and the existence of a solution of a class of control problems is shown.

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1 Introduction

In this work we consider control problems of tracking type with distributed control for two-dimensional stationary, incompressible flow of non-Newtonian fluids with shear-dependent viscosity.
The considered class of fluids are described by a quasi-linear generalization of the Navier-Stokes system. The Laplace operator (i.e. the divergence of the velocity gradient) in the momentum equation is replaced by the divergence of a non-linear function of the symmetrized velocity gradient. In this paper we study a certain class of non-linearities that include both *shear-thinning* and *shear-thickening fluids*, that means fluids whose viscosity decreases or increases when the shear-rate – described by the symmetrized velocity gradient – grows.
Examples for such kind of fluids among others are blood and chemical suspensions. Several applications for control problems may be considered. Here the study of distributed control is only one example, also boundary and shape control problems may be of interest.
For the studied class of fluids a certain monotonicity of the non-linearity is assumed. Under this assumption existence, uniqueness, boundedness, and regularity results can be found in the literature. We want to emphasize the work of J.-L. Lions, Kaplický, Málek, Nečas, Rokyta, Ružička, Stará, Frehse, and Steinhauer, see [12, Chapter 2, section 5], [13], [10], [11], [6]. They mainly study the case of homogeneous Dirichlet boundary conditions, to which we also restrict our work here. Results on the state equations for inhomogeneous boundary conditions can be found in [2]. Numerical simulations were presented by Hron, Málek, and Turek in [9]. Control problems for non-Newtonian fluid flows have only been studied very rarely in the past. We mention the work of Casas and Fernández [3], [4], [5]. In [4] they showed Gâteaux differentiability for quasilinear equations with the same class of nonlinearity as in non-Newtonian fluids, but without the convective term and the divergence condition. Our differentiability proof basically relies on this work, but applies to a wider range of nonlinearities (with exponent $p > \frac{3}{2}$ rather than $p \geq 2$). This is due to the regularity results given in [11]. Moreover we treat the system case and the nonlinear convective term. In [14] a control problem for a scalar equation with a nonlinearity similar to the one in non-Newtonian fluids is analyzed, too. A recent paper by Abraham, Behr, and Heinkenschloss [1] studies numerical shape optimization for a non-Newtonian fluid.
The structure of the paper is as follows. We state the necessary assumptions on the state equation in section 2. In section 3 we give some examples for shear-dependent fluids and show how they fit in the abstract framework. In section 4 we state some preliminary lemmas. Then we summarize the basic existence and regularity results for the state equation, that are mainly based on [11]. We show Lipschitz continuity and thus uniqueness of the solution operator of the state equation in section 6. Afterwards we present the linearized equation, and show under which assumptions

it has a unique solution. In section 8 we present the central part of this work, namely the proof of Gâteaux differentiability. Here we extend the results in [4]. Finally we show the requirements for the existence of a solution in section 9 and formulate the necessary optimality conditions of first order in section 10.

2 State equation and assumptions on the non-linearity

In this section we present the formulation of the state equations and characterize the considered class of non-linearities.

The state equation under consideration is the following form of the quasilinear, stationary, and incompressible Navier-Stokes equation in a bounded domain $\Omega \in \mathbb{R}^2$ with C^2 boundary:

$$
\begin{aligned}
u \cdot \nabla u - \operatorname{div}(T(Du)) + \nabla \pi &= f \quad \text{in } \Omega \\
\operatorname{div} u &= 0 \quad \text{in } \Omega \\
u &= 0 \quad \text{on } \partial\Omega.
\end{aligned} \tag{2.1}
$$

Here u is the velocity vector and π the pressure. The velocity gradient

$$
\nabla u := \left(\frac{\partial u_j}{\partial x_i} \right)_{i,j=1,2} \in \mathbb{R}^{2\times 2} \tag{2.2}
$$

is a (2×2)-matrix (or tensor of second order). Note that we define it such that *the first index corresponds to the differentiation index*. By

$$
(u \cdot \xi) := \left(\sum_j u_j \xi_{ji} \right)_{i=1,2} \in \mathbb{R}^2, \quad u \in \mathbb{R}^2, \xi = (\xi_{ji})_{ji} \in \mathbb{R}^{2\times 2} \tag{2.3}
$$

we denote the scalar product between a vector and a tensor of second order. Here and from now on we omit the limits of the sum which are taken over $\{1, 2\}$. The non-linear convective term in (2.1) is defined as

$$
u \cdot \nabla u := \left(\sum_j u_j \frac{\partial u_i}{\partial x_j} \right)_{i=1,2} \in \mathbb{R}^2
$$

We define the (double) scalar product between tensors, and their norms by

$$
\begin{aligned}
(\xi : \eta) &:= \sum_{ij} \xi_{ij}\eta_{ij} \in \mathbb{R}, \quad |\eta| := (\eta : \eta)^{\frac{1}{2}}, \quad \xi, \eta \in \mathbb{R}^{2\times 2}, \\
(\zeta : \eta) &:= \left(\sum_{kl} \zeta_{ijkl}\eta_{kl} \right)_{i,j=1,2} \in \mathbb{R}^{2\times 2}, \quad \zeta \in \mathbb{R}^{2\times 2\times 2\times 2}, \eta \in \mathbb{R}^{2\times 2},
\end{aligned}
$$

and note that

$$
(\zeta : \eta) : \xi = (\xi : \zeta) : \eta, \qquad \eta, \xi \in \mathbb{R}^{2\times 2}, \zeta \in \mathbb{R}^{2\times 2\times 2\times 2}. \tag{2.4}
$$

By $\mathbb{S}$ we denote the subspace of symmetric tensors in $\mathbb{R}^{2\times 2}$. The non-linear tensor-valued function

$$
T := (T_{ij})_{i,j=1,2} : \mathbb{S} \to \mathbb{S}
$$

appearing in (2.1) is a function of the symmetrized velocity gradient defined by

$$
Du := \frac{1}{2} \left(\nabla u + (\nabla u)^T \right).
$$

Because of $|\eta| = |\eta^T|$ the symmetrized velocity gradient satisfies

$$
|Du| \leq |\nabla u|. \tag{2.5}
$$

We assume that T has a potential F, i.e.

$$T_{ij}(\eta) = \partial_{ij} F(|\eta|^2) = 2F'(|\eta|^2)\eta_{ij}, \quad i,j = 1,2, \quad \eta \in \mathbb{S}, \partial_{ij} := \frac{\partial}{\partial \eta_{ij}}$$

with

$$F \in C^2(\mathbb{R}_0^+, \mathbb{R}_0^+), \quad F(0) \;=\; 0, \quad T(0) = \partial_{ij} F(|\eta|^2)\big|_{\eta=0} = 0. \tag{2.6}$$

Moreover we assume that there exist C_1, C_2 such that

$$\begin{aligned} T'(\eta) : \xi : \xi \;=\; \sum_{ijkl} \partial_{ij} T_{kl}(\eta) \xi_{kl} \xi_{ij} \;&=\; \sum_{ijkl} \partial_{ij} \partial_{kl} F(|\eta|^2) \xi_{kl} \xi_{ij} \\ &\geq\; C_1 (1 + |\eta|^2)^{\frac{p-2}{2}} |\xi|^2 \end{aligned} \tag{2.7}$$

$$|\partial_{ij} T_{kl}(\eta)| \;=\; \left|\partial_{ij} \partial_{kl} F(|\eta|^2)\right| \;\leq\; C_2 (1 + |\eta|^2)^{\frac{p-2}{2}}, i,j,k,l = 1,2 \tag{2.8}$$

for all $\xi, \eta \in \mathbb{S}$ and some $p \in (1, \infty)$.

3 Examples for applications

In this section we present a class of non-linear tensor functions T that are used in applications and satisfy the assumptions made above. We consider

$$T(\eta) \;=\; \nu_0 \left(\mu_0 + |\eta|^2\right)^{\frac{p-2}{2}} \eta + \mu_\infty \eta, \qquad \eta = Du, \tag{3.1}$$

with $\nu_0 > 0, \mu_0, \mu_\infty \geq 0$. The case $p \in (1,2)$ correspond to *shear-thinning* fluids, whereas for $p > 2$ the fluid is called *shear-thickening*. If $\mu_0 = \mu_\infty = 0$ the fluid is said to obey a *Power-Law*. In this case and for $p = 2$ system (2.1) reduces to the well-known incompressible Navier-Stokes equations. We show that (3.1) satisfies the assumptions (2.6)–(2.8) with

$$F(|\eta|^2) \;=\; \frac{\nu_0}{p} \left(\mu_0 + |\eta|^2\right)^{\frac{p}{2}} + \frac{\mu_\infty}{2} |\eta|^2 + C.$$

Clearly (2.6) is satisfied if $C \in \mathbb{R}$ is chosen such that $F(0) = 0$. We obtain

$$\partial_{ij} T_{kl}(\eta) = \nu_0 \Big[(p-2) \left(\mu_0 + |\eta|^2\right)^{\frac{p-4}{2}} \eta_{ij} \eta_{kl} + \left(\mu_0 + |\eta|^2\right)^{\frac{p-2}{2}} \delta_{ik} \delta_{jl} \Big] + \mu_\infty \delta_{ik} \delta_{jl}$$

for $i,j,k,l = 1,2$. Here $(\delta_{ij})_{ij}$ denotes the Kronecker or identity tensor. We note that $|\eta_{ij}\eta_{kl}| \leq |\eta|^2 \leq \mu_0 + |\eta|^2, |\delta_{ij}\delta_{kl}| \leq 1$. For $p \in (\frac{3}{2}, 2)$ this implies

$$|\partial_{ij} T_{kl}(\eta)| \;\leq\; \nu_0 (p-1) \left(\mu_0 + |\eta|^2\right)^{\frac{p-2}{2}} + \mu_\infty \;\leq\; \nu_0 (p-1) + \mu_\infty.$$

Thus (2.8) is satisfied for $\mu_\infty = 0, \mu_0 \geq 1$. The same is true for $\mu_\infty = 0, \mu_0 \in (0,1)$ since

$$|\partial_{ij} T_{kl}(\eta)| \;\leq\; \nu_0 (p-1) \mu_0^{\frac{p-2}{2}} \left(1 + \mu_0^{-1} |\eta|^2\right)^{\frac{p-2}{2}} \;\leq\; c \left(1 + |\eta|^2\right)^{\frac{p-2}{2}}.$$

If $p \in (\frac{3}{2}, 2), \mu_\infty > 0$ then (2.8) is still valid, taking $p = 2$. For $p \in [2, \infty)$ and $\mu_0, \mu_\infty \geq 0$ we get

$$|\partial_{ij} T_{kl}(\eta)| \leq \nu_0 (p-1) \left(\mu_0 + |\eta|^2\right)^{\frac{p-2}{2}} + \mu_\infty^{\frac{2-p}{2}} \left(\mu_\infty + |\eta|^2\right)^{\frac{p-2}{2}} \leq c_1 \left(c_2 + |\eta|^2\right)^{\frac{p-2}{2}}.$$

Thus (2.8) holds since

$$|\partial_{ij} T_{kl}(\eta)| \;\leq\; \begin{cases} c_1 \left(1 + |\eta|^2\right)^{\frac{p-2}{2}}, & c_2 \leq 1, \\ c_1 c_2^{\frac{p-2}{2}} \left(1 + c_2^{-1} |\eta|^2\right)^{\frac{p-2}{2}} \leq c_1 c_2^{\frac{p-2}{2}} \left(1 + |\eta|^2\right)^{\frac{p-2}{2}}, & c_2 > 1. \end{cases}$$

To check (2.7) we note that

$$\begin{aligned}\sum_{ijkl}\eta_{ij}\eta_{kl}\xi_{kl}\xi_{ij} &= \sum_{ij}\eta_{ij}\xi_{ij}\sum_{kl}\eta_{kl}\xi_{kl} = (\eta:\xi)^2 \le |\eta|^2|\xi|^2 \ \le\ \left(\mu_0+|\eta|^2\right)|\xi|^2,\\ \sum_{ijkl}\delta_{ik}\delta_{jl}\xi_{kl}\xi_{ij} &= \sum_{ij}\xi_{ij}\xi_{ij} \ = \ |\xi|^2\end{aligned}$$

and hence

$$T'(\eta):\xi:\xi = \nu_0\Big[(p-2)\left(\mu_0+|\eta|^2\right)^{\frac{p-4}{2}}(\eta:\xi)^2 + \left(\mu_0+|\eta|^2\right)^{\frac{p-2}{2}}|\xi|^2\Big] + \mu_\infty|\xi|^2.$$

For $p \in (\frac{3}{2}, 2)$, i.e. $p-2<0$, and all $\mu_\infty \ge 0$ we may estimate

$$\begin{aligned}T'(\eta):\xi:\xi &\ge \nu_0\Big[(p-2)\left(\mu_0+|\eta|^2\right)^{\frac{p-4}{2}}\left(\mu_0+|\eta|^2\right) + \left(\mu_0+|\eta|^2\right)^{\frac{p-2}{2}}\Big]|\xi|^2\\ &= \nu_0(p-1)\left(\mu_0+|\eta|^2\right)^{\frac{p-2}{2}}|\xi|^2.\end{aligned}$$

This proves (2.7) since

$$\left(\mu_0+|\eta|^2\right)^{\frac{p-2}{2}}\begin{cases}\ge \ \left(1+|\eta|^2\right)^{\frac{p-2}{2}}, & \mu_0<1,\\ = \ \mu_0^{\frac{p-2}{2}}\left(1+\mu_0^{-1}|\eta|^2\right)^{\frac{p-2}{2}} \ \ge \ \mu_0^{\frac{p-2}{2}}\left(1+|\eta|^2\right)^{\frac{p-2}{2}}, & \mu_0>1.\end{cases}$$

For $p \in [2,\infty)$ we may estimate

$$T'(\eta):\xi:\xi \ \ge\ \nu_0\Big[\left(\mu_0+|\eta|^2\right)^{\frac{p-2}{2}}+\mu_\infty\Big]|\xi|^2 \ \ge\ \nu_0\left(\mu_0+|\eta|^2\right)^{\frac{p-2}{2}}|\xi|^2.$$

This proves (2.7) since (as above)

$$\left(\mu_0+|\eta|^2\right)^{\frac{p-2}{2}} \ \ge\ \begin{cases}\left(1+|\eta|^2\right)^{\frac{p-2}{2}}, & \mu_0\ge 1,\\ \mu_0^{\frac{p-2}{2}}\left(1+|\eta|^2\right)^{\frac{p-2}{2}}, & \mu_0\in(0,1).\end{cases}$$

Summarizing we obtain that (2.7) and (2.8) are satisfied by T defined in (3.1) for all $p>1, \mu_\infty \ge 0$, and $\mu_0>0$.

4 Preliminary results

In this section we state some basic results that we will use throughout the paper. From now on we use the notation $p' := \frac{p}{p-1}$, i.e. $\frac{1}{p}+\frac{1}{p'}=1$, and $W^{k,p}(\Omega)$ for Sobolev spaces of functions whose weak derivatives up to order k are in $L^p(\Omega)$ for $k\in\mathbb{N}, p\in[1,\infty]$. We denote by $\|\cdot\|_p$ the $L^p(\Omega)$ norm, and use

$$\|u\|_{1,p} := \|u\|_p + \|\nabla u\|_p$$

as norm on $W^{1,p}(\Omega)$. We will use the following notation:

$$\begin{aligned}(u,v) &:= \int_\Omega u\cdot v\,dx, \quad u\in L^p(\Omega)^2, v\in L^{p'}(\Omega)^2,\\ (\xi,\eta) &:= \int_\Omega \xi:\eta\,dx, \quad \xi\in L^p(\Omega)^{2\times2}, \eta\in L^{p'}(\Omega)^{2\times2}.\end{aligned}$$

For simplicity we omit the space dimension $d=2$ in the function space notation, i.e. $W^{k,p}(\Omega)$ means $W^{k,p}(\Omega)^2$ or $W^{k,p}(\Omega)^{2\times2}$, respectively. The meaning should be clear from the context. We recall the following embedding result.

Lemma 4.1 *For $k\in\mathbb{N}\cup\{0\}, \Omega\subset\mathbb{R}^2$ the embedding $W^{k+1,p}(\Omega)\hookrightarrow W^{k,q}(\Omega)$ is*

- *continuous for* $p \in (\frac{3}{2}, 2)$ *and* $q = \frac{2p}{2-p}$,
- *compact for* $p \in (\frac{3}{2}, 2)$ *and* $q < \frac{2p}{2-p}$,
- *compact for* $p \geq 2$ *and* $1 \leq q < \infty$.

Moreover we will need the following two classical inequalities:

Lemma 4.2 (Poincaré's inequality) *Let* $u \in W_0^{1,q}(\Omega)$ *for* $q \in [1, \infty]$. *Then there exists* $P_q = P_q(\Omega) \leq 1$ *such that*

$$\|u\|_q \quad \leq \quad P_q \|u\|_{1,q} \; \leq \; \frac{P_q}{1-P_q} \|\nabla u\|_q.$$

Proof: See for example [7, Chapter I, Theorem 1.1]. □

Lemma 4.3 (Korn's inequality) *Let* $u \in W_0^{1,q}(\Omega)$ *for* $q \in (1, \infty)$. *Then there exists* $K_q > 0$ *such that*

$$K_q \|u\|_{1,q} \quad \leq \quad \|Du\|_q.$$

Proof: See [13, Chapter 5, Theorem 1.10]. □

As a consequence of (2.6-2.8) we get:

Lemma 4.4 *For all* $\xi, \eta \in \mathbb{S}$ *and some* $C_i > 0$ *the function* T *satisfies:*

$$T(\eta) : \eta \quad \geq \quad C_3 (1 + |\eta|^2)^{\frac{p-2}{2}} |\eta|^2, \quad p \in [2, \infty), \tag{4.1}$$

$$|T(\eta)| \quad \leq \quad C_2 \left(1 + |\eta|^2\right)^{\frac{p-2}{2}} |\eta|, \quad p \in (1, \infty), \tag{4.2}$$

$$(T(\eta) - T(\xi)) : (\eta - \xi) \quad \geq \quad \begin{cases} C_5(\eta, \xi) |\eta - \xi|^2, & p \in (1, \infty), \\ C_6 |\eta - \xi|^p & p \in [2, \infty), \end{cases} \tag{4.3}$$

$$\textit{where} \qquad C_5(\eta, \xi) \quad = \quad C_1 \int_0^1 \left(1 + |\xi + t(\eta - \xi)|^2\right)^{\frac{p-2}{2}} \, dt.$$

Proof: See [13, Chapter 5, Lemma 1.19]. Note that for $p \in [2, \infty)$ condition $(1.8)_2$ in this reference implies $(1.8)_1$ which gives (4.1). For (4.2) and the second estimate in (4.3) see also [11, (1.7), (1.8)]. □

Setting $\eta = Du$ we obtain the following consequences:

Lemma 4.5 *For* $u \in W^{1,p}(\Omega)$ *the tensor function* T *satisfies*

$$T(Du) \quad \in \quad L^{p'}(\Omega) \quad \textit{for } p \in (1, \infty), \tag{4.4}$$

$$T'(Du) \quad \in \quad L^{\infty}(\Omega), \quad \|T'(Du)\|_\infty \; \leq \; C_2 \quad \textit{for } p \in (1, 2], \tag{4.5}$$

$$T'(Du) \quad \in \quad L^{\frac{p}{p-2}}(\Omega), \quad \textit{for } p \in (2, \infty). \tag{4.6}$$

Proof: To show (4.4) we use (4.2) and obtain for $p \leq 2$ that

$$|T(Du)| \quad \leq \quad C_2 \left(1 + |Du|^2\right)^{\frac{p-2}{2}} |Du| \; \leq \; |Du|^{p-1}.$$

Now $Du \in L^p(\Omega)$ gives $T(Du) \in L^{\frac{p}{p-1}}(\Omega) = L^{p'}(\Omega)$.
For $p > 2$ we note that for $f \in L^s, g \in L^{s'}$ the product fg is in $L^{p'}$ if $\frac{1}{s} + \frac{1}{s'} = \frac{1}{p'}$. Since Ω is bounded $Du \in L^p(\Omega)$ implies $(1 + |Du|^2) \in L^{\frac{p}{2}}(\Omega)$ and $(1 + |Du|^2)^{\frac{p-2}{2}} \in L^{\frac{p}{p-2}}(\Omega)$. Now $T(Du) \in L^{p'}(\Omega)$ since $\frac{p-2}{p} + \frac{1}{p} = \frac{p-1}{p} = \frac{1}{p'}$.
For (4.5) assumption (2.8) implies in the case $p \leq 2$ that

$$|\partial_{ij} T_{kl}(Du)| \quad \leq \quad C_2 (1 + |Du|^2)^{\frac{p-2}{2}} \leq C_2$$

for almost all $x \in \Omega$ and $i, j, k, l = 1, 2$. Thus $T'(Du) \in L^\infty(\Omega)$. For $p > 2$ the fact that $(1 + |Du|^2) \in L^{\frac{p}{2}}(\Omega)$ gives $T'(Du) \in L^{\frac{p}{p-2}}(\Omega)$. □

5 Existence, uniqueness, and regularity of weak solutions

In this section we present a weak formulation of problem (2.1). In order to eliminate the pressure π we work in the divergence-free spaces

$$V_p \quad := \quad \{u \in W_0^{1,p}(\Omega), \operatorname{div} u = 0 \text{ in } \Omega\}, \quad p \in [1, \infty].$$

Endowed with the $W^{1,p}(\Omega)$ norm, V_p is a Banach, and for $p = 2$ a Hilbert space.
The proper definition of weak solutions depends on the parameter p.

Weak solutions for $p \in [\frac{3}{2}, \infty)$

For $f \in V_p^*$ and $p \geq \frac{3}{2}$ we call $u \in V_p$ a weak solution to (2.1) if

$$(u \cdot \nabla u, v) + (T(Du), Dv) \quad = \quad \langle f, v \rangle_{V_p^*, V_p} \quad \text{for all } v \in V_p. \tag{5.1}$$

This lower bound on p is required for the existence of the convective term.

Lemma 5.1 *The integral in the convective term $(u \cdot \nabla u, v)$ exists for $u, v \in W^{1,p}(\Omega)$ if $p \geq \frac{3}{2}$.*

Proof: Hölder's inequality implies

$$(u \cdot \nabla u, v) \quad \leq \quad \|u\|_s \|\nabla u\|_p \|v\|_s \;\leq\; \|u\|_s \|u\|_{1,p} \|v\|_s$$

for $\frac{2}{s} = 1 - \frac{1}{p}$, i.e. $s = \frac{2p}{p-1}$. The embedding result in Lemma 4.1 gives $\|u\|_s \leq c\|u\|_{1,p}$ for $s \leq \frac{2p}{2-p}$. Combining both gives $p \geq \frac{3}{2}$. □

The second term on the left-hand side of (5.1) exists for arbitrary $p \in (1, \infty)$ because of $T(Du) \in L^{p'}(\Omega)$ due to Lemma 4.4.
We will need the following anti-symmetry property of the convective term.

Lemma 5.2 *Let $u \in V_p$ and $v, w \in W^{1,p}(\Omega)$. Then $(u \cdot \nabla v, w) = -(u \cdot \nabla w, v)$ and $(u \cdot \nabla w, w) = 0$.*

Proof: The proof in [7, Lemma IV.2.2] can be generalized for $p \geq \frac{3}{2}$. □

We have the following existence and regularity result.

Theorem 5.1 *(i) For all $p \in [\frac{3}{2}, \infty)$ and $f \in V_p^*$ there exists a solution $u \in V_p$ to (5.1).*

(ii) For $p \in (\frac{3}{2}, 2)$ and $f \in L^{p'}(\Omega)$ there exists a solution $u \in V_p \cap W^{2,q}(\Omega) \cap C^{1,\alpha}(\bar{\Omega})$ to (5.1) for some $q > 2, \alpha > 0$.

(iii) For $p \in [2, \infty)$ and $f \in L^s(\Omega), s > 2$, there exists a solution $u \in V_p \cap W^{2,q}(\Omega) \cap C^{1,\alpha}(\bar{\Omega})$ to (5.1) for some $q > 2, \alpha > 0$.

Proof: For (i) see [6, Theorem 1.1]. The assumptions on T made there are weaker then the ones made here. For (ii) see [11, Theorem 5.30], and for (iii) [11, Theorem 6.1] in the case $p > 2$, [11, Theorem 3.19] in the case $p = 2$. □

Uniqueness of the solution is obtained if the inhomogeneity is sufficiently small.

Theorem 5.2 *Let $p \in [\frac{3}{2}, \infty)$ and $f \in V_p^*$ with $\|f\|_{V_p^*}$ sufficiently small. Then there exists a unique solution $u \in V_p$ to (5.1).*

Proof: See [11, Theorem 6.7]. □

Due to this result we may define the solution operator

$$G : V_p^* \supset \mathcal{F} \quad \to \quad V_p, \quad f \mapsto u, \tag{5.2}$$

for a bounded subset $\mathcal{F}$ and $p \in (\frac{3}{2}, \infty)$.
In the next two theorems we show boundedness of the solution.

Theorem 5.3 *Let $p \in (\frac{3}{2}, 2)$ and $f \in L^{p'}(\Omega)$. Then every solution $u \in V_p$ of problem (5.1) satisfies*

$$\begin{aligned} \|u\|_{1,p} &\le C(\|f\|_{V_p^*}) \\ \|u\|_{1,\infty} &\le C_0(\|f\|_{p'}) \end{aligned}$$

with continuous nonnegative functions C, C_0 and $\lim_{s\to 0} C(s) = \lim_{s\to 0} C_0(s) = 0$.

Proof: See the proof of Theorem 6.7, and equation (6.12) in [11]. The continuity that we will use below can be deduced from [11, Sections 3 and 4]. $\square$

Theorem 5.4 *Let $p \in [2, \infty)$ and $f \in V_p^*$. Then every solution $u \in V_p$ of problem (5.1) satisfies*

$$\|u\|_{1,p} \le c\|f\|_{V_p^*}^{\frac{1}{p-1}}$$

where $c > 0$ is independent of f.

Proof: Setting $v = u \in V_p$ in (5.1) and using Lemma 5.2 we obtain

$$(T(Du), Du) = \langle f, u\rangle_{V_p^*, V_p} \le \|f\|_{V_p^*}\|u\|_{1,p}.$$

On the other hand (4.1), the fact that $p - 2 \ge 0$, and Korn's inequality give

$$\begin{aligned} (T(Du), Du) &\ge C_3 \int_\Omega \left(1 + |Du|^2\right)^{\frac{p-2}{2}} |Du|^2 dx \\ &\ge C_3 \int_\Omega |Du|^{p-2}|Du|^2 dx = C_3\|Du\|_p^p \ge c_2\|u\|_{1,p}^p. \end{aligned}$$

$\square$

We finish this section with brief remarks on weak solutions for $p < \frac{3}{2}$.

Weak solutions for $p \in (1, \frac{3}{2})$

In this case the convective term $(u \cdot \nabla u, v)$ is not well-defined for $u, v \in V_p$. A remedy is to write it as

$$(u \cdot \nabla u, v) = -(u \otimes u, Dv)$$

using the tensor $u \otimes u := (u_i u_j)_{ij} \in \mathbb{S}$. Taking test functions in the space

$$C_{0,\sigma}^\infty(\bar\Omega) := \{v \in C_0^\infty(\bar\Omega) : \operatorname{div} v = 0 \text{ in } \Omega\}$$

allows us to define $u \in V_p$ as a weak solution of (2.1) if it satisfies

$$(T(Du), Dv) - (u \otimes u, Dv) = \langle f, v\rangle_{V_p^*, V_p} \quad \text{for all } v \in C_{0,\sigma}^\infty(\Omega). \tag{5.3}$$

This approach is used in [6], and existence of a weak solution is shown for $p > 1$ (in two space dimensions), see [6, Theorem 1.1].
The existence of a strong solution $u \in V_p \cap W_{loc}^{2,q}(\Omega) \cap C_{loc}^{1,\alpha}(\Omega)$ for some $q > 2, \alpha > 0$ is shown for $p > \frac{6}{5}$ and $f \in L^{p'}(\Omega)$ in [11, Theorem 4.26]. The test function space $C^1(\bar\Omega)$, equation (5.3) and an additional energy equation is used for the definition of weak solutions.

6 Lipschitz continuity of the solution

In this section we show Lipschitz continuity of weak solutions with respect to the inhomogeneity. We consider $p \in (\frac{3}{2}, \infty)$ throughout this section.

Theorem 6.1 *Let $p \in (\frac{3}{2}, 2)$ and let $u, \bar{u}$ denote solutions to (5.1) for $f, \bar{f} \in L^{p'}(\Omega)$, respectively, with $\|f\|_{V_p^*}$ sufficiently small. Then there exists $L = L(f) > 0$ such that*

$$\|u - \bar{u}\|_{1,2} \quad \leq \quad L\|f - \bar{f}\|_{p'}.$$

Proof: Equation (5.1) gives

$$(u \cdot \nabla u - \bar{u} \cdot \nabla \bar{u}, v) + (T(Du) - T(D\bar{u}), Dv) \quad = \quad \langle f - \bar{f}, v \rangle_{V_p^*, V_p}$$

for all $v \in V_p$. At first we note that for $z := u - \bar{u}$ we have

$$u \cdot \nabla u - \bar{u} \cdot \nabla \bar{u} \quad = \quad z \cdot \nabla u + u \cdot \nabla z + z \cdot \nabla z. \tag{6.4}$$

We set $v = z \in V_p \cap W^{1,\infty}(\Omega)$ and obtain with Lemma 5.2 that

$$(z \cdot \nabla u, z) + (T(Du) - T(D\bar{u}), Dz) \quad = \quad (f - \bar{f}, z) \tag{6.5}$$

where

$$(z \cdot \nabla u, z) \quad \leq \quad c_1 \|\nabla u\|_p \|z\|_{1,2}^2$$

due to Lemma 5.1 and $p \in (\frac{3}{2}, 2)$. For a.e. $x \in \Omega$ the mean value theorem gives

$$T(Du(x)) - T(D\bar{u}(x)) \quad = \quad \left(\int_0^1 T'(D\bar{u}(x) + tDz(x))\, dt \right) : Dz(x) \;=:\; \beta(x)$$

since $T \in C^1(\mathbb{S})$. From (2.7) we get

$$\beta(x) : Dz(x) \quad \geq \quad C_1 \int_0^1 \left(1 + |T'(D\bar{u}(x) + tDz(x))|^2\right)^{\frac{p-2}{2}} : Dz(x) : Dz(x)\, dt.$$

Moreover (2.8) gives with $p - 2 \leq 0$ that

$$|T'(D\bar{u}(x) + tDz(x))|^2 \quad \leq \quad 4C_2^2 \left(1 + |D\bar{u}(x) + tDz(x))|^2\right)^{p-2} \;\leq\; 4C_2^2$$

for all $x \in \Omega$ and $t \in [0, 1]$. This implies

$$\beta(x) : Dz(x) \geq C_1 \int_0^1 \left(1 + 4C_2^2\right)^{\frac{p-2}{2}} dt |Dz(x)|^2 = C_1 \left(1 + 4C_2^2\right)^{\frac{p-2}{2}} |Dz(x)|^2$$

for all $x \in \Omega$ and thus

$$\begin{aligned}(T(Du) - T(D\bar{u}), Dz) \;=\; (\beta, Dz) \quad &\geq \quad C_1 \left(1 + 4C_2^2\right)^{\frac{p-2}{2}} \|Dz\|_2^2 \\ &\geq \quad C_1 \left(1 + 4C_2^2\right)^{\frac{p-2}{2}} K_2^2 \|z\|_{1,2}^2 \;=:\; c_2 \|z\|_{1,2}^2\end{aligned}$$

using Korn's inequality. From (6.5) we now obtain

$$(c_2 - c_1 \|\nabla u\|_p) \|z\|_{1,2}^2 \quad \leq \quad c\|f - \bar{f}\|_{p'} \|z\|_{1,2}.$$

Thus we have shown Lipschitz continuity if

$$\|\nabla u\|_p \quad < \quad \frac{c_2}{c_1}.$$

By Theorem 5.3 this estimate is fulfilled if $\|f\|_{V_p^*}$ is sufficiently small. □

Theorem 6.2 *Let $p \in [2,\infty)$ and let $u, \bar{u}$ denote solutions to (5.1) for $f, \bar{f} \in V_2^*$, respectively, with $\|f\|_{V_p^*}$ sufficiently small. Then there exists $L = L(f) > 0$ such that*

$$\|u-\bar{u}\|_{1,2} \quad \leq \quad L\|f-\bar{f}\|_{V_2^*}.$$

Proof: We proceed as above up to (6.5) and estimate

$$(z\cdot\nabla u, z) \;\leq\; \|\nabla u\|_p \|z\|_q^2 \quad \leq \quad E_2^2 \|\nabla u\|_p \|z\|_{1,2}^2$$

for $\frac{1}{p}+\frac{2}{q}=1$, where E_2 is the embedding constant $W_0^{1,2}(\Omega) \hookrightarrow L^q(\Omega)$. On the other hand (4.3) and Korn's inequality imply

$$\begin{aligned}(T(Du)-T(D\bar{u}), Dz) \quad &\geq \quad C_1 \int_\Omega \int_0^1 \left(1+|D\bar{u}+tDz|^2\right)^{\frac{p-2}{2}} dt |Dz|^2 dx \\ &\geq \quad C_1\|Dz\|_2^2 \;\geq\; C_1K_2^2\|z\|_{1,2}^2.\end{aligned}$$

From (6.5) we obtain

$$\left(C_1K_2^2 - E_2^2\|\nabla u\|_p\right)\|z\|_{1,2}^2 \quad \leq \quad \|f-\bar{f}\|_{V_2^*}\|z\|_{1,2}.$$

Thus we have shown local Lipschitz continuity if

$$\|\nabla u\|_p \quad < \quad C_1K_2^2E_2^{-2}.$$

By Theorem 5.4 this estimate is fulfilled if $\|f\|_{V_p^*}$ is sufficiently small. □

As a direct consequence the dependency of the solution on the inhomogeneity for $p > 2$ is still continuous (but not Lipschitz) with respect to the $W^{1,p-\varepsilon}(\Omega)$ norm for $\varepsilon > 0$.

Corollary 6.1 *Let $p \in (2,\infty)$ and $f_k \to f$ in V_2^* with $\|f\|_{V_p^*}$ sufficiently small. Let u_k, u denote the solutions to (5.1) with inhomogeneities f_k, f, respectively. Then $u_k \to u$ in $V_{p-\varepsilon}$ for $\varepsilon > 0$.*

Proof: By Theorem 5.4 we know that $\{u_k\}_k$ is bounded in V_p. Thus a subsequence converges weakly in V_p and strongly in $L^p(\Omega)$ to some $\bar{u} \in V_p$. Since $V_2 \hookrightarrow L^p(\Omega)$ Theorem 6.2 implies $\bar{u} = u$. Now it suffices to show that $\nabla u_k \to \nabla u$ in $V_{p-\varepsilon}$. Hölder's inequality gives

$$\|\nabla u_k - \nabla u\|_{p-\varepsilon} \quad \leq \quad \|\nabla u_k - \nabla u\|_2^\theta \, \|\nabla u_k - \nabla u\|_p^{1-\theta}$$

for some $\theta \in (0,1)$. Since the first term on the right tends to zero for $k \to \infty$ and the second one is bounded, the result follows. □

7 The Linearized Equation

To show the differentiability of the solution operator G defined in (5.2) we study the linearized equation in weak form,

$$(z\cdot\nabla u, v) + (u\cdot\nabla z, v) + (T'(Du) : Dz, Dv) \;=\; \langle g, v\rangle_{V_p^*,V_p} \text{ for all } v \in V_p \tag{7.1}$$

with $u \in W^{1,\infty}(\Omega)$ and $g \in V_p^*$ given. We will show that this equation has a unique solution. In the following lemma we show that (7.1) is well-defined.

Lemma 7.1 *For $p \in (\frac{3}{2},\infty)$ and fixed $u \in W^{1,\infty}(\Omega)$ the bilinear form*

$$a_u(w,v) \quad := \quad (w\cdot\nabla u, v) + (u\cdot\nabla w, v) + (T'(Du) : Dw, Dv) \tag{7.2}$$

is continuous on $W^{1,s}(\Omega) \times W^{1,s'}(\Omega)$ for all $s \in [1,\infty]$.

Proof: The result follows immediately from Hölder's inequality and $u \in W^{1,\infty}(\Omega)$ which implies $T'(Du) \in L^{\infty}(\Omega)$ due to the continuity of T'. $\square$

We now show coercivity of the bilinear form a_u.

Lemma 7.2 *For $p \in (\frac{3}{2}, 2)$ and $u \in V_p \cap W^{1,\infty}(\Omega)$ with $\|\nabla u\|_\infty$ sufficiently small the bilinear form a_u defined in (7.2) is coercive on V_2.*

Proof: For $u \in V_p, z \in V_2$ Lemma 5.2 gives

$$a_u(z,z) \quad = \quad (z \cdot \nabla u, z) + (T'(Du) : Dz, Dz).$$

Hölder's and Poincaré's inequalities imply

$$(z \cdot \nabla u, z) \ \leq \ \|\nabla u\|_\infty \|z\|_2^2 \quad \leq \quad P_2^2 \|\nabla u\|_\infty \|z\|_{1,2}^2.$$

Assumption (2.7) implies with (2.5), Poincaré's and Korn's inequality:

$$\begin{aligned} (T'(Du) : Dz, Dz) \quad &\geq \quad C_1 \int_\Omega \left(1 + |Du|^2\right)^{\frac{p-2}{2}} |Dz|^2 \, dx \\ &\geq \quad K_2^2 C_1 \left(1 + \|\nabla u\|_\infty^2\right)^{\frac{p-2}{2}} \|z\|_{1,2}^2. \end{aligned}$$

Thus we obtain

$$a_u(z,z) \quad \geq \quad \Big(K_2^2 C_1 (1 + \|\nabla u\|_\infty^2)^{\frac{p-2}{2}} - P_2^2 \|\nabla u\|_\infty\Big) \|z\|_{1,2}^2.$$

Now a_u is coercive if the term in the brackets is positive. Since $(1 + \|\nabla u\|_\infty^2)^{\frac{p-2}{2}}$ is a positive, decreasing function (for $p \in (\frac{3}{2}, 2)$) with respect to $\|\nabla u\|_\infty$ this is true for $\|\nabla u\|_\infty$ sufficiently small. $\square$

Lemma 7.3 *For $p \in [2, \infty)$ and $u \in V_2$ with $\|\nabla u\|_2$ sufficiently small the bilinear form a_u defined in (7.2) is coercive on V_2.*

Proof: Here we estimate

$$(z \cdot \nabla u, z) \ \leq \ \|\nabla u\|_2 \|z\|_4^2 \quad \leq \quad E_4^2 \|\nabla u\|_2 \|z\|_{1,2}^2$$

where E_4 is the embedding constant $W_0^{1,2}(\Omega) \hookrightarrow L^4(\Omega)$. Since $p \in [2, \infty)$ assumption (2.7) implies

$$(T'(Du) : Dz, Dz) \quad \geq \quad C_1 \int_\Omega |Dz|^2 \, dx \ = \ C_1 \|Dz\|_2^2 \ \geq \ K_2^2 C_1 \|z\|_{1,2}^2.$$

Thus

$$a_u(z,z) \quad \geq \quad \left(K_2^2 C_1 - E_4^2 \|\nabla u\|_2\right) \|z\|_{1,2}^2$$

and a_u is coercive for $\|\nabla u\|_2 < K_2^2 C_1 E_4^{-2}$. $\square$

If u is a solution to (5.1) with sufficiently small inhomogeneity we now deduce uniqueness of the solution of the linearized equation.

Theorem 7.1 *Let $p \in (\frac{3}{2}, 2)$ and $f \in L^{p'}(\Omega)$ or $p \in [2, \infty)$ and $f \in L^s(\Omega), s > 2$, with $\|f\|_{V_p^*}$ sufficiently small in both cases. Let u denote the solution of (5.1). Then for every $g \in V_2^*$ equation (7.1) has a unique solution $z \in V_2$ satisfying*

$$\|z\|_{1,2} \quad \leq \quad c \, \|g\|_{V_2^*}$$

with a constant $c = c(f) > 0$.

Proof: Theorem 5.1 implies $u \in W^{1,\infty}(\Omega) \cap V_p$. We already have shown the continuity and coercivity of the bilinear form a_u if $\|\nabla u\|_\infty$ (for $p \in (\frac{3}{2}, 2)$) or $\|\nabla u\|_2$ (for $p \in [2, \infty)$), respectively, is sufficiently small. By Theorems 5.3 and 5.4 these assumptions are given in both cases for $\|f\|_{V_p^*}$ sufficiently small. Thus the Lax-Milgram Theorem implies existence, uniqueness, and the estimate of the solution to the linearized equation. $\square$

8 Differentiability

In this section we show Gâteaux differentiability of the solution operator G. We follow the proof of Casas and Fernández [4, Theorem 3.1]. As an extension to this work the regularity result stated in Theorem 5.1 enables us to treat also the case $p \in (\frac{3}{2}, 2)$. Moreover we treat a *system* of quasi-linear equations and a different nonlinearity, namely the convective term. We assume

- $p \in (\frac{3}{2}, 2)$ and $f, h \in L^{p'}(\Omega)$ or
- $p \in [2, \infty)$ and $f, h \in L^s(\Omega), s > 2$,

both with $\|f\|_{V_p^*}$ sufficiently small. Note that taking V_2^* instead of $L^2(\Omega)$ would be sufficient in the latter case.
For $t > 0$ we denote by $u = G(f), u_t := G(f + th) \in V_p$ the unique solutions to (5.1), compare the definition of the operator G in (5.2). Subtracting (5.1) for u_t, u, respectively, gives

$$(T(Du_t) - T(Du), Dv) + (u_t \cdot \nabla u_t - u \cdot \nabla u, v) = t(h, v) \text{ for all } v \in V_p. \tag{8.1}$$

Since $u, u_t \in C^1(\bar{\Omega})$ the mean value theorem implies for all $x \in \Omega, t > 0$

$$T(Du_t(x)) - T(Du(x)) = M_t(x) : D(u_t - u)(x) \tag{8.2}$$

$$M_t(x) := \left(\int_0^1 T'(\beta_t(\tau)(x))\, d\tau \right), \tag{8.3}$$

$$\beta_t(\tau)(x) := Du(x) + \tau D(u_t(x) - u(x)), \tau \in [0, 1].$$

For $z_t := \frac{1}{t}(u_t - u), t > 0$, moreover (6.4) implies

$$u_t \cdot \nabla u_t - u \cdot \nabla u = t(z_t \cdot \nabla u + u \cdot \nabla z_t - t z_t \cdot \nabla z_t). \tag{8.4}$$

Thus we obtain

$$(M_t : Dz_t, Dz_t) + (z_t \cdot \nabla u + u \cdot \nabla z_t - t z_t \cdot \nabla z_t, z_t) = (h, z_t). \tag{8.5}$$

We split up the proof of differentiability into several parts. First we show boundedness of the sequence $\{z_t\}_{t>0}$.

Lemma 8.1 *For $p > \frac{3}{2}$ the sequence $\{z_t\}_{t>0}$ is bounded in V_2.*

Proof: By Theorems 6.1 and 6.2 we obtain

$$\|z_t\|_{1,2} = \frac{1}{t}\|u_t - u\|_{1,2} \le L(f)\|h\|_{\bar{p}} \qquad \text{for all } t > 0$$

with $\bar{p} = p'$ for $p \in (\frac{3}{2}, 2)$ and $\bar{p} = s$ for $p \in [2, \infty)$. The continuous embedding $L^s(\Omega) \hookrightarrow V_2^*$ gives the estimate for $p > 2$. □

The differentiability of the nonlinear convective term is obtained very easily.

Lemma 8.2 *If $p \in (\frac{3}{2}, \infty)$ and $z_t \rightharpoonup z$ in V_2, then*

$$\frac{1}{t}(u_t \cdot \nabla u_t - u \cdot \nabla u, v) \to (z \cdot \nabla u, v) + (u \cdot \nabla z, v), \quad t \to 0 \quad \textit{for all } v \in V_p.$$

Proof: By (8.4) we have

$$\frac{1}{t}(u_t \cdot \nabla u_t - u \cdot \nabla u, v) = (z_t \cdot \nabla u + u \cdot \nabla z_t - t z_t \cdot \nabla z_t, v) \quad \text{for all } v \in V_p.$$

Theorem 5.1 gives $u \in W^{1,\infty}(\Omega)$ and thus

$$z_t \cdot \nabla u \to z \cdot \nabla u \quad \text{and} \quad u \cdot \nabla z_t \to u \cdot \nabla z,$$

both weakly in V_2 and strongly in $L^q(\Omega)$ for all $q \in (1,\infty)$, due to the compact embedding, see Lemma 4.1.
With the same argument the boundedness of $\{z_t\}$ in V_2 implies boundedness of $\{\nabla z_t\}$ in $L^r(\Omega)$ for all $r \in (1,\infty)$. Thus the product $\{z_t \cdot \nabla z_t\}$ is bounded in $L^q(\Omega)$ for $\frac{1}{2}+\frac{1}{r}=\frac{1}{q}$. i.e for any $q>2$. Thus

$$tz_t \cdot \nabla z_t \quad \to \quad 0 \qquad \text{strongly in } L^q(\Omega), q>2.$$

Now $q>2$ implies $q'<2$ and Lemma 4.1 gives $V_p \hookrightarrow L^{q'}(\Omega)$. □

As next step we show the differentiability of the nonlinear term $T(Du)$, tested with smooth functions in $C^\infty_{0,\sigma} := \{\phi \in C_0^\infty(\Omega) : \operatorname{div}\phi = 0 \text{ in } \Omega\}$.

Lemma 8.3 *If $p \in (\frac{3}{2},\infty)$ and $z_{t_k} \rightharpoonup z$ in V_2 for any sequence $t_k \to 0$, then*

$$\frac{1}{t}(T(Du_{t_k}) - T(Du), D\phi) \quad \to \quad (T'(Du) : Dz, D\phi) \qquad \textit{for } t_k \to 0, \phi \in C^\infty_{0,\sigma}.$$

Proof: Using (8.2), (2.4), and the definition of z_t we have

$$\frac{1}{t_k}(T(Du_{t_k}) - T(Du)) : D\phi \quad = \quad \frac{1}{t_k}(M_{t_k} : D(u_t - u)) : D\phi \;=\; (D\phi : M_{t_k}) : Dz_t$$

in Ω. For $p \in (\frac{3}{2}, 2]$ the fact that $u_{t_k} \to u$ in V_2 implies

$$D\phi : \beta_{t_k}(\tau) \quad \to \quad D\phi : Du \qquad \text{in } L^2(\Omega) \text{ for all } \tau \in [0,1]$$

and thus for a subsequence

$$D\phi : \beta_{t_k}(\tau) \quad \to \quad D\phi : Du \qquad \text{a.e. in } \Omega \text{ for all } \tau \in [0,1].$$

By continuity of T' we have

$$D\phi : T'(\beta_{t_k}(\tau)) \quad \to \quad D\phi : T'(Du) \qquad \text{a.e. in } \Omega \text{ for all } \tau \in [0,1]$$

and with the definition of M_{t_k} in (8.3)

$$D\phi : M_{t_k} \quad \to \quad D\phi : T'(Du) \qquad \text{a.e. in } \Omega.$$

For all $x \in \Omega$ and all k every element of the tensor $M_{t_k}(x)$ is bounded by

$$(M_{t_k}(x))_{ijlm} \quad \le \quad \left(\int_0^1 \left(1 + |\beta_{t_k}(\tau)(x)|^2\right)^{\frac{p-2}{2}} d\tau \right) \;\le\; C_2, \quad i,j,l,m = 1,2,$$

due to (2.8). Thus there exists $c \in \mathbb{R}$ such that for all k and all $x \in \Omega$

$$(D\phi(x) : M_{t_k}(x))_{lm} \quad = \quad \sum_{ij} (D\phi(x))_{ij} (M_{t_k}(x))_{ijlm} \;\le\; c\,|D\phi(x)|, \quad l,m = 1,2.$$

Since $T'(Du) \in L^\infty(\Omega)$ by (4.5) the dominated convergence theorem implies

$$D\phi : M_{t_k} \quad \to \quad D\phi : T'(Du) \qquad \text{in } L^2(\Omega).$$

Since $Dz_{t_k} \rightharpoonup Dz$ in $L^2(\Omega)$ this completes the proof.
For $p>2$ the fact that $u_{t_k} \to u$ in $V_{p-\varepsilon}$ for $\varepsilon>0$ by Corollary 6.1 implies $\beta_{t_k}(\tau) \to Du$ in $L^{p-\varepsilon}(\Omega)$ and $(1+|\beta_{t_k}(\tau)|^2) \to (1+|Du|^2)$ in $L^{\frac{p-\varepsilon}{2}}(\Omega)$ for all $\tau \in [0,1]$. Thus

$$\left(1+|\beta_{t_k}(\tau)|^2\right)^{\frac{p-2}{2}} \quad \to \quad \left(1+|Du|^2\right)^{\frac{p-2}{2}} \qquad \text{in } L^{\frac{p-\varepsilon}{p-2}}(\Omega) \qquad \text{for all } \tau \in [0,1]$$

and therefore

$$\int_0^1 \left(1+|\beta_{t_k}(\tau)|^2\right)^{\frac{p-2}{2}} d\tau \quad \to \quad \left(1+|Du|^2\right)^{\frac{p-2}{2}} \quad \text{in } L^{\frac{p-\varepsilon}{p-2}}(\Omega)$$

and

$$\left(\int_0^1 \left(1+|\beta_{t_k}(\tau)|^2\right)^{\frac{p-2}{2}} d\tau\right)^{\frac{1}{2}} \quad \to \quad \left(1+|Du|^2\right)^{\frac{p-2}{4}} \quad \text{in } L^{\frac{2(p-\varepsilon)}{p-2}}(\Omega) \hookrightarrow L^2(\Omega)$$

since $\frac{2(p-\varepsilon)}{p-2} = 2\frac{p-\varepsilon}{p-2} \geq 2$ for ε sufficiently small. Thus

$$D\phi : \left(\int_0^1 \left(1+|\beta_{t_k}(\tau)|^2\right)^{\frac{p-2}{2}} d\tau\right)^{\frac{1}{2}} \quad \to \quad D\phi : \left(1+|Du|^2\right)^{\frac{p-2}{4}} \quad \text{in } L^2(\Omega).$$

Because of $Dz_{t_k} \rightharpoonup Dz$ in $L^2(\Omega)$ we obtain

$$\int_\Omega D\phi : \left(\int_0^1 \left(1+|\beta_{t_k}(\tau)|^2\right)^{\frac{p-2}{4}} d\tau\right)^{\frac{1}{2}} : Dz_{t_k} dx \rightharpoonup \int_\Omega D\phi : \left(1+|Du|^2\right)^{\frac{p-2}{4}} : Dz dx.$$

Together with the boundedness we have

$$\left(\int_0^1 \left(1+|\beta_{t_k}(\tau)|^2\right)^{\frac{p-2}{2}} d\tau\right)^{\frac{1}{2}} : Dz_{t_k} \quad \rightharpoonup \quad \left(1+|Du|^2\right)^{\frac{p-2}{4}} : Dz \quad \text{in } L^2(\Omega). \tag{8.6}$$

We define the superposition (or Nemytskij) operator

$$\begin{aligned} H \quad &: \quad L^1([0,1], L^p(\Omega)^{2\times 2}) \to L^2(\Omega)^{2\times 2}, \\ H(\eta)(x) \quad &:= \quad \left(\int_0^1 (1+|\eta(\tau)(x)|^2)^{\frac{p-2}{2}} d\tau\right)^{-\frac{1}{2}} D\phi : \left(\int_0^1 T'(\eta(\tau)(x))d\tau\right), x \in \Omega. \end{aligned}$$

Because of

$$|H(\eta)| \quad \leq \quad C_2|D\phi| \left(\int_0^1 \left(1+|\eta(\tau)|^2\right)^{\frac{p-2}{2}} d\tau\right)^{\frac{1}{2}}$$

H satisfies the Carathéodory condition and is thus continuous (see e.g. [8, Theorem 4]). Now $\beta_{t_k}(\tau) \to Du$ in $L^p(\Omega)$ for $t \to 0$ and all $\tau \in [0,1]$ implies

$$\begin{aligned} H(\beta_{t_k}) \quad &= \quad \left(\int_0^1 (1+|\beta_{t_k}(\tau)|^2)^{\frac{p-2}{2}} d\tau\right)^{-\frac{1}{2}} D\phi : \left(\int_0^1 T'(\beta_{t_k}(\tau))d\tau\right) \\ &\to \quad \left(\int_0^1 (1+|Du|^2)^{\frac{p-2}{2}} d\tau\right)^{-\frac{1}{2}} D\phi : \left(\int_0^1 T'(Du)d\tau\right) \\ &= \quad (1+|Du|^2)^{-\frac{p-2}{4}} D\phi : T'(Du) \; = \; H(Du) \qquad \text{in } L^2(\Omega)^{2\times 2}. \end{aligned}$$

Together with (8.6) and using (2.4) this gives

$$\int_\Omega \left(\int_0^1 T'(\beta_{t_k}(\tau))d\tau\right) : Dz_{t_k} : D\phi \, dx \quad \to \quad \int_\Omega T'(Du) : Dz : D\phi \, dx.$$

$\square$

Combining the last two lemmas and using the density of $C^\infty_{0,\sigma}(\Omega)$ in V_2 we obtain the following result.

Corollary 8.1 *If $p \in (\frac{3}{2}, \infty)$ and $z_{t_k} \rightharpoonup z$ in V_2 for any sequence $t_k \to 0$, then the limit point z is the unique solution to the linearized equation (7.1) with $g = h$.*

Finally we show strong convergence of $z_t \to z$.

Lemma 8.4 *Let $p \in (\frac{3}{2}, \infty)$ and $z_{t_k} \rightharpoonup z$ in V_2 for any sequence $t_k \to 0$, where z is the solution to (7.1) with $g = h$. Then $z_{t_k} \to z$ strongly in V_2.*

Proof: It remains to show that $Dz_{t_k} \to Dz$ in $L^2(\Omega)$. We note that $\mathbb{R}^{2\times 2}$ can be identified with $\mathbb{R}^4$ and using an index transformation $\{1,2\}^2 \to \{1,2,3,4\}$. Similarly $\mathbb{R}^{2\times 2\times 2\times 2}$ can be identified with $\mathbb{R}^{4\times 4}$ if the index transformation is applied to the first two indices and the last two indices separately. Thus we may interpret $M(x) := T'(Du(x))$ and $M_t(x)$ defined in (8.3) as matrices in $\mathbb{R}^{4\times 4}$. Moreover we may write the double scalar product as a quadratic form,

$$T'(Du(x)) : \xi : \xi \quad = \quad \xi^T M(x)\xi, \quad x \in \Omega, \xi \in \mathbb{R}^{2\times 2} \cong \mathbb{R}^4,$$

and similarly for M_{t_k}. Since

$$\xi^T M(x)\xi > 0 \quad \text{ for all } \xi \in \mathbb{R}^4 \setminus \{0\}$$

due to (2.7) and

$$\xi^T M(x)\xi = \xi^T M^s(x)\xi, \quad M^s(x) := \frac{1}{2}(M(x) + M(x)^T), \xi \in \mathbb{R}^4, x \in \Omega$$

there exists a Cholesky factor $L(x) \in \mathbb{R}^{4\times 4}$ of $M^s(x)$, i.e. a lower triangular matrix with positive diagonal elements such that

$$L(x)L^T(x) = M^s(x)$$

and thus

$$\xi^T M(x)\xi = \xi^T L(x)L^T(x)\xi = |L(x)^T\xi|^2 \quad \text{for all } x \in \Omega, \xi \in \mathbb{R}^4.$$

Here $|\cdot|$ denotes the euclidian vector norm. Similar arguments hold for $M_t(x)$, i.e. for all $t > 0$ there exists $L_t(x) \in \mathbb{R}^{4\times 4}$ satisfying

$$L_t(x)L_t^T(x) = M^s(x), \qquad \xi^T M_t(x)\xi = \xi^T L_t(x)L_t^T(x)\xi = |L_t(x)^T\xi|^2.$$

By (8.3) and (8.1) we have

$$\begin{aligned} \|L_{t_k}^T Dz_{t_k}\|_2^2 \ = \ (M_{t_k} : Dz_{t_k}, Dz_{t_k}) \quad &= \quad \frac{1}{t}(T(Du_{t_k}) - T(Du), Dz_{t_k}) \\ &= \quad (h, z_{t_k}) - \frac{1}{t}(u_{t_k} \cdot \nabla u_{t_k} - u \cdot \nabla u, z_{t_k}) \end{aligned} \tag{8.7}$$

and thus by (8.4) and Lemma 8.1

$$\|L_{t_k}^T Dz_{t_k}\|_2^2 \quad = \quad (h, z_{t_k}) - (z_{t_k} \cdot \nabla u + u \cdot \nabla z_{t_k} - t z_{t_k} \cdot \nabla z_{t_k}, z_{t_k}) \ \leq \ c_1 = c_1(f).$$

Thus $\{L_{t_k}^T Dz_{t_k}\}_{t>0}$ is bounded in $L^2(\Omega)$. By (2.8) any matrix norm of $M_t(x)$, denoted by $|\cdot|$, can be estimated by

$$|M_t(x)| \quad \leq \quad c_2 \int_0^1 \left(1 + |\beta_t(\tau)(x)|^2\right)^{\frac{p-2}{2}} d\tau \quad x \in \Omega, t > 0.$$

This gives

$$|L_{t_k}(x)| = |M_{t_k}(x)|^{\frac{1}{2}} \leq \begin{cases} c_3, & p \leq 2, \\ c_3 \left(1 + (|Du(x)| + Du_{t_k}(x)|)^2\right)^{\frac{p-2}{4}} =: H(x), & p > 2, \end{cases}$$

for all $x \in \Omega$. Now $Du, Du_{t_k} \in L^p(\Omega)$ and $\frac{p}{2}\frac{4}{p-2} = \frac{2p}{p-2} > 2$ gives $H \in L^{\frac{2p}{p-2}}(\Omega) \hookrightarrow L^2(\Omega)$. Theorem 6.2 implies that $\{Du_{t_k}\}_{k>k^*}$ can be bounded in $L^2(\Omega)$ uniformly in t. Thus for all p there exists $H \in L^2(\Omega)$ such that

$$|L_{t_k}(x)| \quad \leq \quad H(x) \qquad \text{for all } x \in \Omega, k > k^*.$$

Moreover $u_{t_k} \to u$ in V_2 for all $p > \frac{3}{2}$ implies $\beta_{t_k}(\tau)(x) \to Du(x)$ for a subsequence, a.e. $x \in \Omega$, and all $\tau \in [0,1]$. The continuity of T' then leads to $M_{t_k}(x) \to M(x)$ and thus

$$L_{t_k}(x) \quad \to \quad L(x) \quad \text{for a.e. } x \in \Omega.$$

The dominated convergence theorem then implies

$$L_{t_k} \quad \to L \quad \text{in } L^2(\Omega), \tag{8.8}$$

and the weak convergence of $z_{t_k} \rightharpoonup z$ in V_2 gives

$$L_{t_k}^T Dz_{t_k} \quad \rightharpoonup L^T Dz \quad \text{in } L^2(\Omega).$$

Now (8.7), the weak convergence of z_{t_k} to the solution z of the linearized equation, and the convergence of the convective term (see Lemma 8.2) give

$$\begin{aligned}
\|L^T Dz\|_2^2 \quad &\leq \quad \liminf_{t_k \to 0} \|L_{t_k}^T Dz_{t_k}\|_2^2 \;\leq\; \limsup_{t_k \to 0} \|L_{t_k}^T Dz_{t_k}\|_2^2 \\
&= \quad \limsup_{t_k \to 0} \int_\Omega Dz_{t_k} M_{t_k} Dz_{t_k} dx \;=\; \limsup_{t_k \to 0} (M_{t_k} : Dz_{t_k}, Dz_{t_k}) \\
&= \quad \limsup_{t_k \to 0} \left[(h, z_{t_k}) - \frac{1}{t}(u_{t_k} \cdot \nabla u_{t_k} - u \cdot \nabla u, z_{t_k})\right] \\
&= \quad (h, z) - (u \cdot \nabla z - z \cdot \nabla u, z) \;=\; (T'(Du) : Dz, Dz) \\
&= \quad \int_\Omega DzMDzdx \;=\; \|L^T Dz\|_2^2.
\end{aligned}$$

Weak convergence together with norm convergence implies strong convergence

$$L_{t_k}^T Dz_{t_k} \quad \to L^T Dz \quad \text{in } L^2(\Omega).$$

Thus there exists a new subsequence (also denoted by t_k) satisfying

$$L_{t_k}(x) Dz_{t_k}(x) \quad \to \quad L(x) Dz(x) \quad \text{for a.e. } x \in \Omega, \tag{8.9}$$

and there exists $G \in L^2(\Omega)$ with

$$|L_{t_k}^T(x) Dz_{t_k}(x)| \leq G(x) \quad \text{for a.e. } x \in \Omega \text{ and all } k > k^*.$$

For all $x \in \Omega, t > 0, \tau \in [0,1]$ we have

$$\left(1 + |\beta_t(\tau)(x)|^2\right)^{\frac{p-2}{2}} \quad \geq \quad \begin{cases} \left(1 + (|Du(x)| + |Du_t(x)|)^2\right)^{\frac{p-2}{2}}, & p \in (\frac{3}{2}, 2) \\ 1, & p \in [2, \infty). \end{cases}$$

For $p \in (\frac{3}{2}, 2)$ Theorem 5.3 implies the estimate

$$|Du(x)| + |Du_t(x)| \quad \leq \quad \|u\|_{1,\infty} + \|u_t\|_{1,\infty} \;\leq\; C_0(\|f\|_{p'}) + C_0(\|f + th\|_{p'}).$$

for all $x \in \Omega, t > 0$. Since the function C_0 is continuous we may estimate

$$C_0(\|f + th\|_{p'}) \quad \leq \quad c_4(f, h)$$

uniformly for $t \le t^*$ and some $t^* > 0$. Therefore we obtain

$$\int_0^1 \left(1+|\beta_t(\tau)(x)|^2\right)^{\frac{p-2}{2}} d\tau \quad \ge \quad c_5 \;=\; \left\{ \begin{array}{ll} c_5(f,h), & p \in (\frac{3}{2},2) \\ 1, & p \in [2,\infty). \end{array} \right\}, x \in \Omega, t \le t^*.$$

Now (2.7) gives

$$M_t(x) = \int_0^1 T'(\beta_t(\tau)(x))d\tau \quad \ge \quad C_1 \int_0^1 \left(1+|\beta_t(\tau)(x)|^2\right)^{\frac{p-2}{2}} d\tau \ge C_1 c_5 =: c_6$$

for all $x \in \Omega, t \le t^*$. Thus we may estimate

$$\begin{array}{rcl} |Dz_{t_k}(x)|^2 \;=\; Dz_{t_k}(x) : Dz_{t_k}(x) & \le & c_6^{-1} M_{t_k}(x) : Dz_{t_k}(x) : Dz_{t_k}(x) \\ & = & c_6^{-1} Dz_{t_k}^T(x) M_{t_k}(x) Dz_{t_k}(x) \\ & = & c_6^{-1} |L_{t_k}^T(x) Dz_{t_k}(x)|^2 \\ & \le & c_6^{-1} G(x)^2 \qquad\qquad \text{for a.e. } x \in \Omega, k > k^*. \end{array}$$

Since (8.8) implies $L_{t_k}^{-T} \to L^{-T}$ a.e. in Ω now (8.9) implies

$$Dz_{t_k}(x) = L_{t_k}^{-T}(x) L_{t_k}^T(x) Dz_{t_k}(x) \to L^{-T}(x) L^T(x) Dz(x) = Dz(x) \quad \text{a.e. in } \Omega.$$

The dominated convergence theorem now completes the proof. □

Gâteaux differentiability is now a direct consequence.

Theorem 8.1 *For $p \in (\frac{3}{2},\infty)$ and $\|f\|_{V_p^*}$ sufficiently small the operator G is Gâteaux differentiable from V_p to V_2. The derivative $z = DG(f)h$ at f in direction h is obtained as the unique solution of (7.1) with $g = h$.*

Proof: The boundedness of $\{z_t\}_{t>0}$ in V_2 showed in Lemma 8.1 implies the existence of a weak convergent subsequence. Due to Corollary 8.1 its limit point z is the unique solution to the linearized equation (7.1) with $g = h$. Moreover $z_t \to z$ strongly in V_2 by Lemma 8.4. Lemma 8.1 moreover gives the estimate

$$\|z\|_2 = \lim_{t\to 0} \|z_t\|_2 \le L(f)\|h\|_{\bar{p}}$$

with $\bar{p} = p'$ for $p \in (\frac{3}{2},2)$ and $\bar{p} = s$ for $p \in [2,\infty)$. This implies the continuity of $DG(f)$. □

9 Existence of an optimal solution

In this section we present an existence result for a solution to the optimal control problem

$$\min_{f \in \mathcal{F}_{ad}} J(u,f) \quad \text{s.t.} \quad (5.1) \tag{9.1}$$

where the set of admissible controls $\mathcal{F}_{ad} \subset L^{p'}(\Omega)$ has to be chosen appropriately. We assume that

- J is continuous with respect to the state u in the V_p norm,
- for $p \in (\frac{3}{2},2)$ the functional J is continuous with respect to the control f in the $L^{p'}(\Omega)$ norm,
- for $p \in [2,\infty)$ the functional J is weakly lower semi-continuous with respect to the control f in $L^2(\Omega)$,
- J is bounded from below.

A typical example for the cost J is a tracking type functional

$$J(u,f) \quad := \quad \frac{1}{2}\int_\Omega |u(x)-u_d(x)|^2\,dx + \frac{\alpha}{2}\int_\Omega |f(x)|^2\,dx.$$

Here $\alpha > 0$ is a regularization parameter and $|\cdot|$ denotes the euclidian vector norm.
To show existence of a solution to (9.1) we distinguish between the two cases $p \in (\frac{3}{2}, 2)$ and $p \in [2,\infty)$.

- For $p \in (\frac{3}{2}, 2)$ we choose $\mathcal{F}_{ad}$ as a bounded subset of a space that is compactly embedded in $L^{p'}(\Omega)$. By Theorem 4.1 the embedding $W^{1,q}(\Omega) \hookrightarrow L^{p'}(\Omega)$ is compact for

  $$p' = \frac{p}{p-1} < \frac{2q}{2-q}, \qquad \text{i.e. } q > \frac{2p'}{2+p'} = \frac{p}{3p-2}.$$

  Thus $q = 1$ is sufficient.
- For $p \in [2,\infty)$ Theorems 5.2 and 6.2 imply that if $f_k \rightharpoonup f$ in $L^2(\Omega)$, then the corresponding solutions satisfy $u_k \to u$ in V_p. Thus we may here choose a bounded subspace of $L^s(\Omega), s > 2$, as the set $\mathcal{F}_{ad}$.

We now prove the following existence result for a solution to (9.1).

Theorem 9.1 *Let either*

$$\begin{array}{lllll} & p \in (\frac{3}{2},2) & \textit{and} & \mathcal{F}_{ad} & := & \{f \in W^{1,1}(\Omega) : \|f\|_{1,1} \le M\} \\ \textit{or} & p \in [2,\infty) & \textit{and} & \mathcal{F}_{ad} & := & \{f \in L^s(\Omega) : \|f\|_s \le M\}, \quad s > 2, \end{array}$$

for some $M > 0$ sufficiently small. Then problem (9.1) has a solution in $\mathcal{F}_{ad}$.

Proof: The proof follows the standard way. We use the notation

$$\hat{J}(f) := J(G(f), f), \quad f \in L^{p'}(\Omega), \tag{9.2}$$

where G is the solution operator defined in (5.2). Since we assumed that J is bounded from below we may choose a minimizing sequence $(f_k)_k$ in $\mathcal{F}_{ad}$, i.e.

$$\lim_{k\to\infty} \hat{J}(f_k) = \inf_{f\in\mathcal{F}_{ad}} \hat{J}(f).$$

For $p < 2$ the boundedness of $\mathcal{F}_{ad}$ and the compact embedding $W^{1,1}(\Omega) \hookrightarrow L^{p'}(\Omega)$ a subsequence, denoted again by $(f_k)_k$, converges strongly in $L^{p'}(\Omega)$ to some $\bar{f} \in L^{p'}(\Omega)$, i.e.

$$\lim_{i\to\infty} f_k = \bar{f} \in L^{p'}(\Omega).$$

The continuity of G and J implies continuity of $\hat{J}$ and thus

$$\min_{f\in\mathcal{F}_{ad}} \hat{J}(f) = \inf_{f\in\mathcal{F}_{ad}} \hat{J}(f) = \lim_{i\to\infty} \hat{J}(f_k) = \hat{J}(\bar{f}). \tag{9.3}$$

For $p \ge 2$ the boundedness of $\mathcal{F}_{ad}$ in $L^s(\Omega), s > 2$ implies $f_k \rightharpoonup f$ in V_2 and (9.3) follows using the weakly lower semi-continuity of J with respect to f. $\square$

10 First order optimality conditions

Based on the differentiability of the solution operator proved above we now present the first order optimality conditions for problem (9.1). We introduce the Lagrangian and present the optimality system including the adjoint equation. Let $p \in (\frac{3}{2}, \infty)$ throughout the section.
We now assume that the cost functional J is differentiable with respect to u and f. Moreover J shall satisfy

$$\begin{aligned} D_u J(\bar{u}, \bar{f}) &\in V_p^* \\ D_f J(\bar{u}, \bar{f}) &\in L^p(\Omega) \end{aligned}$$

for a solution $\bar{f} \in \mathcal{F}_{ad}$ of (9.1) and $\bar{u} = G(\bar{f}) \in V_p$.
The Lagrangian associated with (9.1) is given as:

$$\begin{aligned} L \quad &: \quad V_p \times L^r(\Omega) \times V_p \to \mathbb{R} \\ L(u, f, \lambda) \quad &= \quad J(u, f) + (u \cdot \nabla u, v) + (T(Du), D\lambda) - (f, \lambda) \end{aligned}$$

where $\bar{p} := p'$ if $p < 2$ and $\bar{p} := s$ if $p \geq 2$. We compute the derivatives with respect to $u, \lambda \in V_p, f \in L^{\bar{p}}(\Omega)$ in the directions $v \in V_p, g \in L^{\bar{p}}(\Omega)$ and obtain

$$\begin{aligned} \langle D_u L(u, f, \lambda), v \rangle_{V_p^*, V_p} &= \langle D_u J(u, f), v \rangle_{V_p^*, V_p} + (u \cdot \nabla v + v \cdot \nabla u, \lambda) \\ &\quad + (T'(Du) : Dv), D\lambda) \\ (D_f L(u, f, \lambda), g) &= (D_f J(u, f), g) + (g, \lambda) \\ \langle D_\lambda L(u, f, \lambda), v \rangle_{V_p^*, V_p} &= (u \cdot \nabla u, v) + (T(Du), Dv) - (f, v). \end{aligned} \tag{10.1}$$

For a saddle-point $(\bar{u}, \bar{f}, \lambda)$ of L these derivatives have to vanish in all directions. The third equation gives the state equation (5.1), the second one the relation between the Lagrange multiplier λ and the optimal control $\bar{f}$,

$$(\lambda, g) \quad = \quad -(D_f J(\bar{u}, \bar{f}), g) \qquad \text{for all } g \in L^{\bar{p}}(\Omega).$$

Equation (10.1) can be re-written as follows. Lemma 5.2 implies

$$(\bar{u} \cdot \nabla v, \lambda) \quad = \quad -(\bar{u} \cdot \nabla \lambda, v).$$

Moreover using the definitions of the scalar products we get

$$\begin{aligned} (v \cdot \nabla \bar{u}) \cdot \lambda &= \big((\nabla \bar{u})^T \cdot \lambda\big) \cdot v \\ (T'(D\bar{u}) : Dv) : D\lambda &= (T'(D\bar{u}) : D\lambda) : Dv, \end{aligned}$$

i.e. $T'(Du)$ is self-adjoint. Thus we obtain the adjoint equation

$$\begin{aligned} \big((\nabla \bar{u})^T \cdot \lambda - \bar{u} \cdot \nabla \lambda, v\big) + (T'(D\bar{u}) : D\lambda, Dv) \quad &= \quad -\langle D_u J(\bar{u}, \bar{f}), v \rangle_{V_p^*, V_p} \\ &\qquad \text{for all } v \in V_p. \end{aligned}$$

Since by Theorem 5.1 the linearized equation is uniquely solvable, the same is true for the adjoint equation.

Corollary 10.1 *The adjoint equation has a unique solution $\lambda \in V_2$.*

We thus obtain the following optimality system:

Theorem 10.1 *Let $\bar{f} \in \mathcal{F}_{ad}$ be a solution to (9.1). Then there exists a unique pair $(\bar{u}, \lambda) \in V_p \times V_2$ such that*

$$\begin{aligned} (\bar{u} \cdot \nabla u, v) + (T(D\bar{u}), Dv) \quad &= \quad \langle \bar{f}, v \rangle_{V_p^*, V_p} \quad \textit{for all } v \in V_p \\ \big((\nabla \bar{u})^T \cdot \lambda - \bar{u} \cdot \nabla \lambda, v\big) + (T'(D\bar{u}) : D\lambda, Dv) \quad &= \quad -\langle D_u J(\bar{u}, \bar{f}), v \rangle_{V_p^*, V_p} \\ &\qquad \textit{for all } v \in V_p \\ (\lambda, g) \quad &= \quad -(D_f J(\bar{u}, \bar{f}), g) \\ &\qquad \textit{for all } g \in L^{\bar{p}}(\Omega). \end{aligned}$$

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