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Abstract. We investigate optimal control problems with vector-valued controls. As model problem serve the optimal distributed control of the instationary Navier-Stokes equations. We study pointwise convex control constraints, which is a constraint of the form $u(x,t) \in U(x,t)$ that has to hold on the domain Q. Here, U is an set-valued mapping that is assumed to be measurable with convex and closed images. We establish first-order necessary as well as second-order sufficient optimality conditions. And we prove regularity results for locally optimal controls.

1. Introduction

In fluid dynamics the control can be brought into the system by blowing or suction on the boundary. Then the control is a velocity, which is a directed quantity, hence it is a vector in $\mathbb{R}^2$ respectively $\mathbb{R}^3$. That is, the optimal control problem is to find a vector-valued function $u \in L^p((0,T) \times \Omega)^n$. Distributed control can be realized for instance as a force induced by an outer magnet field in a conducting fluid, see e.g. Kunisch and Griesse [14]. There, the control u is a function of class $L^2(Q)^2 = L^2(Q; \mathbb{R}^2)$. This illustrates that the control is a directed quantity: it consists of a direction and an absolute value. Or in other words, the control u at a point (x,t) is a vector in $\mathbb{R}^2$.

The optimization has to take into account that one is not able to realize arbitrarily large controls. To this end, control constraints are introduced. If the control $u(x,t)$ is only a scalar variable such as heating or cooling then there is only one choice of a convex pointwise control constraint: the so-called box constraints

$$u_a(x,t) \le u(x,t) \le u_b(x,t). \tag{1a}$$

For the analysis of optimal control of non-stationary Navier-Stokes equations using this particular type of control constraints, we refer to Hinze and Hintermüller [16], Roubíčcek and Tröltzsch [23], Tröltzsch and Wachsmuth [25], and Wachsmuth [28]. But these box constraints are not the only choice for vector-valued controls. For instance, if one wants to bound the $\mathbb{R}^2$-norm of the control, one gets a nonlinear constraint

$$|u(x,t)| = \sqrt{u_1(x,t)^2 + u_2(x,t)^2} \le \rho(x,t). \tag{1b}$$

What happens if the control is not allowed to act in all possible directions but only in directions of a segment with an angle less than π? Using polar coordinates $u_r(x,t)$ and $u_\phi(x,t)$ for the control vector $u(x,t)$, this can be formulated as

$$0 \le u_r(x,t) \le \psi(u_\phi(x,t), x, t), \tag{1c}$$

where the function ψ models the shape of the set of allowed control actions.

Here, we will use another — and more natural — representation of the constraints. Let us denote by U the set of admissible control vectors. Then we can write the control constraints (1a)–(1c) as an inclusion

$$u(x,t) \in U.$$

The advantage of this approach is that the analysis is based on rather elementary say geometrical arguments, hence there is no need of any constraint qualification. We will impose assumptions on U that allow to apply the common theory of existence and optimality condition: non-emptyness, convexity, and closedness, but no boundedness or further regularity of the boundary. We have to admit that the assumption of convexity gives some inherent regularity, the boundary of convex sets is locally Lipschitz. However, even in the convex case, there can be very irregular situations: one can construct convex sets in $\mathbb{R}^2$ with countably many corners, which lie dense on the boundary, see [10].

The formulation of the control constraint as an inclusion has a further benefit: the set of admissible control vectors can vary over time and space by simply writing

$$u(x,t) \in U(x,t),$$

without causing any additional problems. The main difficulty appears already in the non-varying case, see the discussion in Section 7.1 below.

Optimal control problems with such control constraints are rarely investigated in literature. Second-order necessary conditions for problems with the control constraint $u(\xi) \in U(\xi)$ were proven by Páles and Zeidan [20] involving second-order admissible variations. Second-order necessary as well as sufficient conditions were established in Bonnans [5], Bonnans and Shapiro [8], and Dunn [12]. However, the set of admissible controls has to be polygonal and independent of ξ, i.e. $U(\xi) \equiv U$. This results were extended by Bonnans and Zidani [9] to the case of finitely many convex contraints $g_i(u(\xi)) = 0$, $i = 1, \dots, l$. As already mentioned, we will follow another approach and treat the control constraint as an inclusion $u(x,t) \in U(x,t)$. State constraints of the form $y(x,t) \in C$ are considered in the recent research paper by Griesse and de los Reyes [13].

As a model problem serves the optimal distributed control of the instationary Navier-Stokes equations in two dimensions. We emphasize that the restriction to two dimensions, i.e. $u \in L^2(Q)^2$, is only due to the limitation of the analysis of instationary Navier-Stokes equations. As long as there exists an applicable theory of a state equation in $\mathbb{R}^n$, all results regarding convex control constraints are ready for an extension to the n-dimensional case.

To be more specific, we want to minimize the following quadratic objective functional:

$$J(y,u) = \frac{\alpha_T}{2}\int_\Omega |y(x,T) - y_T(x)|^2\,\mathrm{d}x + \frac{\alpha_Q}{2}\int_Q |y(x,t) - y_Q(x,t)|^2\,\mathrm{d}x\,\mathrm{d}t + \frac{\alpha_R}{2}\int_Q |\operatorname{curl} y(x,t)|^2\,\mathrm{d}x\,\mathrm{d}t + \frac{\gamma}{2}\int_Q |u(x,t)|^2\,\mathrm{d}x\,\mathrm{d}t \tag{2}$$

subject to the instationary Navier-Stokes equations

$$\begin{aligned} y_t - \nu\Delta y + (y\cdot\nabla)y + \nabla p &= u && \text{in } Q,\\ \operatorname{div} y &= 0 && \text{in } Q,\\ y(0) &= y_0 && \text{in } \Omega, \end{aligned} \tag{3}$$

and to the control constraints $u \in U_{ad}$ with set of admissible controls defined by

$$U_{ad} = \{u \in L^2(Q)^2 :\ u(x,t) \in U(x,t) \text{ a.e. on } Q\}. \tag{4}$$

Here, Ω is a bounded domain in $\mathbb{R}^2$, Q denotes the time-space cylinder $Q := \Omega \times (0,T)$. Let us underline the fact that for $(x,t) \in Q$ the control $u(x,t)$ is a vector in $\mathbb{R}^2$.

The conditions imposed on the various ingredients of the optimal control problem are specified in Sections 2.1 and 4.1, see assumptions (A) and (AU).

For the optimal control of the non-stationary Navier-Stokes equations there are several articles about existence of solution and necessary optimality conditions, for instance Abergel and Temam [1], Gunzburger and Manservisi [15]. Sufficient optimality conditions and second-order optimization methods were investigated by Hinze [17], Hinze and Kunisch [18], Ulbrich [26], and Tröltzsch and Wachsmuth [25]. However, in these articles only the box constraints (1a) or even no control constraints are considered.

The plan of the article is as follows. At first we introduce some notation and results concerning the state equation in Section 2. Set-valued mappings are the subject of Section 3. The exact statement of our model problem can be found in Section 4 together with first-order necessary optimality conditions in Section 5. We prove regularity results for locally optimal controls in Section 6. Finally, we discuss sufficient optimality conditions and stability of optimal controls in Sections 7 and 8 respectively.

2. Notations and preliminary results

At first, we introduce some notations and results that we will need later on. To begin with, we define the spaces of solenoidal or divergence-free functions

$$H := \{v \in L^2(\Omega)^2 : \operatorname{div} v = 0\}, \qquad V := \{v \in H_0^1(\Omega)^2 : \operatorname{div} v = 0\}.$$

These spaces are Hilbert spaces with scalar products $(\cdot,\cdot)_H$ and $(\cdot,\cdot)_V$ respectively. The dual of V with respect to the scalar product of H we denote by V' with the duality pairing $\langle \cdot, \cdot\rangle_{V',V}$.

We will work with the standard spaces of abstract functions from $[0,T]$ to a real Banach space X, $L^p(0,T;X)$, endowed with its natural norm,

$$\|y\|_{L^p(X)} := \|y\|_{L^p(0,T;X)} = \left(\int_0^T |y(t)|_X^p \,\mathrm{d}t \right)^{1/p} \quad 1 \le p < \infty,$$

$$\|y\|_{L^\infty(X)} := \operatorname*{ess\,sup}_{t\in(0,T)} |y(t)|_X .$$

In the sequel, we will identify the spaces $L^p(0,T;L^p(\Omega)^2)$ and $L^p(Q)^2$ for $1 < p < \infty$, and denote their norm by $\|u\|_p := |u|_{L^p(Q)^2}$. The usual $L^2(Q)^2$-scalar product we denote by $(\cdot,\cdot)_Q$ to avoid ambiguity.

In all what follows, $\|\cdot\|$ stands for norms of abstract functions, while $|\cdot|$ denotes norms of "stationary" spaces like H and V.

To deal with the time derivative in (3), we introduce the common spaces of functions y whose time derivatives y_t exist as abstract functions,

$$W^\alpha(0,T;V) := \{y \in L^2(0,T;V) : \ y_t \in L^\alpha(0,T;V')\}, \quad W(0,T) := W^2(0,T;V),$$

where $1 \le \alpha \le 2$. Endowed with the norm

$$\|y\|_{W^\alpha(0,T;V)} := \|y\|_{L^2(V)} + \|y_t\|_{L^\alpha(V')},$$

these spaces are Banach spaces. Every function of $W(0,T)$ is, up to changes on sets of zero measure, equivalent to a function of $C([0,T],H)$, and the imbedding $W(0,T) \hookrightarrow C([0,T],H)$ is continuous, cf. [2, 19].

2.1. The state equation

Before we start with the discussion of the state equation, we specify the requirements for the various ingredients describing the optimal control problem. In the sequel, we assume that the following conditions are satisfied:

$$(\mathbf{A}) \left\{ \begin{array}{l} 1.\ \Omega \textit{ has Lipschitz boundary } \Gamma := \partial\Omega, \\ 2.\ y_0,\, y_T \in H,\ y_Q \in L^2(Q)^2, \\ 3.\ \alpha_T,\, \alpha_Q,\, \alpha_R \ge 0, \\ 4.\ \gamma,\, \nu > 0. \end{array} \right.$$

The assumptions on the set-valued mapping U are given in the next section. Now, we will briefly summarize known facts about the solvability of the instationary Navier-Stokes equations (3). First, we define the trilinear form $b : V \times V \times V \mapsto \mathbb{R}$ by

$$b(u,v,w) = ((u\cdot\nabla)v, w)_2 = \int_\Omega \sum_{i,j=1}^2 u_i \frac{\partial v_j}{\partial x_i}\, w_j \ \mathrm{d}x.$$

Its time integral is denoted by b_Q,

$$b_Q(y,v,w) = \int_0^T b(y(t), v(t), w(t))\,\mathrm{d}t.$$

To specify the problem setting, we introduce a linear operator $A : L^2(0,T;V) \mapsto L^2(0,T;V')$ by

$$\int_0^T \langle (Ay)(t),\, v(t)\rangle_{V',V}\,\mathrm{d}t := \int_0^T (y(t),v(t))_V\,\mathrm{d}t,$$

and a nonlinear operator B by

$$\int_0^T \big\langle \big(B(y)\big)(t),\, v(t)\big\rangle_{V',V}\,\mathrm{d}t := \int_0^T b(y(t),y(t),v(t))\,\mathrm{d}t.$$

For instance, the operator B is continuous and twice Frechét-differentiable as operator from $W(0,T)$ to $L^2(0,T;V')$.

Now, we concretize the notation of weak solutions for the instationary Navier-Stokes equations (3) in the Hilbert space setting.

Definition 2.1 (Weak solution)**.** *Let $f \in L^2(0,T;V')$ and $y_0 \in H$ be given. A function $y \in L^2(0,T;V)$ with $y_t \in L^2(0,T;V')$ is called weak solution of* (3) *if*

$$\begin{aligned} y_t + \nu Ay + B(y) &= f, \\ y(0) &= y_0. \end{aligned} \tag{5}$$

Results concerning the solvability of (5) are standard, cf. [24] for proofs and further details.

Theorem 2.2 (Existence and uniqueness of solutions)**.** *For every source term $f \in L^2(0,T;V')$ and initial value $y_0 \in H$, the equation* (5) *has a unique solution $y \in W(0,T)$. Moreover, the mapping $(y_0,f) \mapsto y$ is locally Lipschitz continuous from $H \times L^2(0,T;V')$ to $W(0,T)$.*

It is well-known that the control-to-state mapping is Fréchet-differentiable. The first derivative can be computed as the solution of a linearized equation, cf. [15, 17, 18].

Remark 2.3 (Linearized state equation)**.** *We consider the linearized equation*

$$\begin{aligned} y_t + \nu Ay + B'(\bar{y})y &= f, \\ y(0) &= y_0, \end{aligned} \tag{6}$$

for a given state $\bar{y}$, which is usually the solution of the nonlinear system (5)*. Following the lines of Temam, existence and uniqueness of a weak solution y in the space $W(0,T)$ was proven for instance in* [18, Prop. 2.4]*. See also the discussion in* [15]*.*

3. Set-valued functions

Before we begin with the formulation of the optimal control problem with inclusion constraints, we will provide some background material. Here, we will specify the notation and assumptions for the admissible set $U(\cdot)$. It is itself a mapping from the control domain Q to the set of subsets of $\mathbb{R}^2$, it is a so-called *set-valued mapping.* We will use the notation $U : Q \rightsquigarrow \mathbb{R}^2$.

The optimal control problem is the minimization of the objective functional subject to the state equations and to the control constraint

$$u(x,t) \in U(x,t). \tag{7}$$

The controls are taken from the space $L^2(Q)^2$, so it is natural to require the fulfillment of (7) for (only) almost all $(x,t) \in Q$. And we have to impose at least some measurability conditions on the mapping U. In the sequel, we will work with measurable set-valued mappings. For an excellent — and for our purposes complete — introduction we refer to the textbook by Aubin and Frankowska [4].

Definition 3.1. *A set-valued mapping $F : Q \rightsquigarrow X$ with closed images is called* measurable*, if the inverse of each open set is measurable. In other words, for every open subset $\mathcal{O} \subset X$ the inverse image*

$$F^{-1}(\mathcal{O}) = \{\omega \in Q : F(\omega) \cap \mathcal{O} \neq \emptyset\}$$

has to be measurable.

Observe, that for a single-valued function f the definition of measurability coincides with the definition of measurability for the set-valued function $\tilde{f}$ given by

$$\tilde{f}(\omega) = \{f(\omega)\}.$$

However, this definition does not imply the existence of a measurable selection, which is a single-valued function f satisfying $f(x,t) \in U(x,t)$ almost everywhere on Q. The existence is guaranteed under additional assumptions on U.

Theorem 3.2. [4, Th. 8.1.4] *Let $F : Q \rightsquigarrow \mathbb{R}^2$ be a set-valued mapping with non-empty closed images. Then the following two statements are equivalent:*

1. *F is measurable*
2. *There exists a sequence of measurable selections $\{f_n\}_{n=1}^\infty$ of F such that for all $(x,t) \in Q$ it holds*

$$F(x,t) = \overline{\bigcup_{n \geq 1} \{f_n(x,t)\}}.$$

The theorem gives not only the existence of a measurable selection but also a tool to prove measurability of set-valued mappings based on countable approximations.

It is well-known that every optimal control is the projection of its associated state on the admissible set. Such a characterization is also valid in the set-valued constraint case. But as a first step, we have to make sure that the pointwise projection on the set-valued mapping U preserves measurability.

Theorem 3.3. [4, Cor. 8.2.13] *Let $F : Q \rightsquigarrow \mathbb{R}^2$ be a set-valued measurable mapping with closed, non-empty, and convex images, and $f : Q \mapsto \mathbb{R}^2$ a measurable (single-valued) mapping. Then the projection*

$$g(x,t) = \mathrm{Proj}_{F(x,t)}(f(x,t))$$

is a single-valued measurable function too.

4. The optimal control problem

Here, we will investigate the optimal control problem with the control constraint (7). At first, we have to specify the assumptions to ensure existence of solutions.

4.1. Set of admissible controls

In this section, we want to investigate the convex control constraint, which has to hold pointwise

$$u(x,t) \in U(x,t) \text{ a.e. on } Q.$$

We recall the definition of the set of admissible controls U_{ad},

$$U_{ad} = \{u \in L^2(Q)^2 : \ u(x,t) \in U(x,t) \text{ a.e. on } Q\}.$$

Once and for all, we specify the requirements for the function U, which defines the control constraints.

(AU) *The set-valued function $U : Q \rightsquigarrow \mathbb{R}^2$ satisfies:*

1. *U is a measurable set-valued function.*
2. *The images of U are non-empty, closed, and convex a.e. on Q. That is, the sets $U(x,t)$ are non-empty, closed and convex for almost all $(x,t) \in Q$.*
3. *There exists a function $f_U \in L^2(Q)^2$ with $f_U(x,t) \in U(x,t)$ a.e. on Q.*

Please note, we did not impose any conditions on the sets $U(x,t)$ that are beyond convexity such as boundedness or regularity of the boundaries $\partial U(x,t)$. Assumptions *(i)* and *(ii)* guarantee that there exists a measurable selection of U, i.e. a measurable single-valued function f_M with $f_M(x,t) \in U(x,t)$ a.e. on Q. However, no measurable selection needs to be square-integrable as the following example shows.

Example 4.1. *Set $U(t) = [t^{-1/2}, 1 + t^{-1/2}]$, $0 < t \leq 1$. Assumptions (i) and (ii) are fulfilled. But every function f with $f(t) \in U(t)$ for almost all $0 < t \leq 1$ cannot be in $L^2(0,1)$, since the function $g(t) = t^{-1/2}$ is not square integrable on $[0,1]$.*

The existence of a square integrable, admissible function is then ensured by the third assumption. This implies that the set of admissible control is non-empty.

Corollary 4.2. *The set of admissible controls U_{ad} defined by*

$$U_{ad} = \{u \in L^2(Q)^2 : \ u(x,t) \in U(x,t) \text{ a.e. on } Q\}$$

is non-empty, convex and closed in $L^2(Q)^2$.

The assumption (AU) is as general as the analysis of the second-order condition allows it. In the case that the set-valued function U is a constant function, i.e. $U(x,t) \equiv U_0$, we can give a simpler characterization.

Corollary 4.3. *Let the set-valued function U be a constant function, i.e. $U(x,t) = U_0$ a.e. on Q for some $U_0 \subset \mathbb{R}^2$. Then the assumption (AU) is fulfilled if the set U_0 is non-empty, closed, and convex.*

Assuming (AU) we can derive another interesting result. Conditon *(iii)* allows us to prove that the pointwise projection on U_{ad} of a L^2-function is itself a L^2-function.

Corollary 4.4. *Let be given a function $u \in L^2(Q)^2$. Then the function v defined pointwise a.e. by*

$$v(x,t) = \mathrm{Proj}_{U(x,t)}(u(x,t))$$

is also in $L^2(Q)^2$. Further, if for some $p \geq 2$ the functions u and f_U are in $L^p(Q)^2$, then the projection v is in $L^p(Q)^2$ as well.

Proof. By assumption (AU), the set-valued function U is measurable with closed and convex images, and u is a measurable single-valued function. Then by Theorem 3.3 the function v is measurable as well. By Lipschitz continuity of the pointwise projection, it holds

$$\begin{aligned}|v(x,t) - f_U(x,t)| &= |\,\mathrm{Proj}_{U(x,t)}(u(x,t)) - \mathrm{Proj}_{U(x,t)}(f_U(x,t))| \\ &\leq |u(x,t) - f_U(x,t)|\end{aligned}$$

almost everywhere on Q. Thus, squaring and integrating gives

$$\|v - f_U\|_2^2 \leq \|u - f_U\|_2^2 < \infty,$$

which implies $v \in L^2(Q)^2$. If in addition, u and f_U are in $L^p(Q)^2$ for some $p > 2$, then we can prove analogously that the projection is also in L^p, i.e. $v \in L^p(Q)^2$. □

4.2. Existence of optimal controls

Before we can think about existence of solution, we have to specify which problem we want to solve. We will assume that conditions (A) of Section 2.1 are satified. Moreover, we assume that $U(\cdot)$ fulfills the pre-requisite (AU). So we end up with the following optimization problem

$$\min J(y,u) \tag{8a}$$

subject to the state equation

$$y_t + \nu A y + B(y) = u \quad \text{in } L^2(0,T;V'), \tag{8b}$$

$$y(0) = y_0 \quad \text{in } H, \tag{8c}$$

and the control constraint

$$u \in U_{ad}, \tag{8d}$$

where U_{ad} is given by (7).

Under the assumptions above, the optimal control problem (8) is solvable. We recall that in Section 2.1 the regularization parameter γ is supposed to be greater than zero. One can prove existence even with $\gamma = 0$ under the additional condition of boundedness of U_{ad} in L^2.

Theorem 4.5. *The optimal control problem admits a - global optimal - solution $\bar{u} \in U_{ad}$ with associated state $\bar{y} \in W(0,T)$.*

5. First-order necessary conditions

The necessary optimality conditions for the optimal control problem discussed in the present chapter differ slightly from the conditions that can be found in the literature, see e.g. [25]. However, we will repeat the exact statement for convenience of the reader.

Theorem 5.1 (Necessary condition)**.** *Let $\bar{u}$ be locally optimal in $L^2(Q)^2$ with associated state $\bar{y} = y(\bar{u})$. Then there exists a unique Lagrange multiplier $\bar{\lambda} \in W^{4/3}(0,T;V)$, which is the weak solution of the adjoint equation*

$$
\begin{aligned}
-\bar{\lambda}_t + \nu A\bar{\lambda} + B'(\bar{y})^*\bar{\lambda} &= \alpha_Q(\bar{y} - y_Q) + \alpha_R \operatorname{curl}^*\operatorname{curl}\bar{y} \\
\bar{\lambda}(T) &= \alpha_T(\bar{y}(T) - y_T).
\end{aligned} \tag{9}
$$

Moreover, the variational inequality

$$
(\gamma\bar{u} + \bar{\lambda}, u - \bar{u})_Q \geq 0 \quad \forall u \in U_{ad} \tag{10}
$$

is satisfied.

Similar as in the box-constrained case, we can reformulate the variational inequality (10). The projection representation of the optimal control is now realized using the admissible sets $U(\cdot)$

$$
\bar{u}(x,t) = \operatorname{Proj}_{U(x,t)}\left(-\frac{1}{\gamma}\bar{\lambda}(x,t)\right) \qquad \text{a.e. on } Q. \tag{11}
$$

Here, it will be a little bit more difficult to prove regularity results for the optimal control using the regularity of the adjoint state. The projection formula is used in connection with Lipschitz stability of optimal controls [16, 23, 28]. The ideas there cannot be transferred to the case of set-valued constraints, see the discussion in Section 8 below.

Necessary optimality conditions of second order for optimal control problems with set-valued constraints were developped in [20]. It involves the use of the concept of second-order tangent, see e.g. [11].

6. Regularity of optimal controls

Let us comment on the regularity of a locally optimal control $\bar{u}$. By (11), it inherits some regularity from the associated adjoint state $\bar{\lambda}$. Here, we will show, how the regularities $\bar{\lambda} \in L^p(Q)^2$ respectively $\bar{\lambda} \in C(\bar{Q})^2$ can be carried over to the control $\bar{u}$. However, it is not clear whether and how it is possible to prove $\bar{u} \in W^{1,p}(Q)^2$ if $\bar{\lambda} \in W^{1,p}(Q)^2$, and what assumptions on U are needed.

6.1. Optimal controls in L^p

Corollary 4.4 gives a hint, how we can prove the regularity $\bar{u} \in L^p(Q)^2$ provided $\bar{\lambda} \in L^p(Q)^2$ holds. We have to assume only the existence of an admissible L^p-function.

Theorem 6.1. *Let $\bar{u}$ be a locally optimal control of the optimal control problem* (8) *with associated adjoint state $\lambda \in L^p(Q)^2$, $p \le \infty$. If there is an admissible function $f_p \in L^p(Q)^2 \cap U_{ad}$ for $p \le \infty$ then the optimal control $\bar{u}$ is in that $L^p(Q)^2$, too.*

Proof. The proof follows immediately from the projection representation (11) and Corollary 4.4. □

We will complete this short section with the following corollary, which states the precise regularity assumptions on the problem data, such that the pre-requisites of the previous theorem are fulfilled, see also [28].

Corollary 6.2. *Let be given $y_0, y_T \in V$, $y_Q \in L^2(Q)^2$. Let the set-valued mapping U satisfy the assumption (AU). Further, we assume the existence of an admissible L^p-function $f_p \in L^p(Q)^2 \cap U_{ad}$ for $2 \le p < \infty$.*

Then every locally optimal control of problem (8) *is in $L^p(Q)^2$, $2 \le p < \infty$.*

The method of proof applied here does not work to obtain continuity of an optimal control. This is investigated in the next section.

6.2. Continuity of optimal controls

Now, we are going to prove continuity of an locally optimal control. We will rely in our considerations again on the projection formula (11), which says that the optimal control is the pointwise projection of a continuous function on the admissible sets. Hence, this admissible sets $U(x,t)$ vary over space and time. Here, we have to impose some continuity assumptions on the set-valued mapping U.

There are two equivalent characterizations of continuous *single*-valued functions:

1. the image of a converging sequence is also a converging sequence,
2. the preimages of open sets are open sets.

In the set-valued case, however both definitions of a continuous function are no longer equivalent. They define two independent kinds of semicontinuity.

Definition 6.3. *A set-valued mapping $F : D \subset X \rightsquigarrow Y$ is called* lower semicontinuous*, if for all $x \in D$, $y \in F(x)$, and any sequence $\{x_n\} \subset D$ converging to x there is a sequence of elements $y_n \in F(x_n)$ converging to y.*

Definition 6.4. *A set-valued mapping $F : D \subset X \rightsquigarrow Y$ is called* upper semicontinuous*, if for all $x \in D$ and all open sets $O \supset F(x)$ there exists $\delta = \delta(O)$ such that $F(x') \subset O$ for all x' with $|x - x'| \le \delta$.*

Both definition are not equivalent and are independent. There are set-valued mappings, which are lower semicontinuous but not upper and vice-versa. It is natural to define a continuous mapping to have both semicontinuous properties.

Definition 6.5. *A set-valued mapping $F : D \subset X \rightsquigarrow Y$ is called* continuous*, if U is both lower and upper semicontinuous.*

The assumption (AU) on the set-valued mapping U contains the condition that $U(x,t)$ is non-empty, closed and convex almost everywhere on Q. Do these properties of the images hold everywhere provided U is continuous? At first, we want to show the improvement from 'non-empty almost everywhere' to 'non-empty everywhere' in the continuous case.

Lemma 6.6. *Let U fulfill (AU). Further let $U : \bar{Q} \rightsquigarrow \mathbb{R}^2$ be upper semicontinuous. Then $U(x,t)$ is non-empty for all $(x,t) \in \bar{Q}$.*

Proof. We will proof it by contradiction. Let $\xi = (x,t) \in \bar{Q}$ such that $U(\xi)$ is empty. Then we take a sequence of points $\xi_n = (x_n, t_n) \in Q$ with $U(\xi_n) \neq \emptyset$ converging to ξ. Now, take points $u_n \in U(\xi_n)$ and set $O = \mathbb{R}^2 \setminus \overline{\cup_{n=1}^{\infty} u_n}$. Then $u_n \in U(\xi_n) \not\subset O$ holds for all n, which is a contradiction to upper semicontinuity. □

Unfortunately, the property 'closedness of the images' cannot be transferred from 'almost everywhere' to 'everywhere' for continuous U as the following counterexample shows.

Example 6.7. *Define $F : [0,1] \rightsquigarrow \mathbb{R}$ by*

$$F(t) = \begin{cases} (0,1] & \text{if } t = 0 \\ [t, 1+t] & \text{otherwise.} \end{cases}$$

Clearly, F is lower semicontinuous. It is also upper semicontinuous: every open set that contains $F(0)$ contains also $F(\varepsilon)$ for sufficiently small ε. Hence F is continuous. It has closed images almost everywhere but not everywhere.

Now, let us prove an lemma, which will help us later on.

Lemma 6.8. *Let U fulfill (AU). In addition, we assume that U is upper semicontinuous on $\bar{Q}$ with closed images $U(x,t)$ for all $(x,t) \in \bar{Q}$. Then for given sequences (x_n, t_n) converging to $(x,t) \in \bar{Q}$ and $y_n \in U(x_n, t_n)$ converging to y the limit y lies in $U(x,t)$, $y \in U(x,t)$.*

Proof. We will use again the notation $\xi = (x,t)$ and $\xi_n = (x_n, t_n)$. Let us assume $y \notin U(\xi)$. Set $\varepsilon = \operatorname{dist}(y, U(\xi))$, which is positive since $U(\xi)$ is closed. Then there exists N such that for all $n > N$ it holds $y_n \notin U(\xi)$ and $\operatorname{dist}(y_n, U(\xi)) \geq \frac{2}{3}\varepsilon$. Now, we construct an open set by $O := \{v : \operatorname{dist}(v, U(\xi)) < \frac{1}{3}\varepsilon\}$. It implies $y_n \notin O$ and $U(\xi_n) \not\subset O$ for $n \geq N$. This yields a contradiction to upper semicontinuity, since we have $O \supset U(\xi)$. Hence it holds $y \in U(\xi)$. □

Furthermore, it turns out that the assumption of closed images is essential to prove the convexity of the images of U.

Lemma 6.9. *Let U fulfill (AU). In addition, let U be continuous on $\bar{Q}$ with closed images $U(x,t)$ for all $(x,t) \in \bar{Q}$. Then $U(x,t)$ is convex for all $(x,t) \in \bar{Q}$.*

Proof. Let $\xi = (x,t) \in \bar{Q}$ be given with $y_1, y_2 \in U(\xi)$, $\lambda \in (0,1)$. We have to show that $\lambda y_1 + (1-\lambda) y_2$ is in $U(\xi)$. We take a sequence of points $\xi_n = (x_n, t_n) \in Q$, for which $U(\xi_n)$ is non-empty and convex, converging to ξ.

By lower semicontinuity there exists sequences of points $y_1^n, y_2^n \in U(\xi_n)$ converging to y_1 respectively y_2. The points $y^n := \lambda y_1^n + (1-\lambda)y_2^n$ are in $U(\xi_n)$ and converge to $y := \lambda y_1 + (1-\lambda)y_2$ for $n \to \infty$. The previous Lemma 6.8 implies that the limit $y = \lambda y_1 + (1-\lambda)y_2$ is in $U(\xi)$. Hence $U(\xi)$ is convex. □

Assuming the continuity of the set-valued mapping U and the adjoint state λ we can prove continuity of an optimal control.

Theorem 6.10. *Let U satisfies the assumption (AU). Furthermore, let $U : \bar{Q} \rightsquigarrow \mathbb{R}^n$ be continuous with closed images everywhere. Suppose $\bar{u}$ satisfies the first-order necessary optimality conditions together with the state $\bar{y}$ and adjoint $\bar{\lambda}$. If the adjoint is continuous, $\bar{\lambda} \in C(\bar{Q})$, so is the control as well, $\bar{u} \in C(\bar{Q})$.*

Proof. We will show that the projection

$$\operatorname{Proj}_{U(x,t)}\left(-\frac{1}{\gamma}\bar{\lambda}(x,t)\right) = \bar{u}(x,t)$$

results in a continuous function. We abbreviate $v(x,t) := -\bar{\lambda}(x,t)/\gamma$, which is a continuous function by assumption.

Let $\xi = (x,t) \in \bar{Q}$ be given. Take a sequence $\xi_n = (x_n, t_n) \in Q$ that converges to ξ. We have to show the convergence $\bar{u}(\xi_n) \to \bar{u}(\xi)$. We will give the proof in several steps.

Step 1: $U(x,t)$ is non-empty, closed and convex everywhere on $\bar{Q}$. This follows by the preceding Lemmata 6.6 and 6.9.

Step 2: U_{ad} contains a continuous function. Define the function $m : \bar{Q} \to \mathbb{R}^n$ as

$$m(x,t) = \arg\min \{|v| : v \in U(x,t)\},$$

which gives the elements of $U(x,t)$ with the smallest norm. Since $U(x,t)$ is non-empty, closed and convex, the function m is well-defined. By [3, Chapt. 3, Sect. 1, Prop. 23, p. 120], the single-valued function m is continuous. It is also called a continuous selection of U.

Step 3: Boundedness of $\{\bar{u}(\xi_n)\}$. Using Lipschitz continuity of the projection, we can estimate

$$\begin{aligned} |\bar{u}(\xi_n) - m(\xi_n)| &= |\operatorname{Proj}_{U(\xi_n)}(v(\xi_n)) - \operatorname{Proj}_{U(\xi_n)}(m(\xi_n))| \\ &\le |v(\xi_n) - m(\xi_n)| \le \|v - m\|_{C(\bar{Q})} < \infty, \end{aligned}$$

which proves boundedness of the set $\{\bar{u}(\xi_n)\}$.

Step 4: Every accumulation point of $\{\bar{u}(\xi_n)\}$ is in $U(\xi)$. Since $\{\bar{u}(\xi_n)\}$ is bounded in $\mathbb{R}^n$, we can select a subsequence $\{\bar{u}(\xi_{n'})\}$ converging to some element $\tilde{u}$. By Lemma 6.8, we find that $\tilde{u}$ is in $U(\xi)$.

Step 5: There is exacty one accumulation point of $\{\bar{u}(\xi)\}$. Take an arbitrary element $z \in U(\xi)$. By lower semicontinuity, there is a sequence of elements $z_{n'} \in F(\xi_{n'})$ converging to $z \in U(\xi)$. Since $u(\xi_{n'}) = \operatorname{Proj}_{U(\xi_{n'})} v(\xi_{n'})$, we find

$$\big(u(\xi_{n'}) - v(\xi_{n'}),\, z_{n'} - u(\xi_{n'})\big) \ge 0 \quad \forall n'.$$

Hence

$$\big(u(\xi_{n'}) - v(\xi_{n'}),\, z_{n'} - z\big) + \big(u(\xi_{n'}) - v(\xi_{n'}),\, z - u(\xi_{n'})\big) \geq 0 \quad \forall n'.$$

Letting $n' \to \infty$, we find

$$\big(\tilde{u} - v(\xi),\, z - \tilde{u}\big) \geq 0.$$

Since $z \in U(x)$ was arbitrary, it holds

$$\tilde{u} = \mathrm{Proj}_{U(\xi)}\, v(\xi)$$

for every accumulation point of $\{\bar{u}(\xi_n)\}$. The projection is unique hence the set $\{u(\xi_n)\}$ has exactly one accumulation point.

Conclusion. By the previous step, we find the convergence $\bar{u}(\xi_n) \to \bar{u}(\xi)$. Hence the prove is complete, and $\bar{u}$ is a continuous function on $\bar{Q}$. □

For an exact statement, which regularity of the data is sufficient for $\lambda \in C(\bar{Q})^2$, we refer to [28].

Remark 6.11. *The projection formula* (11) *remains true if one replaces $U(x,t)$ by its closure $\bar{U}(x,t)$, provided $U(\cdot)$ is closed almost everywhere on Q. Furthermore, one can show that for continuous $U : Q \rightsquigarrow \mathbb{R}^2$ the closure $\bar{U} : Q \rightsquigarrow \mathbb{R}^2$ is also a continuous set-valued mapping. In this way, we can construct a continuous representation of a locally optimal control $\bar{u}$ without the assumption of closedness of the images of U.*

7. Second-order sufficient optimality conditions

7.1. Normal directions

Before we start with the formulation of the sufficient optimality conditions, let us recall some notations of convex set theory. Let be given a convex set C. Then $\mathcal{N}_C(u)$ and $\mathcal{T}_C(u)$ are the normal and tangent cones of C at some point u. The space of normal directions is written $N_C(u) = \mathrm{span}\,\mathcal{N}_C(u)$ with its orthogonal complement $T_C(u)$.

Now, we want to use these notations with $C = U_{ad}$. Let be given an admissible control $u \in U_{ad}$. It is well-known, that the sets $\mathcal{N}_{U_{ad}}(u)$, $\mathcal{T}_{U_{ad}}(u)$, $N_{U_{ad}}(u)$, and $T_{U_{ad}}(u)$ admit a pointwise representation as U_{ad} itself, cf. [4, 22]. For instance, for $u \in L^2(Q)^2$ the set $\mathcal{N}_{U_{ad}}(u)$ is given by

$$\mathcal{N}_{U_{ad}}(u) = \left\{v \in L^2(Q)^2 :\, v(x,t) \in \mathcal{N}_{U(x,t)}(u(x,t)) \text{ a.e. on } Q\right\}.$$

In a while, we will need the projection of a test function w on the space of normal directions and its complement. We will denote the resulting functions by w_N and w_T respectively. They are defined pointwise by

$$w_N(x,t) = \mathrm{Proj}_{\left[N_{U(x,t)}(u(x,t))\right]}(w(x,t)) \tag{12}$$

and

$$w_T(x,t) = \mathrm{Proj}_{\left[T_{U(x,t)}(u(x,t))\right]}(w(x,t)). \tag{13}$$

It is not easy to prove that the functions w_N and w_T are measurable. At this point the method of Dunn [12] requires that the admissible set U is polyhedric and independent of (x,t). However, these restrictions can be overcome using the results for set-valued mappings. Let us sketch the method of the measurability proof. Behind the projections there are the following mappings:

1. $Q \ni (x,t) \mapsto \mathcal{N}_{U(x,t)}(u(x,t)) =: \mathcal{N}(x,t)$
2. $Q \ni (x,t) \mapsto \mathrm{span}\{\mathcal{N}(x,t)\} = \mathrm{span}\left\{\mathcal{N}_{U(x,t)}(u(x,t))\right\} =: N(x,t)$
3. $Q \ni (x,t) \mapsto \mathrm{Proj}_{N(x,t)}(w(x,t)) =: w_N(x,t)$.

Here, one can see, what happens if $U(x,t)$ is constant over Q: the mapping $\mathcal{N}$ is even in this case a set-valued mapping, which is not constant. Even the dimension of $\mathcal{N}(x,t)$ varies. So we would not have any advantage if we assume constant admissible sets $U(x,t) = U_0$.

Now, the measurability can be proven as follows. The mapping $(x,t) \rightsquigarrow \mathcal{T}_{U(x,t)}(u(x,t))$ is measurable if U has closed and convex images, cf. [4, Cor. 8.5.2]. The normal cone $\mathcal{N}$ is then the dual of $\mathcal{T}$, and one can prove that the dual cone operation does preserve measurability. The same can be done for the linear hull N. Here, the proof is based on the countability argument already stated in Theorem 3.2.

Now, the projection of measurable function on measurable set-valued mappings — here N — results in a measurable function, see Theorem 3.3. Altogether, both of the projections (12) and (13) results in measurable function. Let us remark that it is very difficult to prove this by hand: here one has to go step by step from regular convex sets and constant U to more irregular ones.

As a second point here, let us define some more notations in connection to convex sets. The relative interior of a convex set is defined by

$$\mathrm{ri}\, C = \{x \in \mathrm{aff}\, C : \exists \varepsilon > 0, B_\varepsilon(x) \cap \mathrm{aff}\, C \subset C\},$$

its complement in C is called the relative boundary

$$\mathrm{rb}\, C = C \setminus \mathrm{ri}\, C.$$

The distance of a point $u \in \mathbb{R}^n$ to a set $C \subset \mathbb{R}^n$ is defined by

$$\mathrm{dist}(u, C) = \inf_{x \in C} |u - x|.$$

7.2. Sufficiency

Let us come back to optimization with convex constraints. To motivate the following, we will investigate the finite-dimensional problem $\min_{x\in C} f(x)$ with $C \subset \mathbb{R}^n$ first. It is known that a local minimizer x of the function f over a convex set C fulfills

$$-\nabla f(x) \in \mathcal{N}_C(x).$$

A sufficient condition is then given by

$$-\nabla f(x) \in \operatorname{ri} \mathcal{N}_C(x) \tag{14}$$

and

$$f''(x)[y,y] > 0 \qquad \forall y \in T_C(x). \tag{15}$$

It consists of a first-order part: strict complementarity and a second-order part: coercivity. Now we want to adapt this formulation to the optimal control problem considered here. Condition (14) would become

$$-(\gamma\bar{u}(x,t) + \bar{\lambda}(x,t)) \in \operatorname{ri} \mathcal{N}_{U(x,t)}(\bar{u}(x,t)) \quad \text{ a.e. on } Q. \tag{16}$$

However, this is not enough for optimal control problems, we need the satisfaction of this condition in a uniform sense. We have to assume not only that $-(\gamma\bar{u}(x,t) + \bar{\lambda}(x,t))$ lies in the relative interior of the normal cone, we need moreover that $-(\gamma\bar{u}(x,t) + \bar{\lambda}(x,t))$ has a positive distance to the relative boundary of $\mathcal{N}_{U(x,t)}(\bar{u}(x,t))$. But this cannot be assumed for all (x,t): if $\bar{u}(x,t)$ is in the interior of the admissible set $U(x,t)$ then the normal cone consists only of the origin and has no relative interior. Therefore, we introduce the set of strongly active constraints as the set of points, where this condition is fulfilled,

$$Q_\varepsilon = \left\{(x,t) \in Q : \operatorname{dist}\left(-(\gamma\bar{u}(x,t) + \bar{\lambda}(x,t)), \operatorname{rb}\mathcal{N}_{U(x,t)}(\bar{u}(x,t))\right) > \varepsilon\right\}. \tag{17}$$

That is, we assume that (16) is only fulfilled on a subset of the domain Q. Consequently, we have to require the coercivity assumption for more directions than included in $T_{U_{ad}}(\bar{u})$. Furthermore, the inequality > 0 in (15) has to be replaced by a norm-square, since the proof in finite dimensions that '> 0' suffices is tied to compactness of the unit sphere, which does not hold in the infinite dimensional case.

Altogether, we require that the following is fulfilled. We assume that the reference pair $(\bar{y}, \bar{u})$ satisfies the coercivity assumption on $\mathcal{L}''(\bar{y}, \bar{u}, \bar{\lambda})$, in the sequel called second-order sufficient condition:

(SSC)
There exist $\varepsilon > 0$ and $\delta > 0$ such that

$$\mathcal{L}''(\bar{y}, \bar{u}, \bar{\lambda})[(z,h)]^2 \geq \delta \, \|h\|_2^2 \tag{18a}$$

holds for all pairs $(z,h) \in W(0,T) \times L^2(Q)^2$ with

$$h \in \mathcal{T}_{U_{ad}}(\bar{u}), \; h_N = 0 \text{ on } Q_\varepsilon, \tag{18b}$$

and $z \in W(0,T)$ being the weak solution of the linearized equation

$$\begin{aligned} z_t + Az + B'(\bar{y})z &= h \\ z(0) &= 0. \end{aligned} \tag{18c}$$

In (18b) h_N denotes the pointwise projection of h on the subspaces $N \subset \mathbb{R}^2$, compare (12). We required in (SSC) the coercivity of $\mathcal{L}''$ for more test functions than in (15). The set $T_{U_{ad}}(\bar{u})$, which was user there, is only a subset of $\mathcal{T}_{U_{ad}}(\bar{u})$. However, the space of test functions in (SSC) can be reformulated as: $h \in \mathcal{T}_{U_{ad}}(\bar{u})$ with $h(x,t) \in T_{U(x,t)}(\bar{u}(x,t))$ on Q_ε. We can use test functions with values in the spaces T only on the strongly active set, due to the strong complementarity, which holds there. On the rest of the domain, the values of the test function has to lie in the tangent cones $\mathcal{T}$.

Now, the next theorem states the sufficiency of (SSC).

Theorem 7.1. *Let $(\bar{y}, \bar{u})$ be admissible for the optimal control problem and suppose that $(\bar{y}, \bar{u})$ fulfills the first order necessary optimality conditions with associated adjoint state $\bar{\lambda}$. Assume further that (SSC) is satisfied at $(\bar{y}, \bar{u})$. Then there exist $\alpha > 0$ and $\rho > 0$ such that*

$$J(y, u) \geq J(\bar{y}, \bar{u}) + \alpha \|u - \bar{u}\|_2^2$$

holds for all admissible pairs (y, u) with $\|u - \bar{u}\|_\infty \leq \rho$.

The proof can be found in [27].

There are a number of sufficient second-order optimality conditions for finite-dimensional optimization problems with convex constraints, see for instance [7, 8, 21]. They all use the second-order tangent sets, and it is not clear how those results are related to the condition presented here. Also, the extension of the finite-dimensional results to optimal control problems is not a trivial exercise and requires further research.

Remark 7.2. *Let us comment on the definition of the strongly active set in* (17) *if U is formed by box constraints. Since this particular constraint is formed by two independent inequalities, one can refine the definition of strongly active sets, see* [25]*, to contain more points than the active set introduced here.*

8. Stability of optimal controls

Usually, the fulfillment of a second-order sufficient condition implies stability of locally optimal controls under small perturbations. This is demonstrated in a great variety of articles for optimal control problems with box constraints. However, in the case of general convex control constraints the sufficient condition (SSC) is too weak to get stability of optimal controls. This is due to the fact, that tangent variations of the control are not necessarily admissible directions, which is an essential ingredient in the proofs for the box-constrained case.

In finite-dimensional optimization, there are a few publications concerning stability of solutions to convex constrained optimization problems, see [6, 8]. They use the assumption of second-order regular sets. The extension of that conditions to the infinite-dimensional case considered here is not obvious, since the proofs argue by contradiction and rely on the finite-dimensionality i.e. on compactness of the unit sphere. That means, one has to use methods which differ from the indirect methods of [6] as well as from the direct proofs from e.g. [16, 23, 25].

Obviously, if we assume coercivity of $\mathcal{L}''$ for all test directions, we can prove such stability results. Since this would be only a technical exercise, we do not proceed in this direction.

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