Superefficient estimation of the intensity of a stationary Poisson point process via the Stein method

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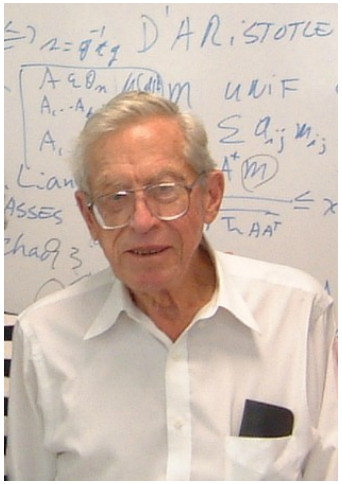
Back to the initial ideas of Charles Stein



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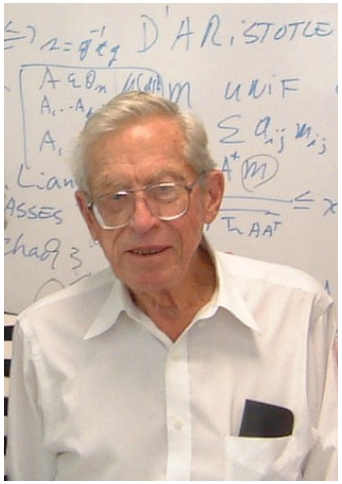

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- Let $X \sim \mathcal{N}(\theta, \sigma^2 \mathbf{I}_d)$
  where I_d is the d-dimensional identity matrix.
- Objective: estimate θ based on a **single** (for simplicity) observation X.

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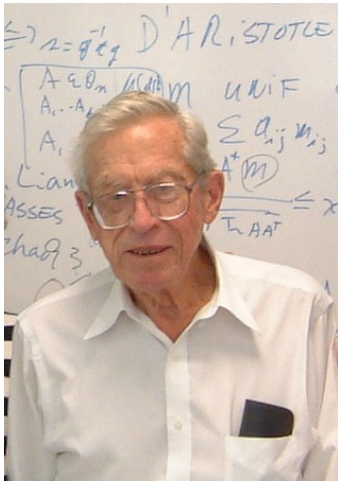

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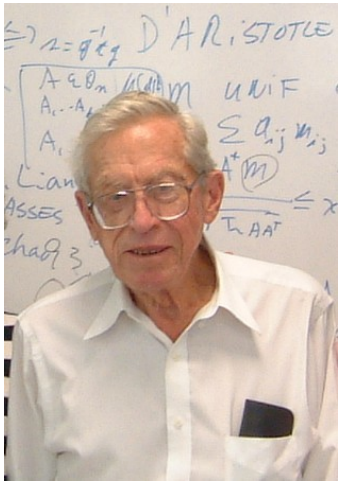

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- Stein (1956)

  $$\widehat{\theta}_S = \left((1 - b(a + X_i^2)^{-1})X_i\right)_{i=1,\dots,d} \Rightarrow \mathrm{MSE}(\widehat{\theta}_S) \leq \mathrm{MSE}(\widehat{\theta}^{mle}) \text{ when } d \geq 3$$

- James-Stein (1961)

  $$\widehat{\theta}_{JS} = X(1 - (d-2)/\|X\|^2) \Rightarrow \mathrm{MSE}(\widehat{\theta}_{JS}) \leq \mathrm{MSE}(\widehat{\theta}^{mle}) \text{ when } d \geq 3$$

- Stein (1981) key-ingredients for the class: $\widehat{\theta} = X + g(X), \quad g : \mathbb{R}^d \to \mathbb{R}^d$.

MSE of $\widehat{\theta} = X + g(X)$ ($X \sim \mathcal{N}(\theta, \sigma^2 \mathbf{I}_d$, σ^2 known)

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$$\mathrm{MSE}(\widehat{\theta}) = \mathrm{MSE}(\widehat{\theta}^{mle}) + 4\sigma^2 \mathrm{E}\left(\frac{\nabla\nabla\sqrt{f(X)}}{\sqrt{f(X)}}\right) \leq \mathrm{MSE}(\widehat{\theta}^{mle}) \quad \text{if} \quad \nabla\nabla\sqrt{f} \leq 0.$$

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Goal: mimic both steps, derive a Stein estimator for the intensity of a Poisson point process [extension Privault-Réveillac (2009), $d = 1$]

Point processes

- S : Polish state space of the point process (equipped with the σ-algebra of Borel sets $\mathcal{B}$).
- A configuration of points is denoted $x = \{x_1, \cdots, x_n, \cdots\}$. For $B \subset S : x_B = x \cap B$.
- N_{lf} : space of locally finite configurations, i.e.

$$\{x,\ n(x_B) = |x_B| < \infty\}, \forall B \text{ bounded } \in S\}$$

equipped with
$\mathcal{N}_{lf} = \sigma(\{x \in N_{lf},\ n(x_B) = m\}; B \in \mathcal{B}, B \text{ bounded}, m \geq 1)$.

Definition

A point process X defined on S is a measurable application defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with values on N_{lf}.

Measurability of $X \Longleftrightarrow N(B) = |X_B|$ is a r.v. for any bounded $B \in \mathcal{B}$.

Poisson point processes

Definition of a Poisson point process with intensity $\rho(\cdot)$.

- $\forall m \geq 1$, $\forall$ bounded and disjoint $B_1, \cdots, B_m \subset S$, the r.v. $X_{B_1}, \cdots, X_{B_m}$ are independent.
- $N(B) \sim \mathcal{P}(\int_B \rho(u)du)$
- $\forall B \subset S, \forall F \in N_{lf}$

$$\mathbb{P}(X_B \in F) = \sum_{n \geq 0} \frac{e^{-\int_B \rho(u)du}}{n!} \int_B \cdots \int_B 1(\{(x_1, \cdots, x_n) \in F\}) \prod_{i=1}^{n} \rho(x_i)dx_i$$

Notation : $X \sim \text{Poisson}(S, \rho)$.

Our framework

- Case considered here $\rho(u) \equiv \theta$.
- X homogeneous Poisson point process with intensity θ .
- We assume observing X on $W \subset \mathbb{R}^d$.
- Given $N(W) = n$, we denote $X_1, \ldots, X_n$ the n points in W.
- The MLE estimate of the intensity θ is $\widehat{\theta} = N(W)/|W|$.
- Construction of a Stein estimator of θ?

Poisson functionals

$\mathcal{S}$: space of Poisson functionals F defined on Ω by

$$F = f_0 \mathbf{1}(N(W) = 0) + \sum_{n \geq 1} \mathbf{1}(N(W) = n) f_n(X_1, \ldots, X_n) \ ,$$

$f_0 \in \mathbb{R}$, $f_n : W^n \to \mathbb{R}^d$ measurable symmetric functions called form functions of F.

Towards a Stein estimator (1)

- MLE is defined by $\widehat{\theta}^{mle} = N(W)/|W|$.
- Aim : define $\widehat{\theta}$ of the form $\widehat{\theta} = \widehat{\theta}^{mle} + \frac{1}{|W|}\zeta$ where $\zeta = \nabla \log(F)$.
- Relation $\mathrm{MSE}(\widehat{\theta}) < \mathrm{MSE}(\widehat{\theta}^{mle})$ satisfied?

$$\mathrm{MSE}(\widehat{\theta}) = \mathrm{E}\left[\left(\widehat{\theta}^{mle} + \frac{1}{|W|}\nabla \log F - \theta\right)^2\right]$$

$$= \mathrm{MSE}(\widehat{\theta}^{mle}) + \frac{1}{|W|^2}\left(\mathrm{E}[(\nabla \log F)^2] + 2\mathrm{E}[(\nabla \log F)(N(W) - \theta|W|)]\right)$$

$\Longrightarrow$ Need to transform $2\mathrm{E}[G(N(W) - \theta|W|)]$ with $G = \nabla \log F$ using a IbP formula.

- Notion of derivative ?

Malliavin derivatives (1)

Differential operator: let $\pi : W^2 \to \mathbb{R}^d$

$$D_x^\pi F = -\sum_{n\geq 1} \mathbf{1}(N(W) = n) \sum_{i=1}^{n} (\nabla_{x_i} f_n)(X_1, \ldots, X_n)\pi(X_i, x) \ ,$$

where $\mathcal{S}' = \{F \in \mathcal{S} :$ 'the f_n are cont. diff. in any variable x_i'$\}$ and where $\nabla_{x_i} f_n$ gradient of $x_i \mapsto f_n(\ldots, x_i, \ldots)$.

Malliavin derivatives (2)

Lemma [product and chain rules]

For any $x \in W$, for all $F, G \in \mathcal{S}'$, $g \in C_b^1(\mathbb{R})$ we have

$$D_x^\pi(FG) = (D_x^\pi F)G + F(D_x^\pi G) \qquad \text{and} \qquad D_x^\pi g(F) = g'(F) D_x^\pi F \ .$$

Malliavin derivatives (2)

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To get an IbP type formula, we need to introduce $\mathrm{Dom}(D^\pi)$ of $\mathcal{S}'$ as

$$\mathrm{Dom}(D^\pi) = \Big\{ F \in \mathcal{S}' : \forall n \geq 1 \text{ and } z_1, \ldots, z_n \in \mathbb{R}^d$$

$$\mathbf{f_{n+1}}\Big|_{z_{n+1} \in \partial W}(z_1, \ldots, z_{n+1}) = \mathbf{f_n}(z_1, \ldots, z_n),\ \mathbf{f_1}\Big|_{\mathbf{z} \in \partial \mathbf{W}}(z) = \mathbf{0} \Big\} \ , \qquad (1)$$

<u>Remark</u>: **compatibility conditions** important to derive a correct Stein estimator.

Integration by parts formula

Theorem

Let $G \in \mathrm{Dom}(\overline{D}^{\pi})$, $V : \mathbb{R}^d \to \mathbb{R}$, $V \in \mathcal{C}^1(W, \mathbb{R}^d)$

$$\underbrace{\mathrm{E}\left[\int_W D_x^{\pi} G \cdot V(x)\mathrm{d}x\right]}_{:=\nabla^{\pi,V}F} = \mathrm{E}\left[F\left(\sum_{u\in\mathbf{X}_W} \nabla\cdot\mathcal{V}(u) - \theta\int_W \nabla\cdot\mathcal{V}(u)\mathrm{d}u\right)\right]$$

where $\mathcal{V} : W \to \mathbb{R}^d$ is defined by $\mathcal{V}(u) = \int_W V(x)\pi(u,x)\mathrm{d}x$.

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Main application: let $\pi(u,x) = u^{\top}V(x)$, we can find some V (omit details) such that $\mathcal{V}(u) = u/d$ and $\nabla\cdot\mathcal{V}(u) = 1$. Then

$$\nabla G = \nabla^{\pi,V}G = -\frac{1}{d}\sum_{n\geq 1}\mathbf{1}(N(W)=n)\sum_{i=1}^{n}\nabla_{x_i}g_n(X_1,\ldots,X_n)\cdot X_i$$

$$\boxed{\Rightarrow \qquad \mathrm{E}[\nabla G] = \mathrm{E}\left[G(N(W)-\theta|W|)\right].}$$

Towards a Stein estimator (2) : end of the proof

Theorem

Let $\widehat{\theta} = \widehat{\theta}^{mle} + \frac{1}{|W|}\zeta$ where $\zeta = \nabla \log(F)$ is such that $\zeta \in \mathrm{Dom}(\overline{D}^{\pi})$ then

$$\mathrm{MSE}(\widehat{\theta}) = \mathrm{MSE}(\widehat{\theta}^{mle}) + \frac{4}{|W|^2}\mathrm{E}\left(\frac{\nabla\nabla\sqrt{F}}{\sqrt{F}}\right).$$

<u>Proof</u>:

$$\begin{aligned}
\mathrm{MSE}(\widehat{\theta}) &= \mathrm{E}\left[\left(\widehat{\theta}^{mle} + \frac{1}{|W|}\nabla \log F - \theta\right)^2\right] \\
&= \mathrm{MSE}(\widehat{\theta}^{mle}) + \frac{1}{|W|^2}\left(\mathrm{E}[(\nabla \log F)^2] + 2\mathrm{E}[(\nabla \log F)(N(W) - \theta|W|)]\right) \\
&= \mathrm{MSE}(\widehat{\theta}^{mle}) + \frac{1}{|W|^2}\left(\mathrm{E}[(\nabla \log F)^2] + 2\mathrm{E}[\nabla\nabla \log F]\right) \\
&= \ldots
\end{aligned}$$

Non-uniqueness of the integration by parts formula

- Natural and easier to define a isotropic Stein estimator. With

$$\nabla \log F = -\frac{1}{d} \sum_{n \geq 1} \mathbf{1}(N(W) = n) \sum_{i=1}^{n} \nabla_{x_i} (\log f_n)(X_1, \ldots, X_n) \cdot X_i$$

$\log F$ is isotropic $\Rightarrow \nabla \log F$ is isotropic (and so will be $\widehat{\theta}$).

- Other possible choices to get $\operatorname{div} \mathcal{V}(y) = 1$: $V(x) = (d|W|)^{-1/2} \mathbf{1}(x \in W) 1^\top$, $\pi(y, x) = y^\top V(x)$. New gradient operator :

$$\nabla \log F = -\sum_{n \geq 1} \mathbf{1}(N(W) = n) \sum_{i=1}^{n} (\operatorname{div}_{x_i} \log f_n)(X_1, \ldots, X_n) \times \overline{X_i}$$

- Formula $\mathrm{E}[\nabla \log F] = \mathrm{E}\left[\log F(N(W) - \theta|W|)\right]$ **still** holds. But....

1. non isotropic.
2. can induce some discontinuity problems when computing $\nabla \log F$ and $\nabla\nabla \log F$ ...

Example in the d-dimensional euclidean ball $W = B_d(0,1)$

- For $1 \leq k \leq n$, $x_{(k),n}$ kth closest (wrt $\|\cdot\|$) point of $\{x_1, \ldots, x_n\}$ to zero.
- X_k kth closest point to 0 of the PPP X (defined on $\mathbb{R}^d$)
- We define

$$\varphi(t) = e^{\gamma(1-t)^\kappa}\, \mathbf{1}(t \leq 1), \qquad \gamma \in \mathbb{R}, \quad \kappa > 2$$

$$F_k = \mathbf{1}(N(W) < k) + \sum_{n \geq k} \mathbf{1}(N(W) = n)\, \varphi(\|X_{(k),n}\|^2)^2.$$

$$Gain(\widehat{\theta}_k) = 1 - \mathrm{MSE}(\widehat{\theta}_k)/\mathrm{MSE}(\widehat{\theta}^{mle})$$

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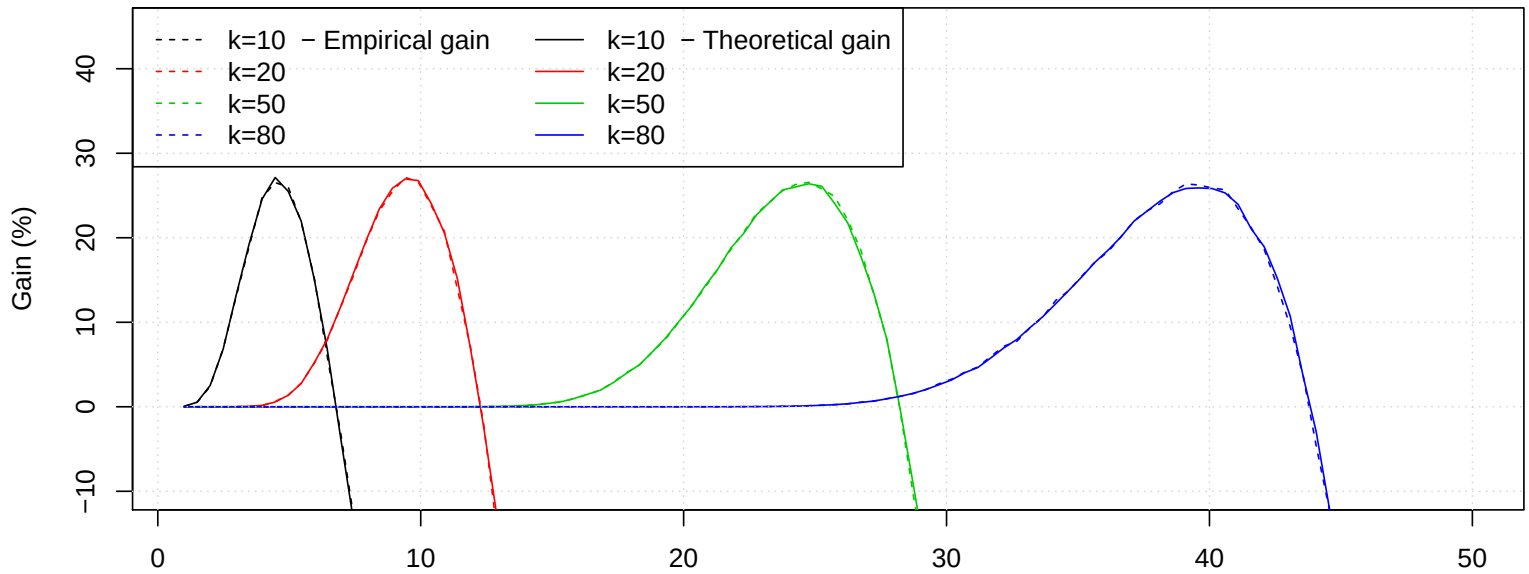
Theorem

$\zeta_k = \nabla \log(F_k) \in \mathrm{Dom}(\overline{D}^\pi)$ [$\varphi > 0, \varphi'(1) = 0$] and

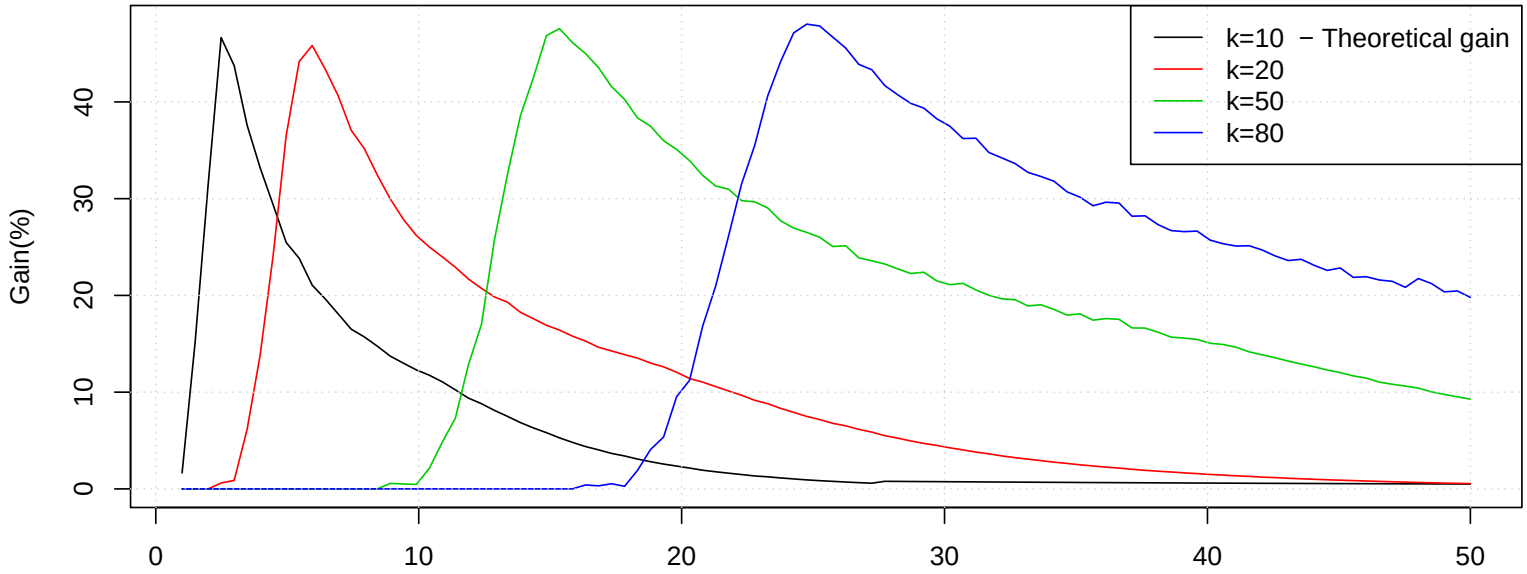
$$\widehat{\theta}_k = \widehat{\theta}^{mle} - \frac{4d}{|W|} \frac{\varphi'(\|X_{(k)}\|^2)}{\varphi(\|X_{(k)}\|^2)} = \widehat{\theta}^{mle} - \frac{4d}{|W|} \left\{ \gamma\kappa \left(1 - \|X_{(k)}\|^2\right)^{\kappa-1} \right\}$$

$$Gain(\widehat{\theta}_k) = \mathrm{E}[\, \mathcal{G}(\|X_{(k)}\|^2)\,] \quad \text{where} \quad \mathcal{G}(t) = -\frac{16}{d^2\theta|W|} \frac{t\,(\varphi'(t) + t\varphi''(t))}{\varphi(t)}$$

Theoretical gains with $\varphi(t) = e^{\gamma(1-t)^\kappa}$; $\kappa = 3$; $\gamma = -3$



- $m = 50000$ replications of $PPP_\theta(B(0,1), \theta)$, $d = 2$.
- Empirical and Monte-Carlo approximations of theoretical gains, for different parameters k, κ, γ.
- <u>General comments</u>:
  1. The IbP formula is empirically checked.
  2. The parameters k, κ, γ and θ are strongly connected. A bad choice can lead to **negative** gains [$\varphi'(t) + t\varphi''(t)$ may be negative for some values of t].



- $m = 50000$ replications of $PPP(B_\theta(0,1), \theta)$, $d = 2$.
- Monte-Carlo approximations of theoretical gains for different values of k. The parameters κ and γ optimize $\text{Gain}(\widehat{\theta_k})$ for each value of θ.
- <u>General comments</u>:
  1. For any k, if we optimize in terms of κ and γ, the gain becomes always positive.
  2. Still, if we want interesting values of gains, k needs to be optimized.

- Simulation based on $m = 50000$ replications.
- For each value of θ, d

$$(k^\star, \gamma^\star, \kappa^\star) = \text{argmax}_{(k,\gamma,\kappa)} \text{Gain}(\widehat{\theta}_k) = \text{argmax}_{(k,\gamma,\kappa)} \text{E}[\mathcal{G}(\|X_{(k)}\|^2)].$$

	MLE			STEIN				Gain (%)
	mean	sd	mse	$k^\star$	mean	sd	mse	
$\theta = 5, d = 1$	5	1.6	2.52	11	4.4	1.0	1.44	43.0
$d = 2$	5	1.3	1.58	18	4.6	0.8	0.86	45.6
$d = 3$	5	1.1	1.19	22	4.6	0.7	0.64	46.1
$\theta = 10, d = 1$	10	2.2	5.03	22	9.2	1.4	2.73	45.8
$d = 2$	10	1.8	3.18	34	9.4	1.2	1.72	46.0
$d = 3$	10	1.5	2.37	44	9.5	1.0	1.27	46.3
$\theta = 20, d = 1$	20	3.1	9.91	42	18.8	2.0	5.31	46.4
$d = 2$	20	2.5	6.38	66	19.1	1.6	3.41	46.5
$d = 3$	20	2.2	4.72	84	19.1	1.3	2.47	47.5
$\theta = 40, d = 1$	40	4.5	20.09	84	38.5	2.9	10.61	47.2
$d = 2$	40	3.6	12.79	125	38.6	2.2	6.78	46.9
$d = 3$	40	3.1	9.58	169	38.8	1.9	4.95	48.3

Data-driven estimator: replace θ by $\widehat{\theta}_{mle}$ in the optimization

- Simulation based on $m = 5000$ replications.
- For each value of θ, d, let $\Theta(\theta, \rho) = \left[\theta - \rho\sqrt{\theta/|W|}, \theta + \rho\sqrt{\theta/|W|}\right]$. Then, we suggest define $\kappa^\star$, $\gamma^\star$ as the maximum of

$$\int_{\Theta(\widehat{\theta}_{MLE},\rho)} \mathrm{Gain}(\widehat{\theta}_k)\mathrm{d}\theta = \frac{16}{d^2|W|}\mathrm{E}\int_{\Theta(\widehat{\theta}_{MLE},\rho)} \frac{\mathcal{G}(Y_{(k)})}{\theta}\mathrm{d}\theta. \tag{2}$$

	Gain (%)			
	$\rho = 0$	$\rho = 1$	$\rho = 1.6449$	$\rho = 1.96$
$\theta = 5, d = 1$	48.8	47.9	36.4	30.1
$d = 2$	38.6	42.4	37.1	31.4
$d = 3$	39.4	42.6	37.0	31.7
$\theta = 10, d = 1$	40.3	43.8	36.7	30.1
$d = 2$	36.2	38.8	33.7	27.9
$d = 3$	31.6	36.6	32.0	28.3
$\theta = 20, d = 1$	37.3	38.6	34.5	28.0
$d = 2$	27.3	33.1	31.0	26.5
$d = 3$	20.8	28.6	28.1	23.8
$\theta = 40, d = 1$	22.3	30.8	29.2	23.9
$d = 2$	16.3	24.0	28.2	24.4
$d = 3$	12.7	19.0	24.5	22.0

A few more comments

- Even if the results are done under the Poisson assumption, if the simulated model
  - is clustered (e.g. Thomas, LGCP) the empirical gains (compared to $N(W)/|W|$) are significant .
  - is regular the empirical gains seems to be close to zero (not really worse than $N(W)/|W|$)

Perspectives

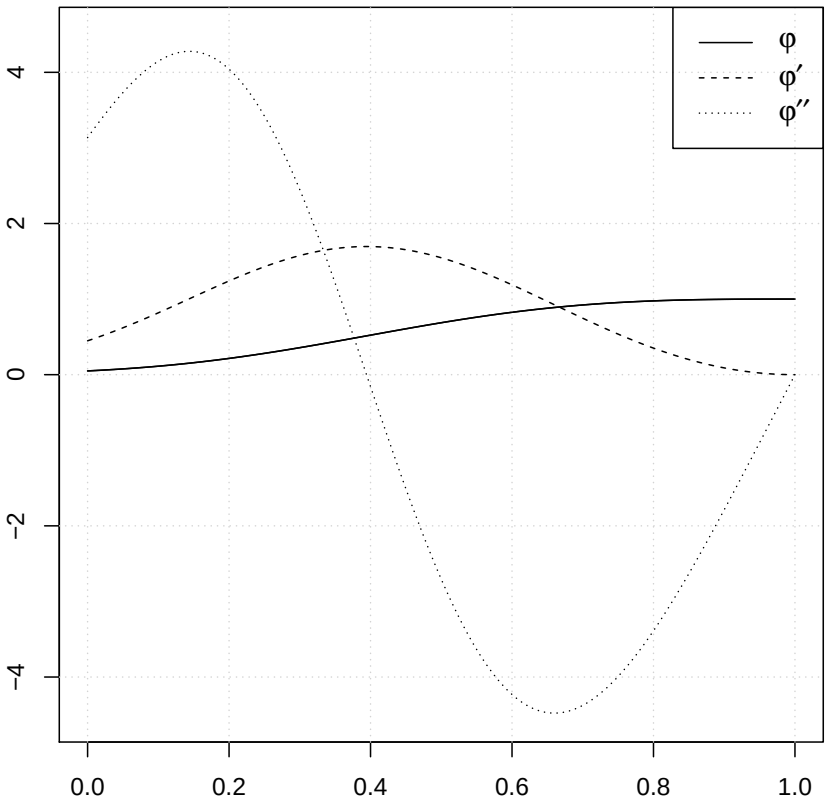
1. Deriving a general IbP formula for inhomogeneous Poisson point processes or Cox point processes seems reasonable.
2. Exploit the IbP for other statistical methodologies.

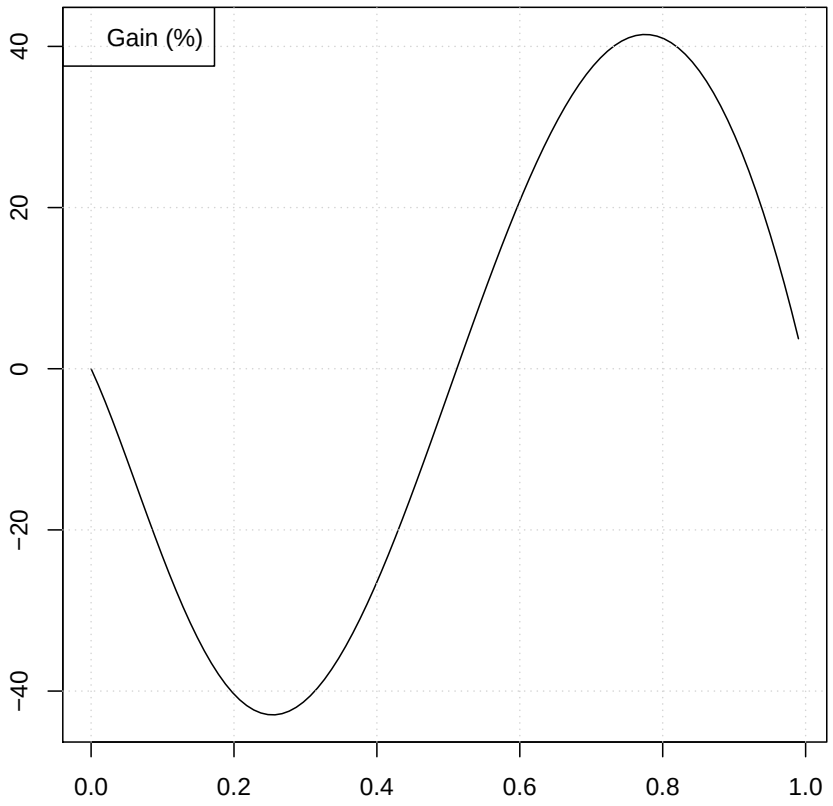
References

- W. James and C. Stein. *Estimation with quadratic loss.* (1961).
- C. Stein. *Estimation of the mean of a multivariate normal distribution.* (1981).
- N. Privault and A. Réveillac. *Stein estimation of Poisson process intensities.* (2009).
- M. Clausel, J.F. Coeurjolly, J. Lelong. *Stein estimation of the intensity of a spatial homogeneous Poisson point process.* (2014) `http://arxiv-web3.library.cornell.edu/abs/1407.4372`

Plots of $\varphi, \varphi', \varphi'', \mathcal{G}$

$\varphi(t) = e^{\gamma(1-r)^{\kappa}}$; $\kappa = 3$; $\gamma = -3$





$\Rightarrow$ $\mathcal{G}(t)$ is not positive everywhere but when t is large (i.e. when $\|X_{(k)}\|$ is large, i.e. when k is large), then $\mathcal{G}(\cdot)$ is positive and can reach high values.

Comparison with Privault-Réveillac's estimator when $d = 1$

- Assume $\mathbf{X}$ is observed on $\widetilde{W} = [0, 2]$.
- Let X_1 be the closest point of $\mathbf{X}$ to 0, then $\widehat{\theta}_{pr}$ is defined for some $\kappa > 0$ by

$$\widehat{\theta}_{pr} = \widehat{\theta}_{mle} + \frac{2}{\kappa}\mathbf{1}(N(\widetilde{W}) = 0) \; + \; \frac{2X_1}{2(1+\kappa) - X_1}\,\mathbf{1}(0 < X_1 \leq 2).$$

  Note that $X_1 \sim \mathcal{E}(\theta)$.
- The gain writes

$$\mathrm{Gain}(\widehat{\theta}_{pr}) = \frac{2}{\theta\kappa^2}\exp(-2\theta) \; - \; \frac{2}{\theta}\mathrm{E}\left(\frac{X_1}{2(1+\kappa) - X_1}\mathbf{1}(X_1 \leq 2)\right).$$

Gain optimized in κ in terms of θ.