Continuum Widom-Rowlinson model

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Definition 1

Let S be a metric space embedded with the borel σ-algebra $\mathcal{S}$ (Think $S{=}\mathbb{R}^2$).
Let $M := M_S$ be the space of locally finite subsets of S.
M is equipped with the smallest σ-algebra $\mathcal{M}$ for which the counting variables

$$\gamma \in M \mapsto \#(\gamma \cap \Delta)$$

are measurable for all bounded subset Δ of S.

Definition 2

Let m be a positive measure in S. A Poisson point process of intensity measure m is a measurable function $\mathbf{X}$ from a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ to $(M, \mathcal{M})$ which satisfies : for all integer n, for all $\Delta_1, \ldots, \Delta_n$ disjoint bounded subsets of S and for all integers $k_1, \ldots, k_n$, we have

$$\mathbb{P}(\#(\mathbf{X} \cap \Delta_1) = k_1, .., \#(\mathbf{X} \cap \Delta_n) = k_n) = \prod_{i \in \{1,..,n\}} e^{-m(\Delta_i)} \frac{m(\Delta_i)^{k_i}}{k_i!}.$$

Simulation of a Poisson Point Process in a region Δ :

- Choose the number of points according to a Poisson distribution of mean $m(\Delta)$.
- Given the number of points, each point is independently distributed by the probability law $\frac{m(.\cap\Delta)}{m(\Delta)}$.

Notations

- $S = \mathbb{R}^2 \times \mathbb{R}^+ \times \{1, 2\}$. A element of this space will be seen has a colored ball.

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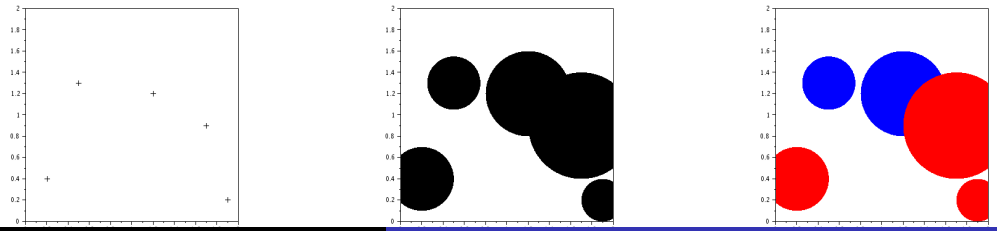
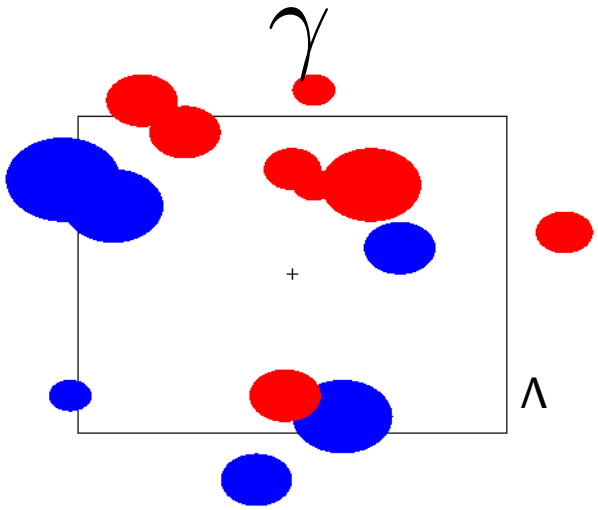
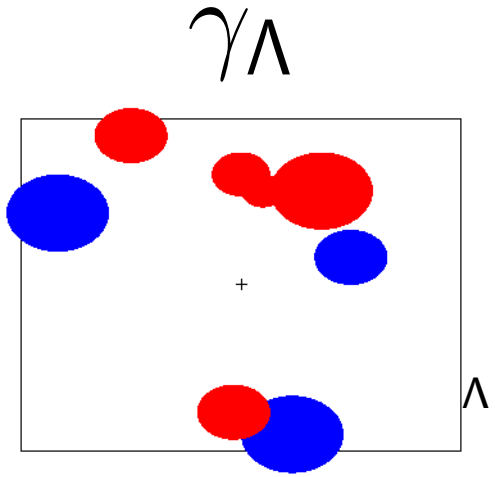
- $m = z \times$ lebesgue $\otimes\, Q \otimes \mathcal{U}(\{1, 2\})$, where $z > 0$, Q is a probability law on $\mathbb{R}^+$ and $\mathcal{U}(\{1, 2\}) = \frac{1}{2}(\delta_1 + \delta_2)$.
  Simulation of poisson point process in the bounded box $\Lambda \subseteq \mathbb{R}^2$:
  1. Choose the number of points $N \sim \mathcal{P}(z|\Lambda|)$.
  2. Given the number of points, each are uniformly and independently distributed in Λ.
  3. In each point, choose a random radius independently and according to Q.
  4. In each point, choose a color independently and uniformly over the 2 colors.



- For a subset Λ of $\mathbb{R}^2$ and a element γ of M, γ_Λ denote the element of M which contain all the balls of γ centered inside Λ.





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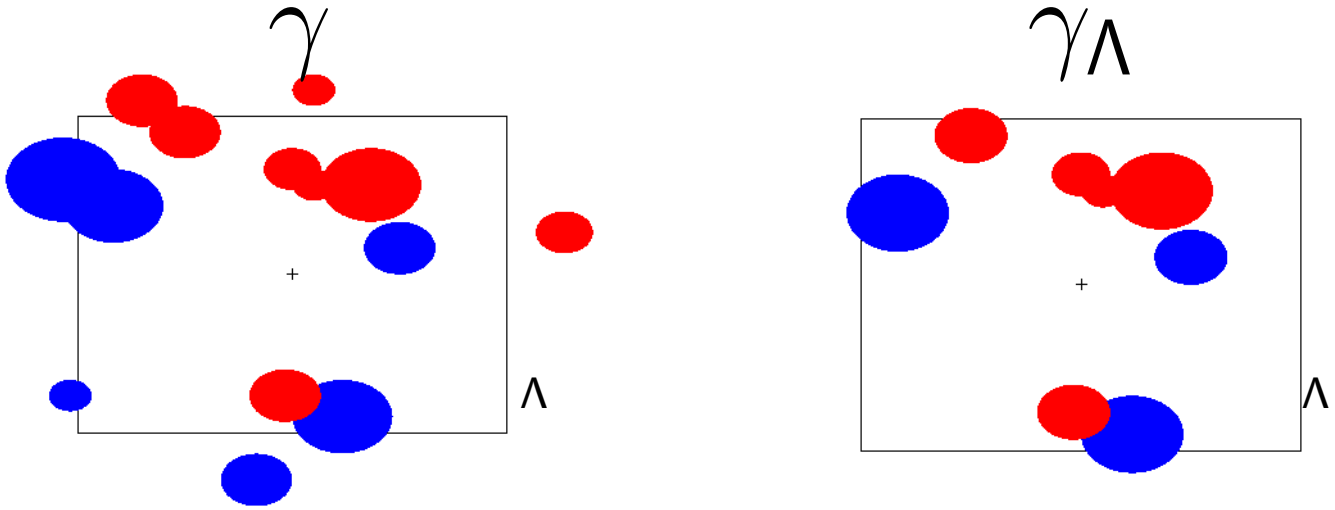

- We will denote by $\pi^{z,Q}$ the probability law on $(M, \mathcal{M})$ of a Poisson point process of intensity measure m.
- $\pi_\Lambda^{z,Q}$ will denote the projection of $\pi^{z,Q}$ in $\Lambda \times \mathbb{R}^+ \times \{1,2\}$, i.e. $\int_M f(\gamma) d\pi_\Lambda^{z,Q}(\gamma) = \int_M f(\gamma_\Lambda) d\pi^{z,Q}(\gamma)$. (We only look at the balls centred inside Λ)

We want to construct a probability measure on M for which balls of different color can't intersect. So we define

$$\mathcal{A} = \{\gamma \in M, \forall (x, R, k), (x', R', k') \in \gamma,\ k \neq k' \Rightarrow ||x - x'|| > R + R'\}$$

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Proposition 3

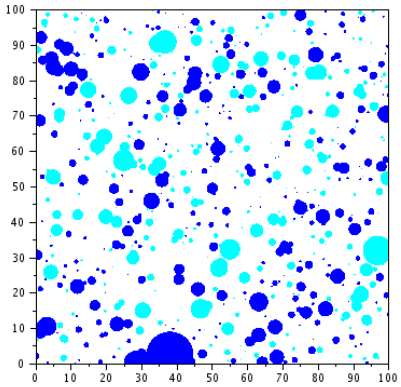
- $\pi^{z,Q}(\mathcal{A}) = 0$, *so we can't conditioned* $\pi^{z,Q}$ *by the event* $\mathcal{A}$.
- *for all bounded set* Λ *of* $\mathbb{R}^2$, $\pi^{z,Q}(\mathcal{A}) > 0$.

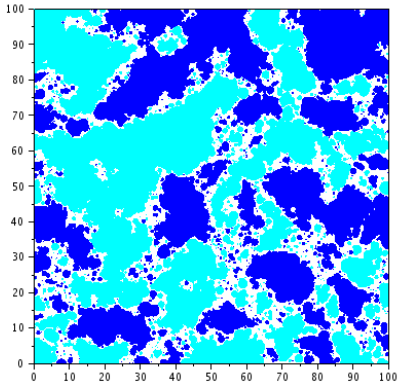
Definition 4

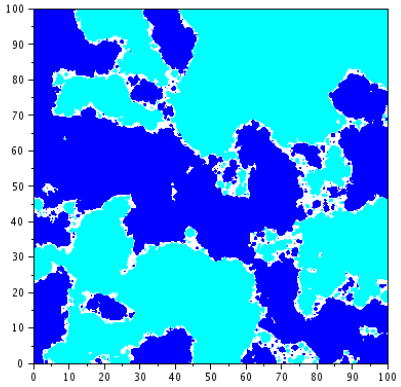
The widom-Rowlinson measure of parameter z, Q on the bounded region Λ is

$$P_\Lambda^{z,Q}(d\gamma) = \frac{I_{\mathcal{A}}(\gamma)}{Z_\Lambda}\pi_\Lambda^{z,Q},$$

where $I_{\mathcal{A}}$ is the indicator fonction of the set $\mathcal{A}$ and $Z_\Lambda = \pi_\Lambda^{z,Q}(\mathcal{A})$ is the normalizing constant.







Definition 5

A probability measure μ on M is called an (infinite volume) Widom-Rowlinson measure of parameters z and Q if it satisfies $\mu(\mathcal{A}) = 1$ and the following DLR (Dobrushin, Landford, Ruelle) equations : For all bounded region Λ of $\mathbb{R}^2$ and for all bounded function f

$$\int_M f d\mu = \int_M \int_M f(\gamma'_\Lambda + \gamma_{\Lambda^c}) \frac{I_{\mathcal{A}}(\gamma'_\Lambda + \gamma_{\Lambda^c})}{Z(\gamma_{\Lambda^c})} \pi_\Lambda^{z,Q}(d\gamma')\mu(d\gamma).$$

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Theorem 6 (H-Dereudre)

- *If $\int r^2 Q(dr) < \infty$, then for all positive z there exist a Widom-Rowlinson measure of parameter z, Q.*
- *In the case $\int r^2 Q(dr) = \infty$, there exist a critical value $z_c > 0$ such that, for all $0 < z < z_c$, there exist a Widom-Rowlinson measure of parameters z, Q.*

Step 1 : find a good candidate

- Let Λ_n be the region $]-n,n]^2$
- $\mu_n(d\gamma) := \mu_n^{z,Q}(d\gamma) = \frac{I_{\mathcal{A}}(\gamma)}{Z_n}\pi_{\Lambda_n}^{z,Q}(d\gamma) = P_{\Lambda_n}^{z,Q}(d\gamma)$

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- Let $\tilde{\mu}_n$ be the probability measure on M such that the configuration on each box $\Lambda_n + x$, $x \in 2n\mathbb{Z}^2$ are independent and identically distribute according to μ_n.
- Finally $\bar{\mu}_n(d\gamma) = \frac{1}{\#(\mathbb{Z}^2\cap\Lambda_n)} \times \sum_{i\in\mathbb{Z}^2\cap\Lambda_n} \tilde{\mu}_n \circ v_i^{-1}(d\gamma)$, where v_i is the translation of the vector i.

Using the good tools, we will show that the sequence $(\bar{\mu}_n)$ have a cluster point, for a good topology.

Definition 7

Let $\mathcal{P}$ denote the set of probability measures on M. We say that a sequence (ν_n) in $\mathcal{P}$ converge to ν for the local convergence topology if, for all local bounded function f, we have

$$\int_M f \, d\nu_n \to \int_M f \, d\nu,$$

where a local function is a function that satisfies, for a bounded Λ, $f(\gamma) = f(\gamma_\Lambda)$.

Definition 8

Let $\mathcal{P}_\theta$ denote the set of probability measure on M which are invariant for the translations $(v_x)_{x\in\mathbb{Z}^2}$.
For ν in $\mathcal{P}_\theta$, we define the specific entropy of ν,
$\mathcal{I}(\nu) = \lim\limits_{n\to\infty} \frac{1}{|\Lambda_n|}\mathcal{I}_{\Lambda_n}(\nu|\pi^{z,Q})$, *where*

$$\mathcal{I}_{\Lambda_n}(\nu|\pi^{z,Q}) = \begin{cases} \int_M f\ln(f)d\pi^{z,Q}_{\Lambda_n} & \text{if} \quad \nu_{\Lambda_n} \ll \pi^{z,Q}_{\Lambda_n},\ f = \frac{d\nu_{\Lambda_n}}{d\pi^{z,Q}_{\Lambda_n}} \\ +\infty & \text{sinon} \end{cases}$$

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Proposition 9 (Georgii 1988)

- *The specific entropy is affine, ie*
  $\mathcal{I}(\alpha\nu + (1-\alpha)\tilde{\nu}) = \alpha\mathcal{I}(\nu) + (1-\alpha)\mathcal{I}(\tilde{\nu})$
- *For all C>0, the set $\{\nu \in \mathcal{P}_\theta, \mathcal{I}(\nu) \leq C\}$ is compact for the local convergence topology.*

Lemma 10

For all n, $\mathcal{I}(\bar{\mu}_n) \leq z$, So the sequence admits a cluster point $\bar{\mu}$.

Lemma 11

- $\bar{\mu} \in \mathcal{P}_\theta$.
- $\bar{\mu}(\mathcal{A}) = 1$.

We want to show that, for a given bounded Λ and for a given bounded function, we have :

$$\int_M f \; d\bar{\mu} = \int_M \int_M f(\gamma'_\Lambda + \gamma_{\Lambda^c}) \frac{I_{\mathcal{A}}(\gamma'_\Lambda + \gamma_{\Lambda^c})}{Z(\gamma_{\Lambda^c})} \pi_\Lambda^{z,Q}(d\gamma')\bar{\mu}(d\gamma).$$

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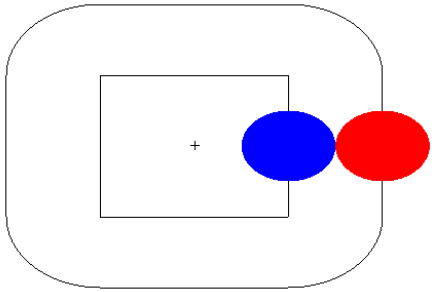
Because of the σ-field structure, we can restrict ourself to the case of f being local.

Case 1 : Bounded radii

In the case of radii being bounded by a constant C>0, and because we have $\bar{\mu}(\mathcal{A}) = 1$, we have

$$I_{\mathcal{A}}(\gamma'_\Lambda + \gamma_{\Lambda^c}) = I_{\mathcal{A}}(\gamma'_\Lambda + \gamma_{\Lambda \oplus B(0,C)\setminus\Lambda}) \text{ for } \bar{\mu} - \text{almost every } \gamma.$$



So we can directly use the local convergence of $\bar{\mu}_n$ to $\bar{\mu}$.

Case 2 : $\int r^2 Q(dr) < \infty$

- Using stochastic domination methods, we can show that $\bar{\mu}(\mathcal{W}) = 1$ where $\mathcal{W} = \bigcup_{k \in \mathbb{N}} \mathcal{W}_k$ and

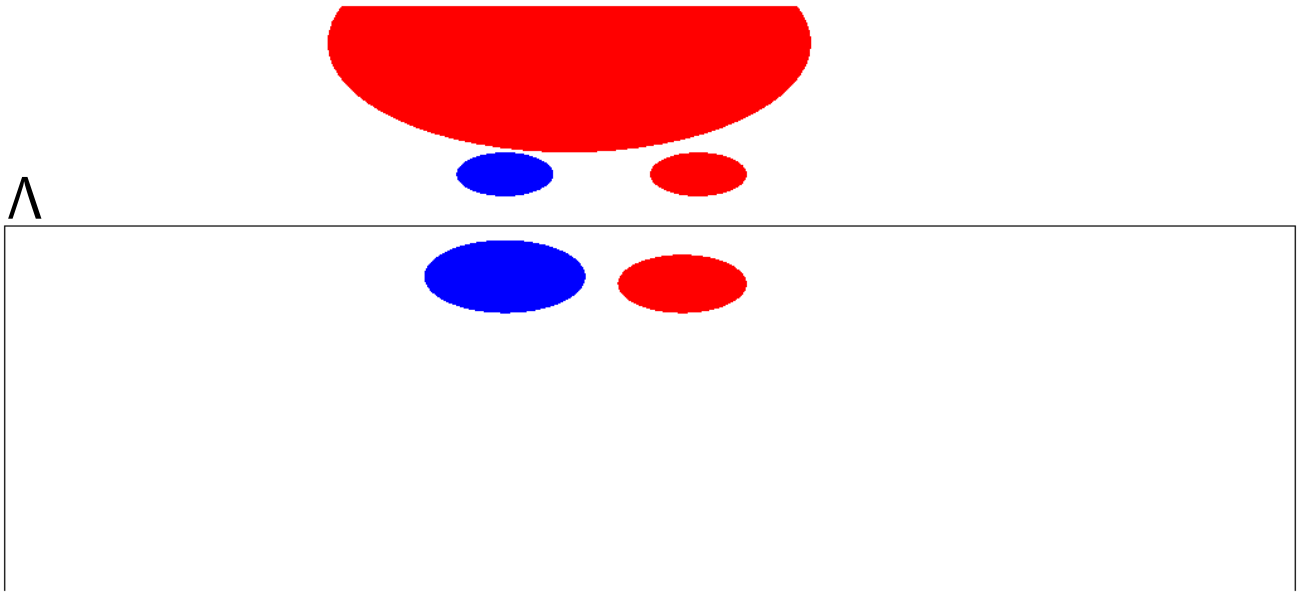
$$\mathcal{W}_k = \{\gamma \in \mathcal{M},\ \forall (x, R, k) \in \gamma,\ R \leq \frac{|x|}{2} + k\}.$$

- So we can use the same idea as before to show the DLR equation.

Case 3 : $\int r^2 Q(dr) = \infty$

In this case the stochastic domination methods don't apply any more.

Idea : To "make local" the term $I_{\mathcal{A}}(\gamma'_\Lambda + \gamma_{\Lambda^c})$, we want to protect de balls in γ'_Λ by putting a "shield" which will prevent the balls far to intersect the one in Λ.



For this shield to work, we need to have

$$\bar{\mu}\Big(\{\gamma \in M, \gamma \text{ got at least a ball of each color}\}\Big) = 1.$$

Problem : I don't know if that's true !
Idea : Show that this probability is positive and conditioned on this event.

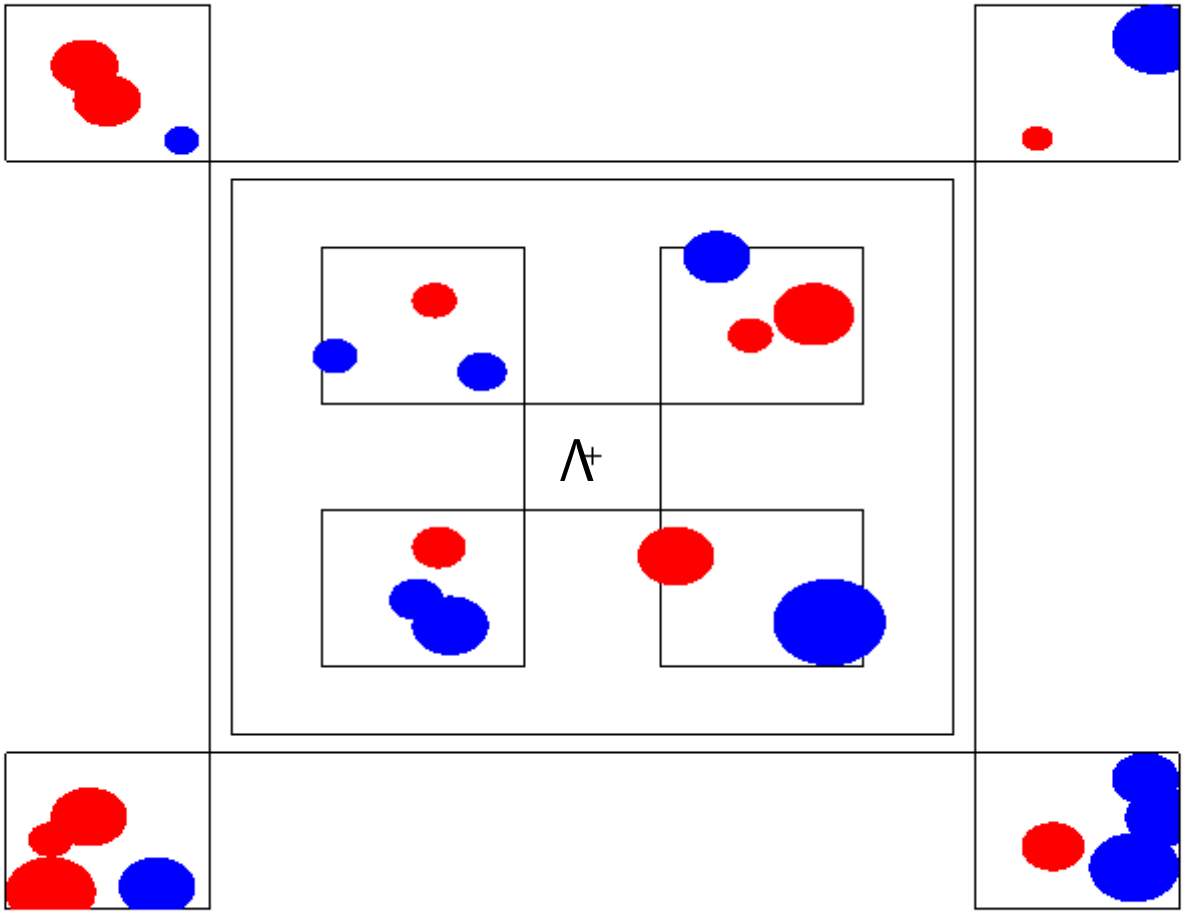
Comparing the specific entropy of $\bar{\mu}$ to the entropy of probabilities giving "monochromatic" configurations, we get

Proposition 12

There is a $z_c > 0$ such that, for all $z < z_c$, we have

$$\bar{\mu}\Big(\{\gamma \in M, \gamma \text{ got at least a ball of each color}\}\Big) > 0.$$

So by conditioning on this event, we get a probability for which every configuration has at least one ball of each color.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