Simple Smiles For The Mixing Setup

Joint work with D. Sloth

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Few Words of Introduction

Analytical approximations of implied volatility have been and continue to be proposed, even for solvable models, for the need of

- Transparency
- Robustness
- Speed

In this work, we propose an approximation of the implied volatility which can be used for a large variety of well-established models and is

- Transparent, as it decomposes the /smile into meaningful quantities associated with higher-order option risks.
- Simple, fast and easy to implement.
- Quite accurate where it matters.

The Mixing Setup

The Mixing Setup

- The mixing setup is a natural generalization of the Black-Scholes model

$$S_t = S_0 \exp\left(-\frac{1}{2}\sigma^2 t + W_{\sigma^2 t}\right).$$

obtained by randomizing the spot S_0 and the total variance $\sigma^2 t$ via their stochastic counterparts.

- The risk-neutral dynamics of the asset price are given by

$$S_t = S_t^{eff} \exp\left(-\frac{1}{2}V_t^{eff} + W_{V_t^{eff}}\right), \qquad V_0^{eff} = 0,\ S_0^{eff} = S_0,$$

where S^{eff} is a positive martingale, V^{eff} is an increasing process and W is a Brownian motion, independent of (S^{eff}, V^{eff}).

- The price process S is a conditionally log-normal martingale.

Embedded subclasses: SV models

- The mixing setup contains stochastic volatility models of the following type

$$\begin{aligned} dS_t &= S_t\sqrt{v_t}\left(\sqrt{1-\rho^2}dW + \rho dB\right) \\ dv_t &= \mu(v_t,t)dt + \sigma(v_t,t)dB_t, \qquad v_0 > 0, \end{aligned}$$

  where $W \perp B$ are Brownian motions and ρ is the correlation parameter.

- The Heston model, the 3/2 model, and the quadratic class specification are examples of solvable specifications.

- The mixing representation follows by setting

$$\begin{aligned} V_t^{eff} &= (1-\rho^2)\int_0^t v_s ds, \\ S_t^{eff} &= S_0 \exp\left(-\frac{\rho^2}{2}\int_0^t v_s ds + \rho\int_0^t \sqrt{v_s}dB_s\right), \end{aligned}$$

Purely jumping models

- Purely jumping models are obtained by setting V^{eff} as an increasing and purely jumping semimartingale.

- The price dynamics are then specified as

$$\begin{aligned} S_t &= S_t^{eff} \exp\left(-\tfrac{1}{2} V_t^{eff} + W_{V_t^{eff}}\right), \\ S_t^{eff} &= S_0 \exp\left(-K_t(c) + c V_t^{eff}\right), \end{aligned}$$

  where the real parameter c allows for correlation between S and V^{eff}, and $K(c)$ is the cumulant exponent process.

- Exponential Lévy models are obtained by modeling V^{eff} as a drift-less Lévy subordinator.

- Models of this class include, for instance, the VG model, the NIG model, and the CGMY model.

The Two Series Expansions

The $\langle S, V \rangle$-expansion

- By conditional log-normality, the price $C(S_0, K, \tau)$ of a call with strike K and expiry $\tau > 0$ admits the mixing representation

$$C(S_0, K, \tau) = \mathbb{E}[(S_\tau - K)^+] = \mathbb{E}[C_{BS}(S_\tau^{eff}, V_\tau^{eff})],$$

where $C_{BS}(S, V)$ denotes the BS call-price in terms of total variance $V = \sigma\tau$.

- First, we apply a 2-dimensional Taylor series expansion around the point $(\mathbb{E}S_\tau^{eff}, \mathbb{E}V_\tau^{eff}) = (S_0, \mathbb{E}V_\tau^{eff})$

$$\begin{aligned}\mathbb{E}[C_{BS}(S_\tau^{eff}, V_\tau^{eff})] \;=\; & C_{BS}(S_0, \mathbb{E}V_\tau^{eff}) \;+\; \frac{1}{2!}\mathbb{E}[(S_\tau^{eff} - S_0)^2]\frac{\partial^2 C_{BS}}{\partial S^2} \\ & + \frac{1}{2!}\mathbb{E}[(V_\tau^{eff} - \mathbb{E}V_\tau^{eff})^2]\frac{\partial^2 C_{BS}}{\partial V^2} \\ & + \mathbb{E}[(S_\tau^{eff} - S_0)(V_\tau^{eff} - \mathbb{E}V_\tau^{eff})]\frac{\partial^2 C_{BS}}{\partial S \partial V} \;+\; \cdots\end{aligned}$$

The $\langle\Sigma\rangle$-expansion

- Next, recall that by definition

$$C(S_0, K, \tau) \equiv C_{BS}(S_0, \Sigma).$$

where $\Sigma = \tau I^2$ denotes the *implied total variance*.

- Then expand this expression in the second variable Σ around $\mathbb{E}V_\tau^{eff}$ using a one-dimensional Taylor series.

$$C(S_0, K, \tau) = C_{BS}(S_0, \mathbb{E}V_\tau^{eff}) + \left(\Sigma - \mathbb{E}V_\tau^{eff}\right) \frac{\partial C_{BS}}{\partial V} + \frac{1}{2!}\left(\Sigma - \mathbb{E}V_\tau^{eff}\right)^2 \frac{\partial^2 C_{BS}}{\partial V^2} + \cdots$$

From Call Prices to Implied Vols

Finally truncate the $\langle\Sigma\rangle$-expansion to the first order and the $\langle S, V\rangle$-expansion to the q-th order, to approximate I^2 as

$$I^2 \approx \frac{\mathbb{E}V_\tau^{eff}}{\tau} + \frac{1}{\tau}\sum_{k=2}^{q}\sum_{l=0}^{k}\frac{\mathfrak{D}_{s^l v^{k-l}}}{l!(k-l)!}\mathbb{E}[(S_\tau^{eff}-S_0)^l(V_\tau^{eff}-\mathbb{E}V_\tau^{eff})^{k-l}],$$

where $\mathfrak{D}_{s^m v^n} \equiv \left(\frac{\partial C_{BS}}{\partial V}\right)^{-1}\frac{\partial^{m+n}C_{BS}}{\partial S^m \partial V^n}$ denote the Vega-normalized Black-Scholes derivatives.

- Application demands that $\mathbb{E}\left[\left(V_t^{eff}\right)^m\left(S_t^{eff}\right)^n\right]$ are easy to compute.
- This is a simple task whenever $\mathcal{L}_t^{eff}(u,w) = \mathbb{E}\left[e^{uX_t^{eff}+wV_t^{eff}}\right]$ with $X^{eff} = \log S^{eff}$, is available in (semi) closed-form. In this case

$$\mathbb{E}\left[\left(V_t^{eff}\right)^m\left(S_t^{eff}\right)^n\right] = \frac{\partial^m \mathcal{L}_t^{eff}(u,w)}{\partial w^m}\Big|_{u=n,w=0},$$

A Simple Quadratic Approximation

- The 2nd-order expansion yields a simple approximation $I_Q^2(x,\tau)$ of the implied variance which is quadratic in $x = \log K/S$.
- Specifically

$$I^2 \approx I_Q^2(x,\tau) = \frac{\mathbb{E}V_\tau^{eff}}{\tau} + \frac{\mathbb{V}ar[S_\tau^{eff}]}{2\,\tau}\,\mathfrak{D}_{ss} + \frac{\mathbb{C}ov[S_\tau^{eff}, V_\tau^{eff}]}{\tau}\,\mathfrak{D}_{sv} + \frac{\mathbb{V}ar[V_\tau^{eff}]}{2\,\tau}\,\mathfrak{D}_{vv}\,,$$

where the normalized Gamma $\mathfrak{D}_{ss}$, Vanna $\mathfrak{D}_{sv}$ and Volga $\mathfrak{D}_{vv}$ are

$$\mathfrak{D}_{ss} = \frac{2}{S^2}\,, \qquad \mathfrak{D}_{sv} = \frac{x}{SV} + \frac{1}{2S}\,, \qquad \mathfrak{D}_{vv} = \frac{x^2}{2V^2} - \frac{1}{8} - \frac{1}{2V}\,.$$

- We see that
  - the Gamma risk $\mathfrak{D}_{ss}$ only contributes to the level of smile,
  - the Vanna term $\mathfrak{D}_{sv}$ determines the slope
  - the Volga term $\mathfrak{D}_{vv}$ introduces convexity.

A first look at ATM term-structure

Re-arranging the terms we obtain

$$I_Q^2(x,\tau) = \mathfrak{I}_0(\tau) + \mathfrak{I}_1(\tau)\,x + \mathfrak{I}_2(\tau)\,x^2,$$

where $\mathfrak{I}_\cdot(\tau)$ describe the term structure of the approximate smile:

- ATM Variance:

$$\mathfrak{I}_0(\tau) = \frac{\mathbb{E}V_\tau^{eff}}{\tau} + \frac{\mathbb{V}ar[S_\tau^{eff}]}{S_0^2\tau} + \frac{\mathbb{C}ov[S_\tau^{eff}, V_\tau^{eff}]}{2S_0\tau} - \frac{\mathbb{V}ar[V_\tau^{eff}]}{4\tau\mathbb{E}V_\tau^{eff}}\left(1 + \frac{1}{4}\mathbb{E}V_\tau^{eff}\right)$$

- ATM Skew:

$$\mathfrak{I}_1(\tau) = \frac{1}{\tau}\,\frac{\mathbb{C}ov[S_\tau^{eff}, V_\tau^{eff}]}{S_0\,\mathbb{E}V_\tau^{eff}}$$

- ATM Curvature:

$$\mathfrak{I}_2(\tau) = \frac{1}{\tau}\,\frac{\mathbb{V}ar[V_\tau^{eff}]}{4(\mathbb{E}V_\tau^{eff})^2}$$

Does it work?

Illustration for SV Models

Naturally, we consider the Heston (1993) model

$$dv_t = \kappa(\theta - v_t)dt + \varepsilon v_t^{1/2} dB_t.$$

We also consider the 3/2 model with the instantaneous variance

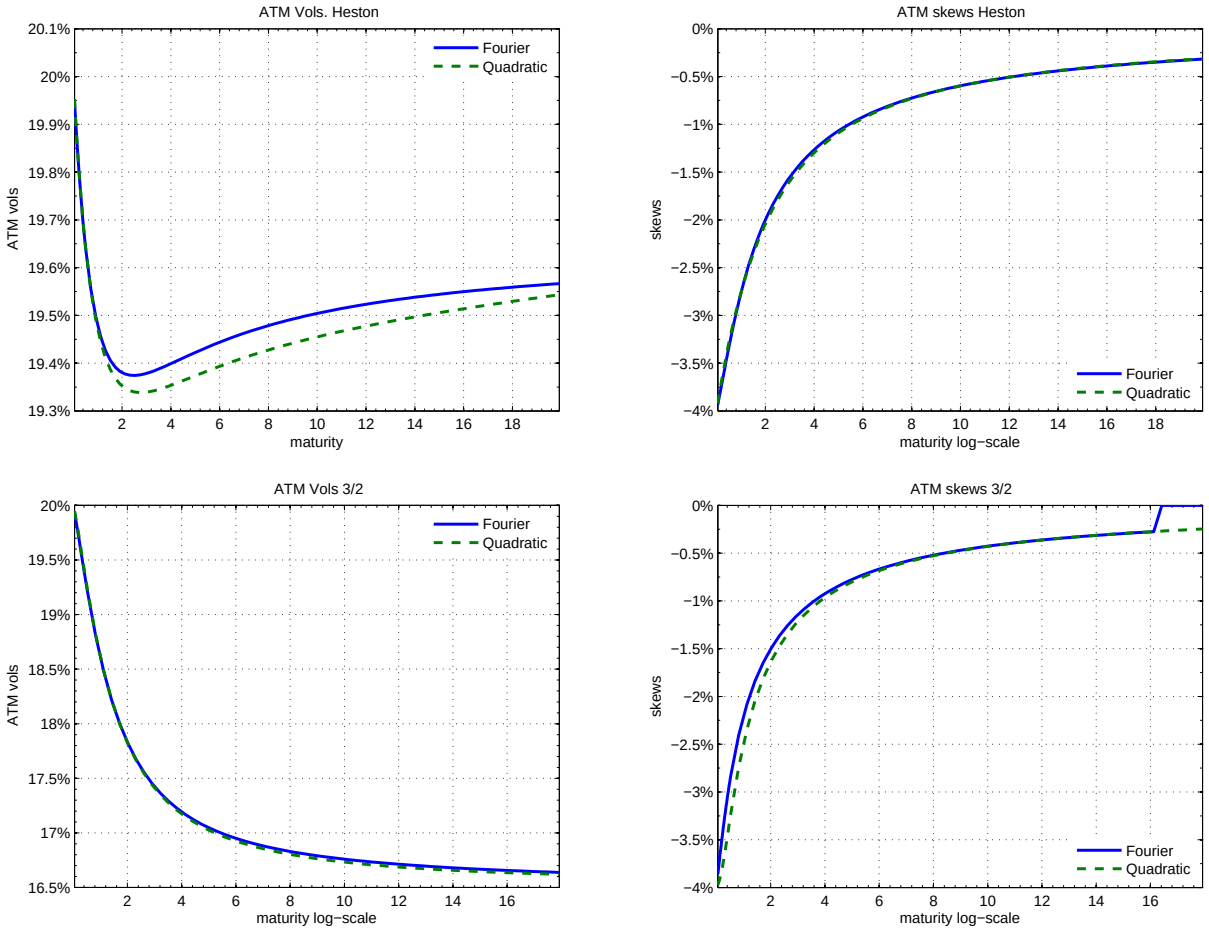
$$dv_t = v_t \kappa(\theta - v_t)dt + \varepsilon v_t^{3/2} dB_t.$$

- Both models are solvable, as the joint Laplace transform $\mathcal{L}_t^{XV}$ of $X = \log S$ and $V = \int_0^{\cdot} v_s ds$ has closed-form.
- Also the relevant moments are computable, since it holds that

$$\mathcal{L}_t^{eff}(u, w) = \mathcal{L}_t^{XV}\left(u\,, (1-\rho^2)(w + \frac{1}{2}u - \frac{1}{2}u^2)\right)\,,$$

  between the "standard" and the "effective" transforms.
- However, Fourier inversion is numerically not trivial in the 3/2 model, due to complex evaluations of the confluent hypergeometric function.

ATM Implied Vols and Skews



ATM vols (left) and skews (right) for the Heston (top) and the 3/2 (bottom). The maturity ranges from $\tau = 0.05$ up to $\tau = 18$ years. Parameters are as in Forde et al. (2012).

SV Models: The Smile at Short Maturities

The ATM accuracy of $I_Q^2(x,\tau)$ at short maturities is not coincidental.

- For models with time-independent coefficients

$$dv_t = a(v_t)dt + b(v_t)dB_t.$$

  it holds that

$$\lim_{\tau\to 0} I^2(0,\tau) = v_0 \qquad \text{and} \qquad \lim_{\tau\to 0} \frac{\partial I^2}{\partial x^2}\Big|_{x=0} = \frac{1}{2}\frac{\rho b(v_0)}{\sqrt{v_0}}$$

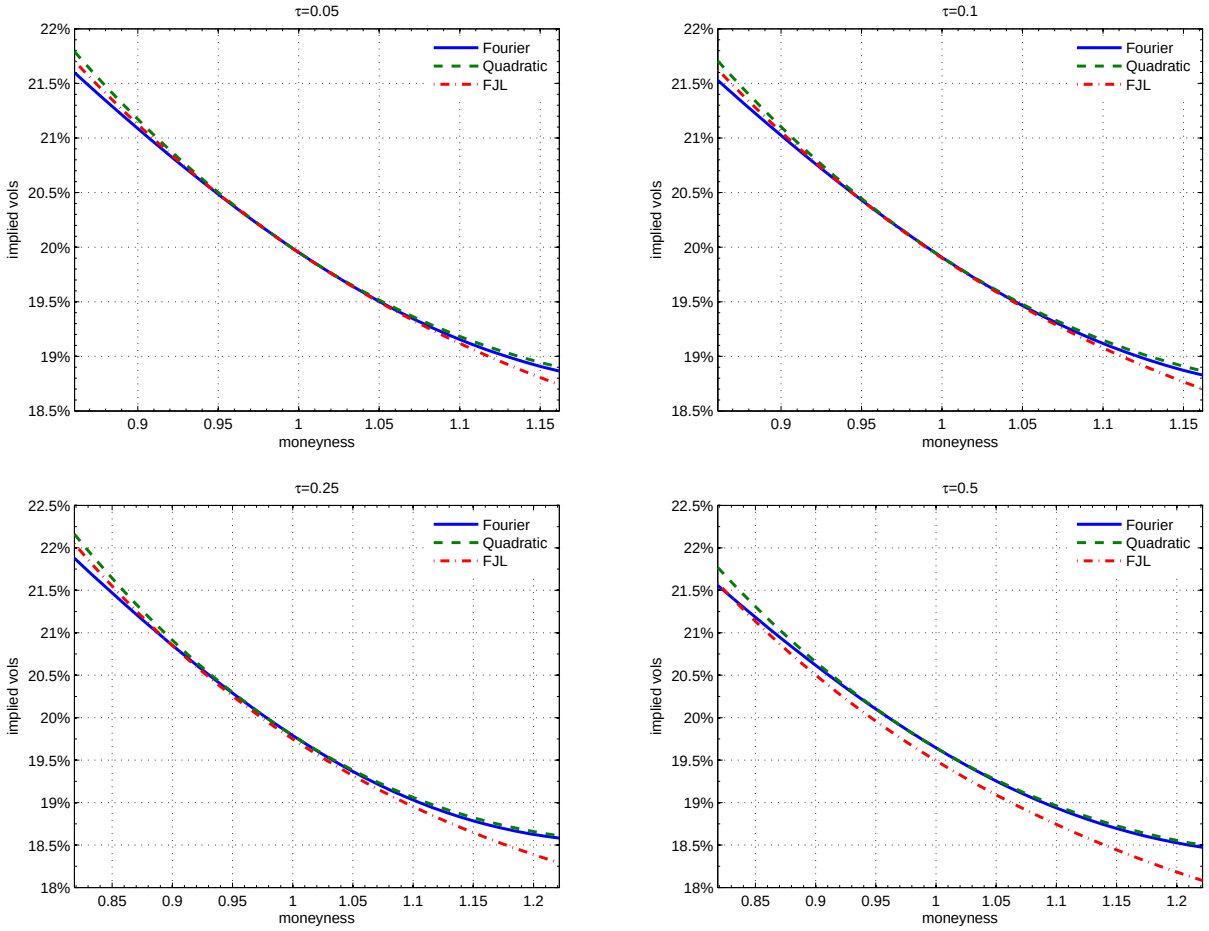
  see e.g., Lewis (2000), Lee (2001) and Medvedev and Scaillet (2007).

- The quadratic approximation is consistent with these results

$$\lim_{\tau\to 0} \mathfrak{I}_0(\tau) = v_0 \qquad \text{and} \qquad \lim_{\tau\to 0} \mathfrak{I}_1(\tau) = \frac{1}{2}\frac{\rho b(v_0)}{\sqrt{v_0}}$$

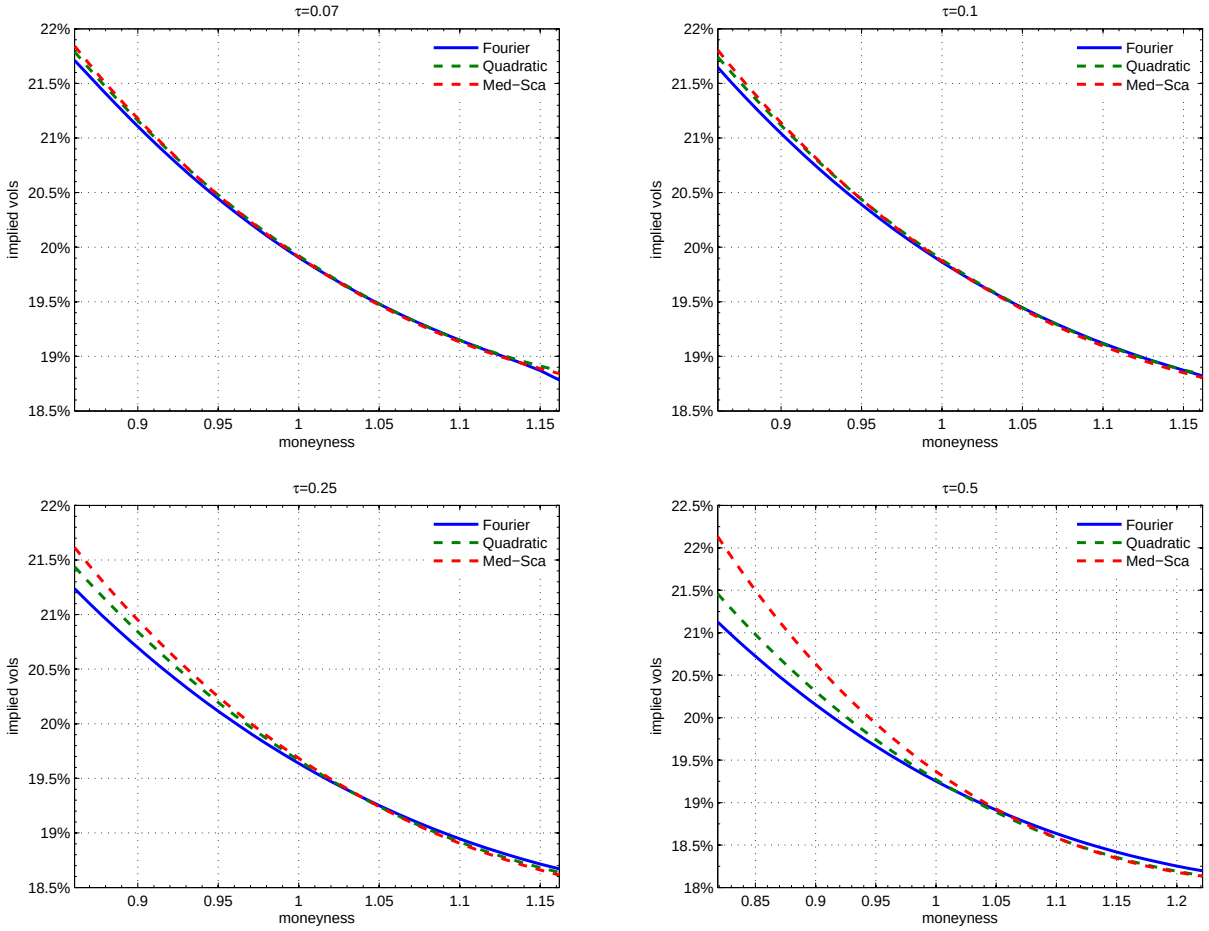
- It is therefore tempting to compare $I_Q^2(x,\tau)$ with well-established asymptotic results.

Heston model at short maturities



- I_Q^2 vs Forde et al. (2012), for Heston model at short maturities.
- Maturities: $\tau = 0.05$ to $\tau = 0.5$. Moneyness: $\pm 15\%$.

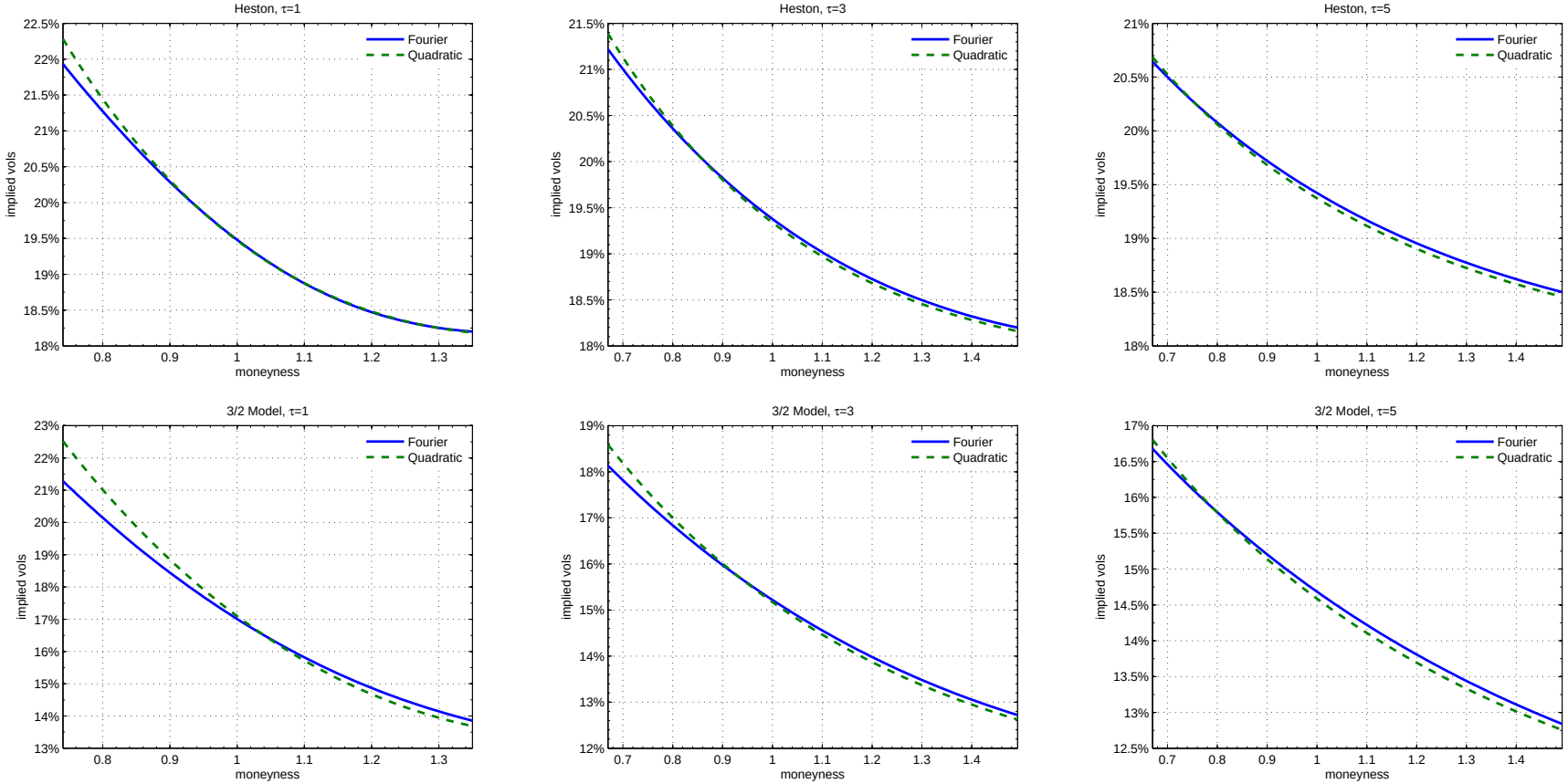
3/2 Model at Short Maturities



I_Q^2 vs Medvedev and Scaillet (2007), for general SV (plus jumps), short-maturities/small strikes. Maturities: $\tau = 0.07$ to $\tau = 0.5$. Moneyness: $\pm 15\%$.

Mid-Long Maturities

- Top: Heston model. Bottom: 3/2 model
- Maturities $\tau = 1, 3, 5$. Moneyness: from $\pm 30\%$ to $\pm 40\%$.



SV models: The smile at long maturities

- At large maturities, the asymptotic behavior is

$$\lim_{\tau\to\infty} I^2(x,\tau) = 8\lambda(k_0) \quad \text{and} \quad \left.\frac{\partial I^2}{\partial x}\right|_{x=0} \approx -\frac{8ik_0+4}{\tau} + O(1/\tau^2) \text{ as } \tau \to \infty$$

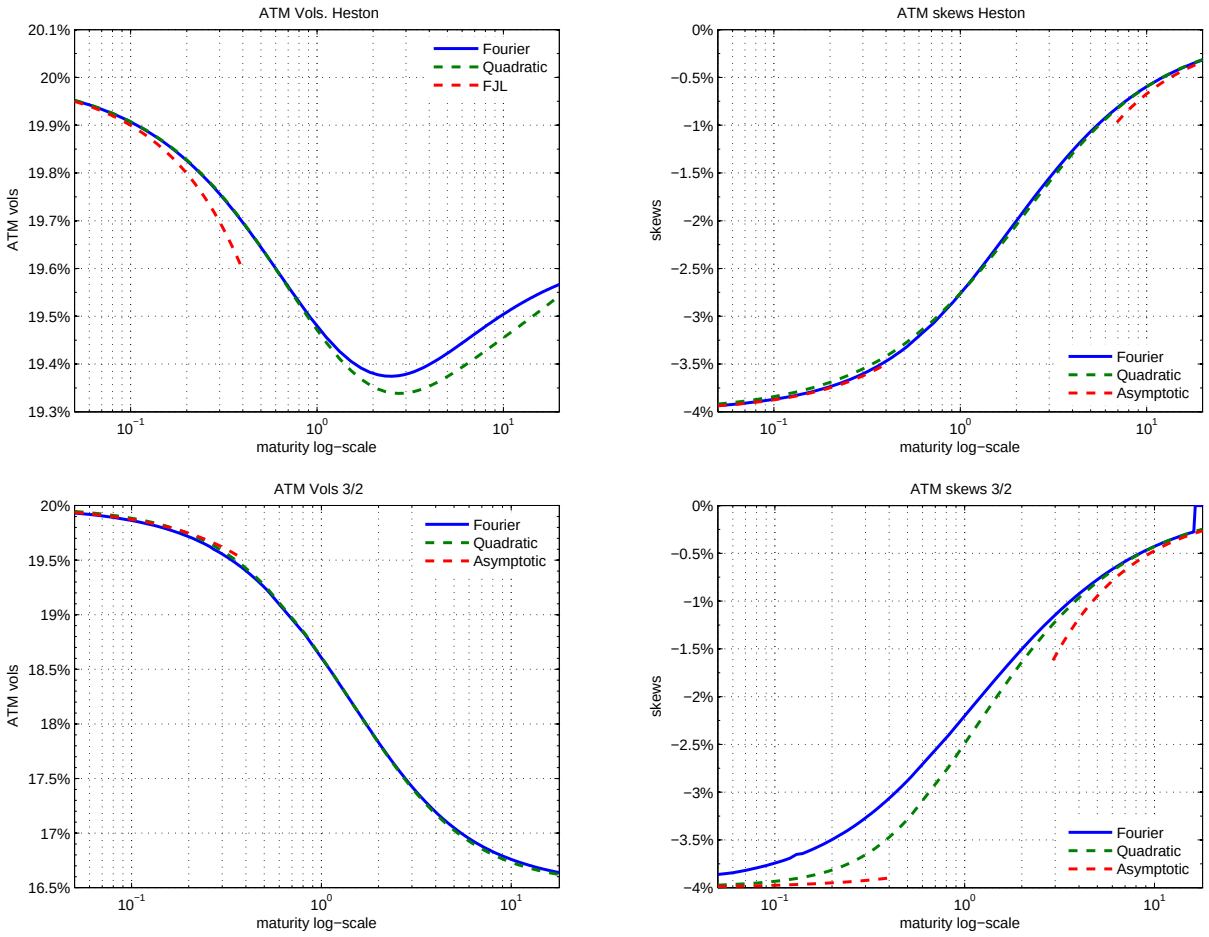
  where λ is the first eigenvalue of a differential operator and k_0 is a complex number. See Lewis (2000), Jaquier (2007) and Tehranchi (2009).

- For I_Q^2, if $\rho \neq 0$, it holds that

$$\lim_{\tau\to\infty} \mathfrak{I}_0(\tau) = \infty \qquad \lim_{\tau\to\infty} \mathfrak{I}_1(\tau) = 0 \qquad \lim_{\tau\to\infty} \mathfrak{I}_2(\tau) = 0$$

- So the accuracy of the quadratic approximation is bound to deteriorate as the maturity increases.
- Luckily, this happens at a very slow rate, and the mismatch becomes observable only at extremely long expiries.

ATM Implied Vols and Skews - Revisited



ATM Vols (left) and Skews (right), for Heston (top) and 3/2 (bottom), but with added ATM asymptotic behaviors. Maturity from $\tau = 0.05$ to $\tau = 18$ years.

Exponential Lévy models

Exponential Lévy Models

- True short-maturity behavior

$$\lim_{\tau\to 0} I^2(0,\tau) = 0 \quad \text{and} \quad \lim_{\tau\to 0} I^2(x,\tau) = +\infty, \quad \text{for } x \neq 0$$

- True long-maturity behavior

$$\lim_{\tau\to\infty} I^2(x,\tau) = A \quad \text{and} \quad I^2(x,\tau) \approx A + \frac{B}{\tau} + \frac{C}{\tau}x \quad \text{for large } \tau$$

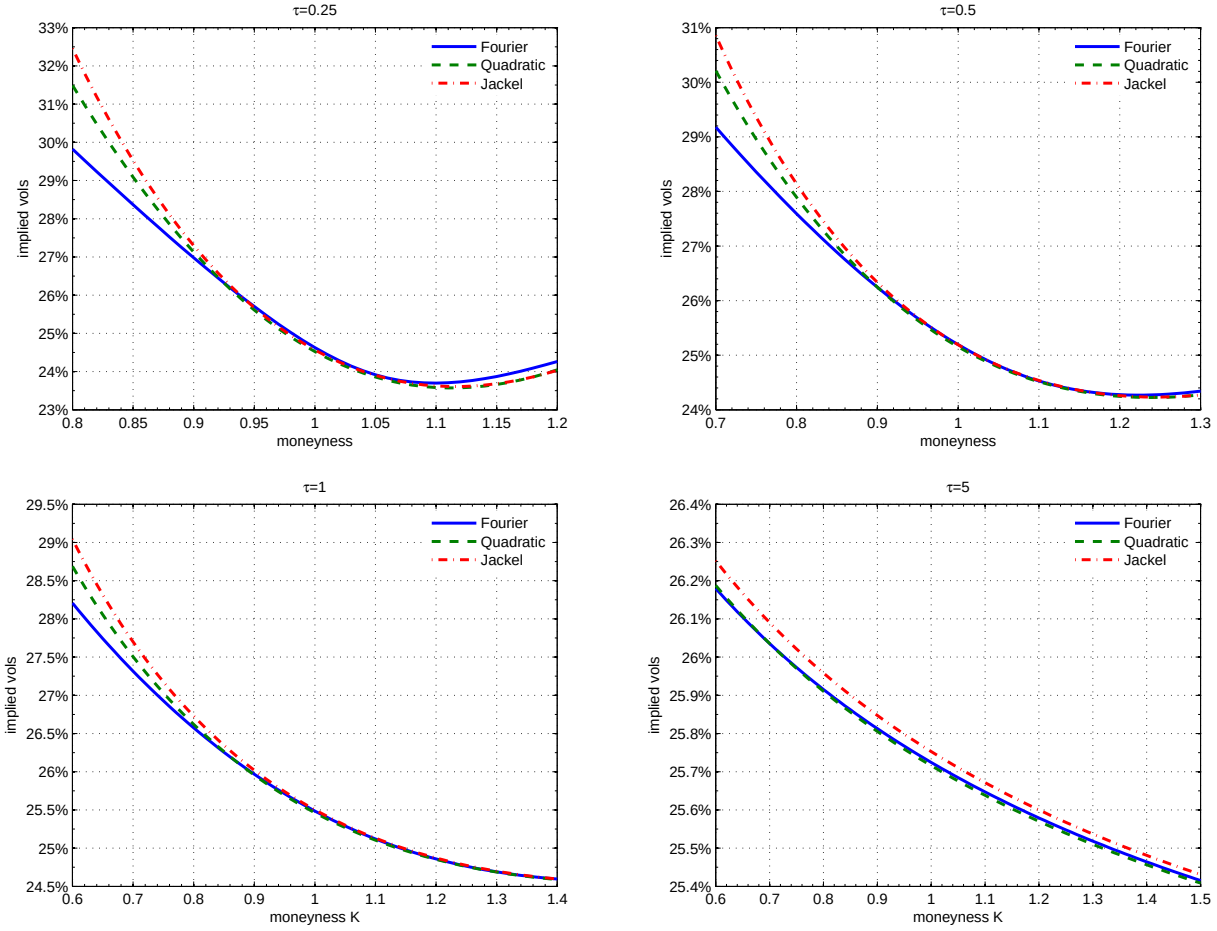
- In case of exponential Lévy models, $I_Q^2(x,\tau)$ takes the form

$$I_Q^2(x,\tau) = \mathcal{A}(\tau) + \frac{\mathcal{B}}{\tau} + \frac{\mathcal{C}}{\tau}x + \frac{\mathcal{D}}{\tau^2}x^2,$$

with $\mathcal{B} < 0$, $\lim_{\tau\to 0}\mathcal{A}(\tau) = \widetilde{\mathcal{A}}$ and $\lim_{\tau\to\infty}\mathcal{A}(\tau) = \infty$ (unless $c = 0$).

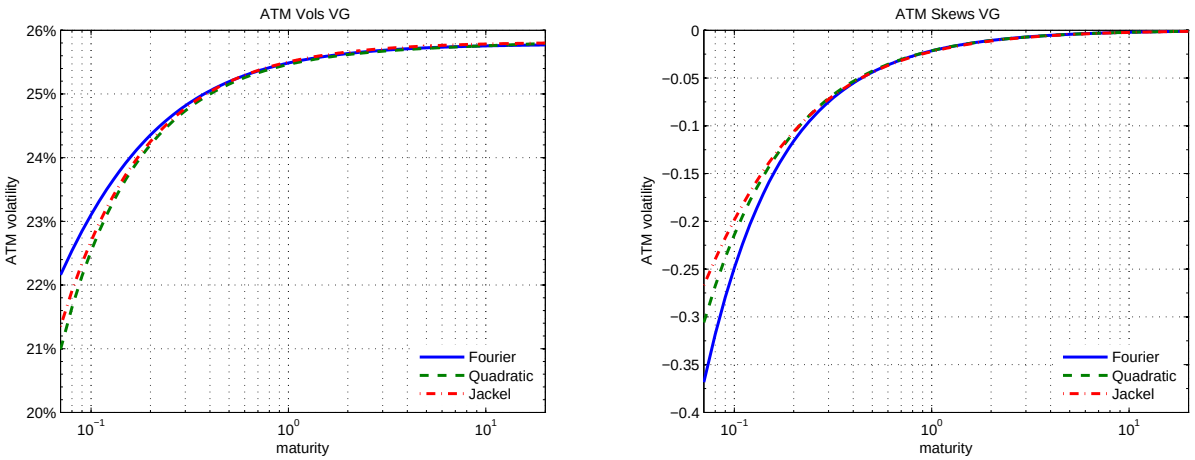
- In spite of this "anti-asymptotic" behavior, $I_Q^2(x,\tau)$ is nevertheless quite useful.

VG Model smiles



$I_Q^2(x,\tau)$ vs Jäckel (2009) singular approximation. Maturity: $\tau = 0.25$ up to $\tau = 5$. Moneyness: $\pm 30\%$ up to $\pm 50\%$. Jäckel (2009) parameters.

VG Model Term Structure



While both approximations are singular around the expiry-date, they both capture the overall behavior of the surface for a large relevant region of the smile.

Concluding Remarks

- The approximation is simple, easy to implement, and quite accurate where it matters (i.e. liquid moneyness).
- The approximation decomposes the volatility smile into meaningful quantities associated with higher-order option risks.
- The approximation is largely generic in the sense that it may be used for a large variety of option pricing models.
- Finally, in the paper we explore two domains of application of the approximation.
  1. We suggest to use the approximation as a control variate in Fourier option pricing.
  2. We propose a 'speedy', approximation-based approach for model calibration to at-the-money volatilities and skews.