Approximation of the invariant measure of an IFS

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Approximation of the invariant measure of an IFS

1. Iterated Function System (IFS)
2. The transfer operator $T$ for an IFS
3. Eigenfunctions of $T$ for affine IFS
4. Approximation for the invariant measure
1. Iterated Function System (IFS)
IFS

Let $X = \mathbb{R}^d$ or $\mathbb{C}^d$.

An **IFS on** $X$ consists of

$$f_1, \ldots, f_N$$ mappings $X \rightarrow X$

and a corresponding

$$\left( p_1, \ldots, p_N \right)$$ probability vector

Assume there is a non-empty **compact set** $K \subset X$ such that

$$f_i(K) \subset K$$ for all $i$. 
IFS as a stochastic dynamical system

\( f_1, \ldots, f_N \) mappings \( X \rightarrow X \)

(\( p_1, \ldots, p_N \)) probability vector

Start at a point \( x_0 \in X \). Given a point \( x_n \), choose a function \( f \) according to \( \mathbb{P}(f = f_i) = p_i \) and set

\[ x_{n+1} = f(x_n) \]

random trajectory in \( X \)

\( x_0, x_1, x_2, \ldots \)
IFS – right-angled triangle

$D$ right-angled triangle

mappings $D_1 = f_1(D)$, $D_2 = f_2(D)$

with prob. $p_1 = p_2 = \frac{1}{2}$
IFS – random trajectory

Points $x_0, x_1, \ldots, x_{100000}$
IFS – right-angled triangle

D right-angled triangle
mappings $D_1 = f_1(D)$ and $D_2 = f_2(D)$

with prob. $p_1 = \frac{|D_1|}{|D|} = \frac{1}{4}$ and $p_2 = \frac{|D_2|}{|D|} = \frac{3}{4}$
IFS – random trajectory

Points $x_0, x_1, \ldots, x_{100000}$ of a random trajectory
Bernoulli IFS

IFS on $\mathbb{R}$ with mappings

\[
f_1(x) = \lambda x - 1 \quad \text{with prob.} \quad \frac{1}{2}
\]
\[
f_2(x) = \lambda x + 1 \quad \text{with prob.} \quad \frac{1}{2}
\]

parameter $\lambda \in [\frac{1}{2}, 1)$

\[
I_\lambda = \left[ -\frac{1}{1-\lambda}, \frac{1}{1-\lambda} \right]
\]

Interval $I_\lambda = [-\frac{1}{1-\lambda}, \frac{1}{1-\lambda}]$ with $I_\lambda = f_1(I_\lambda) \cup f_2(I_\lambda)$
Bernoulli IFS – trajectories

\[ f_1(x) = \lambda x - 1 \quad f_2(x) = \lambda x + 1 \]

\[ n^{th} \text{ iteration: } \pm 1 \pm \lambda \pm \lambda^2 \pm \lambda^3 \pm \ldots \pm \lambda^n x^n \]
Bernoulli IFS - random trajectory

First $10^5$ points (histogram)

$\lambda = 0.5$

$\lambda = 0.6$

$\lambda = 0.7$

$\lambda = 0.8$

$\lambda = 0.9$
2. The transfer operator $T$ for an IFS
The transfer operator $T$

$\mathcal{C}(K)$ space of continuous functions $K \to K$

$$T : \mathcal{C}(K) \to \mathcal{C}(K)$$

$$Th(x) = \sum_{i=1}^{N} p_i h(f_i(x))$$

For a trajectory with starting point $x_0 \in K$ we have

$$\mathbb{E} h(x_1) = Th(x_0)$$

$$\mathbb{E} h(x_n) = T^n h(x_0)$$
Dual operator of $T$

Let $\mathcal{M}(K)$ be the dual space of $C(K)$ i.e. the space of Borel regular measures on $K$.

The Hutchinson operator

$$H : \mathcal{M}(K) \rightarrow \mathcal{M}(K)$$

$$H\mu = \sum_{i=1}^{N} p_i \mu \circ f_i^{-1}$$

is dual to the transfer operator, i.e.:

$$(H\mu, h) = (\mu, Th) \quad \text{duality}$$
Hutchinson operator

\[ H: \mathcal{M}(K) \to \mathcal{M}(K) \]

\[ H\mu(A) = \sum_{i=1}^{N} p_i \mu(f_i^{-1}(A)) \]

Start with a distribution \( \mu_0 \) auf \( K \). The mass in \( A \subset K \) after one step comes with prob. \( p_i \) from the set \( f_i^{-1}(A) \), so that

\[ \mu_1(A) = H \mu_0(A) \]

Distribution after \( n \) steps:

\[ \mu_n = H^n \mu_0 \]
Implications of duality

**Spectra:**
The transfer operator $T$ and the Hutchinson operator $H$ have the **same spectra**.

**Invariant measure of the IFS:**
$\nu = H\nu$ is **orthogonal to all eigenfunctions of $T$** with eigenvalue $\neq 1$. 
Bernoulli IFS - Hutchinson Operator

$$\lambda = 0.9$$
Bernoulli IFS - Hutchinson Operator

\[ \lambda = 0.8 \]
Bernoulli IFS - Hutchinson Operator

$\lambda = 0.7$

$\nu_0$

$\nu_1$

$\nu_2$

$\nu_3$

$\nu_4$

$\nu_{10}$
Bernoulli IFS - Hutchinson Operator

\( \lambda = 0.6 \)
Bernoulli IFS - Hutchinson Operator

\[ \lambda = 0.55 \]
Bernoulli convolution problem

\( \nu \) distribution of the random sum \( \sum_{n=0}^{\infty} \pm \lambda^n \)

Does \( \nu \) have a density?

\( \lambda = 0.55 \)

\( \lambda = 0.618 \)

\( \lambda = 0.68 \)

\( \lambda = 0.80 \)
Results on Bernoulli convolutions

\( \nu \) distribution of the random sum \( \sum_{n=0}^{\infty} \pm \lambda^n \)

**Jessen, Winter 1935**: \( \nu \) is either absolutely continuous or purely singular with respect to the Lebesgue measure.
Results on Bernoulli convolutions

\[ S = \{ \lambda \in \left[ \frac{1}{2}, 1 \right) : \nu \text{ singular} \} \]

**Erdös 1939:** countably many examples in \( S \).

**Garsia 1962:** countably many examples in \( S \cap \left[ \frac{1}{2}, 1 \right) \).

**Solomyak 1995:** \( S \) has Lebesgue measure 0.

**Shmerkin 2013:** \( S \) has Hausdorff dimension 0.
3. Eigenfunctions of $T$ for affine IFS
Transfer operator for an IFS on $K \subset \mathbb{K}^d$ with affine contractions $f_i(x) = A_i x + v_i$

$$T : C(K) \rightarrow C(K)$$

$$Th(x) = \sum_{i=1}^{N} p_i h(A_i x + v_i)$$

with matrix $A_i$ and translation vector $v_i$. 
Let \( \mathcal{P}_n(K) \) be the space of polynomials \( p : K \rightarrow K \) of degree \( \leq n \). Then,

\[
T : \mathcal{P}_n(K) \rightarrow \mathcal{P}_n(K)
\]

is well-defined, i.e.

\( \mathcal{P}_n(K) \) is an invariant eigenspace of \( T \).
Consider an IFS on $\mathbb{K} \subset \mathbb{K}$ with mappings

$$f_i(x) = \lambda_i x + v_i, \quad i = 1, \ldots, N$$

$$\lambda_i, v_i \in \mathbb{K}$$

The operator

$$T : \mathcal{P}_n(\mathbb{K}) \to \mathcal{P}_n(\mathbb{K})$$

is represented by a matrix $T_n \in \mathbb{K}^{(n+1) \times (n+1)}$. 
Matrix for $T_n : \mathcal{P}_n(K) \rightarrow \mathcal{P}_n(K)$

$$T_n = \begin{pmatrix}
1 & * & * & * & \ldots & * \\
0 & \sum_i p_i \lambda_i & * & * & \ldots & * \\
0 & \sum_i p_i \lambda_i & * & \ldots & * \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \ldots & 0 & \sum_i p_i \lambda_i^n
\end{pmatrix}$$

with respect to the basis $1, x, x^2, \ldots, x^n$. 
Eigenvalues of $T_n$

\begin{align*}
\omega_0 &= 1 \\
\omega_1 &= \sum_i p_i \lambda_i \\
\omega_2 &= \sum_i p_i \lambda_i^2 \\
\vdots \\
\omega_n &= \sum_i p_i \lambda_i^n
\end{align*}

$i = 1, \ldots, N$
Eigenvectors of $T_n$

The eigenvectors of $T_n$ correspond to the polynomial eigenfunctions of

$$T : \mathcal{P}_n(K) \rightarrow \mathcal{P}_n(K).$$

The constant function 1 is always an eigenfunction with eigenvalue 1, since

$$T1 = \sum_{i=1}^{N} p_i = 1.$$
Basis of eigenfunctions

**Theorem.** The transfer operator for an IFS with equal contraction factors $\lambda$

$$T : \mathcal{P}_n(K) \rightarrow \mathcal{P}_n(K)$$

$$Th(x) = \sum_{i=1}^{N} p_i h(\lambda x + v_i)$$

has eigenvalues $\lambda^k$ for $0 \leq k \leq n$ and the eigenfunctions build a basis of $\mathcal{P}_n(K)$. 
Basis of eigenfunctions

Theorem. The transfer operator

$$T : C(K) \rightarrow C(K)$$

$$Th(x) = \sum_{i=1}^{N} p_i \cdot h(\lambda x + v_i)$$

has eigenvalues $$\{\lambda^k : k \in \mathbb{N}_0\}$$ and their eigenfunctions build a basis of $$\mathcal{P}(K)$$. 
Bernoulli IFS – Transfer operator

\[ T : \mathcal{C}([-1, 1]) \rightarrow \mathcal{C}([-1, 1]) \]

\[ T h(x) = \frac{1}{2} h \left( \lambda x - 1 + \lambda \right) + \frac{1}{2} h \left( \lambda x + 1 - \lambda \right) \]

with \( \lambda \in \left[ \frac{1}{2}, 1 \right) \).
Bernoulli IFS – Transfer operator

We get the eigenpolynomials of

\[ T h(x) = \frac{1}{2} h(\lambda x - 1 + \lambda) + \frac{1}{2} h(\lambda x + 1 - \lambda) \]

of degree \( \leq 3 \) from the matrix

\[
T_3 = \begin{pmatrix}
1 & 0 & (1 - \lambda)^2 & 0 \\
0 & \lambda & 0 & 3\lambda(1 - \lambda)^2 \\
0 & 0 & \lambda^2 & 0 \\
0 & 0 & 0 & \lambda^3
\end{pmatrix}
\]

\[ q_0(x) = 1, \; q_1(x) = x, \]

\[ q_2(x) = x^2 + \frac{\lambda - 1}{\lambda + 1}, \; q_3(x) = x^3 + 3 \frac{\lambda - 1}{\lambda + 1} \]
Bernoulli IFS – eigenfunctions of $T$

Polynomial eigenfunctions $q_0, \ldots, q_5$ for $\lambda = 0.7$
Bernoulli IFS

Theorem. The transfer operator $T$ has the eigenfunctions

$$q_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} a_{n,k} x^{n-2k} \quad n \in \mathbb{N}_0$$

with eigenvalues $\omega_n = \lambda^n$. The coefficients are given recursively by

$$a_{n,k} = \frac{1}{\lambda^{2k} - 1} \sum_{j=0}^{k-1} \binom{n-2j}{n-2k} (1 - \lambda)^{2k-2j} a_{n,j}$$

for $k \geq 1$ and $a_{n,0} = 1$ else.
4. Approximation of the invariant measure
Consider an IFS on $K \subset \mathbb{K}$ with invariant measure $H\nu = \nu$.

**Assumption:** the eigenfunctions of $T$

$$q_k \in \mathcal{P}_k(K), \quad k \in \mathbb{N}_0$$

build a **basis of** $\mathcal{P}(K)$ *(this is the case for the Bernoulli IFS)*
Approximating densities $v_n$

Duality implies:

$$\nu \perp q_k \quad \text{for} \quad k = 1, 2, \ldots$$

We get a sequence of polynomial probability densities $v_n \in \mathcal{P}_n(K)$ by solving

$$v_n \perp q_k \quad \text{for} \quad 1 \leq k \leq n$$

or equivalently,

$$\langle v_n, x^k \rangle = m_k \quad \text{for} \quad 1 \leq k \leq n$$

$m_k$ is the $k^{th}$ moment of the invariant measure $\nu$. 
Linear system of equations for $v_n$

**Theorem.** The approximation $v_n(x) = \sum_{k=0}^{n} u_k x^k$ satisfies

$$G(u_0, u_1, \ldots, u_n)' = (m_0, m_1, \ldots, m_n)'$$

with the **Hilbert matrix** $G \in \mathbb{K}^{(n+1) \times (n+1)}$

$$G_{ij} = \int_{K} x^{i+j} \, dx.$$  

$m_k = \int_{K} x^k \, d\nu$ are the moments of $\nu$. 
Theorem. The approximating measures $\nu_n$

$$\nu_n(A) = \int_A v_n(x) \, dx$$

converge

$$\nu_n \to \nu \quad \text{weakly}$$

to the invariant measure of the IFS.
Bernoulli IFS – Densities

The approximation $v_n(x) = \sum_{k=0}^{n} u_k x^k$ satisfies

$$G(u_0, u_1, \ldots, u_n)' = (m_0, m_1, \ldots, m_n)'$$

with the **Hilbert matrix** $G \in \mathbb{K}^{(n+1) \times (n+1)}$.

$$G = 2 \begin{bmatrix}
1 & 0 & \frac{1}{3} & 0 & \frac{1}{5} & \cdots \\
0 & \frac{1}{3} & 0 & \frac{1}{5} & 0 & \cdots \\
\frac{1}{3} & 0 & \frac{1}{5} & 0 & \frac{1}{7} & \cdots \\
0 & \frac{1}{5} & 0 & \frac{1}{7} & 0 & \cdots \\
\vdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{bmatrix}$$

$m_k$ are the moments of $\nu$
Bernoulli IFS – Densities

$\lambda = 0.8$

histogram $2^{20}$ points
Bernoulli IFS – Densities

$\lambda = 0.6$
References

- Peres, Schlag, Solomyak. *Sixty years of Bernoulli convolutions*. Springer 2000
- Solomyak. *On the random series $\sum \pm \lambda^n$*. Ann. of Math., 1995
References

• Kato. *Perturbation theory for linear operators.* Springer 1980
• Hilbert. *Ein Beitrag zur Theorie des Legendre’schen Polynoms.* 1894
Polynomial eigenfunctions of $T$

Let $\lambda = 0.5 - 0.5i$. We get the eigenpolynomials

$$T_h(z) = \frac{1}{2} h(\lambda z) + \frac{1}{2} h(\lambda z + 1)$$

of degree $\leq 3$ from the matrix

$$T_3 = \begin{pmatrix}
1 & 1 & 1 & 1 \\
0 & 0.5 - 0.5i & 1 - i & 1.5 - 1.5i \\
0 & 0 & -0.5i & -1.5i \\
0 & 0 & 0 & -0.25 - 0.25i
\end{pmatrix}$$
Polynomial eigenfunctions of $T$ are orthogonal to a measure supported on the set \( \text{dragon curve} \).
Polynomial eigenfunctions of $T$

degree 0

left: real part, right: imaginary part
Polynomial eigenfunctions of $T$

degree 1

left: real part, right: imaginary part
Polynomial eigenfunctions of $T$

degree 2

left: real part, right: imaginary part
Polynomial eigenfunctions of $T$

degree 3

left: real part, right: imaginary part
Polynomial eigenfunctions of $T$

degree 4

left: real part, right: imaginary part
Polynomial eigenfunctions of $T$

degree 5

left: real part, right: imaginary part
Recursions

Moments of $\nu_\lambda$

$$m_{2k} = - \sum_{i=1}^{k} a_{2k,i} m_{2k-2i}$$

with $m_0 = 1$ and

$$a_{n,i} = \frac{1}{\lambda^{2k-1}} \sum_{j=0}^{i-1} \binom{n-2j}{n-2i} (1 - \lambda)^{2i-2j} a_{n,j}$$

the coefficient of $x^i$ in the eigenpolynomial $p_n$ of $A$ with $a_{n,0} = 1$. 
Legendre moments of $\nu_\lambda$

$$(\nu_\lambda, L_{2n}) = \sum_{k=0}^{n} (-1)^{n-k} 4^{-n} \binom{2n + 2k}{2k} \binom{2n}{n + k} m_{2k}$$

where

$$m_{2k} = (\nu_\lambda, x^{2k})$$

are the moments of the convolution measure.
Moments of $\nu_\lambda$, another recursion:

$$m_{2k} = \frac{1}{1 - \lambda^{2k}} \sum_{j=0}^{k-1} b_{2k,\lambda}(2j)m_{2j}$$

where $b_{n,\lambda}(\cdot)$ are the weights of the binomial distribution with parameters $n$ and $\lambda$ and $m_0 = 1$