Approximation of the invariant measure of an IFS

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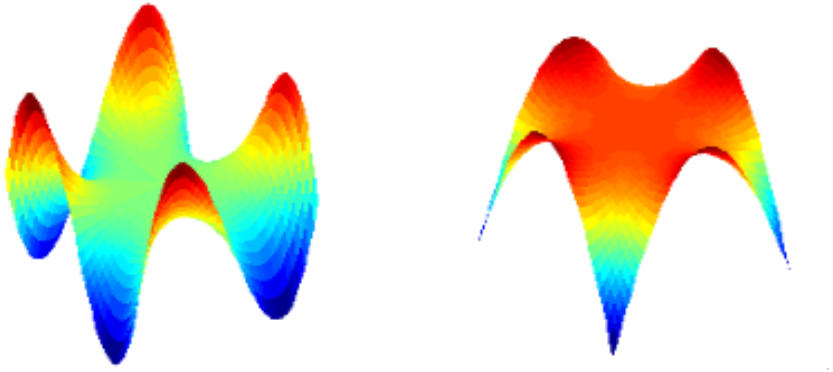


Approximation of the invariant measure of an IFS

1. Iterated Function System (IFS)
2. The transfer operator T for an IFS
3. Eigenfunctions of T for affine IFS
4. Approximation for the invariant measure

1. Iterated Function System (IFS)

IFS

Let $\mathbf{X} = \mathbb{R}^{\mathbf{d}}$ or $\mathbb{C}^{\mathbf{d}}$.

An **IFS on X** consists of

$$\mathbf{f_1, \ldots, f_N} \quad \text{mappings } \mathbf{X} \to \mathbf{X}$$

and a corresponding

$$(\mathbf{p_1, \ldots, p_N}) \quad \text{probability vector}$$

Assume there is a non-empty **compact set** $\mathbf{K} \subset \mathbf{X}$ such that

$$\mathbf{f_i(K)} \subset \mathbf{K} \quad \text{for all } \mathbf{i}.$$

IFS as a stochastic dynamical system

$\mathbf{f_1, \ldots, f_N}$ **mappings** $\mathbf{X \to X}$

$(\mathbf{p_1, \ldots, p_N})$ **probability vector**

Start at a point $\mathbf{x_0} \in \mathbf{X}$. Given a point $\mathbf{x_n}$, choose a function $\mathbf{f}$ according to $\mathbb{P}(\mathbf{f} = \mathbf{f_i}) = \mathbf{p_i}$ and set

$$\mathbf{x_{n+1}} = \mathbf{f}(\mathbf{x_n})$$

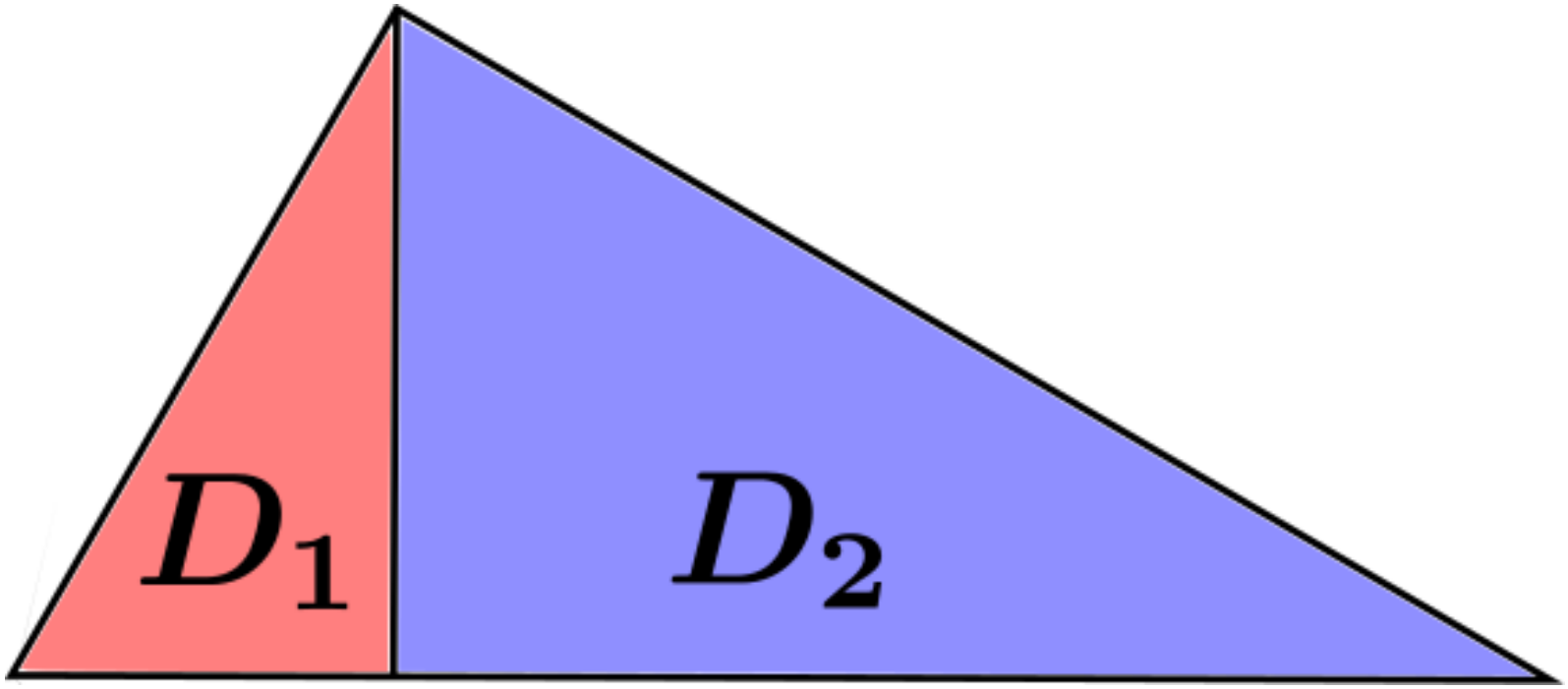
random trajectory in X

$\mathbf{x_0, x_1, x_2, \ldots}$

IFS – right-angled triangle

$\mathbf{D}$ right-angled triangle

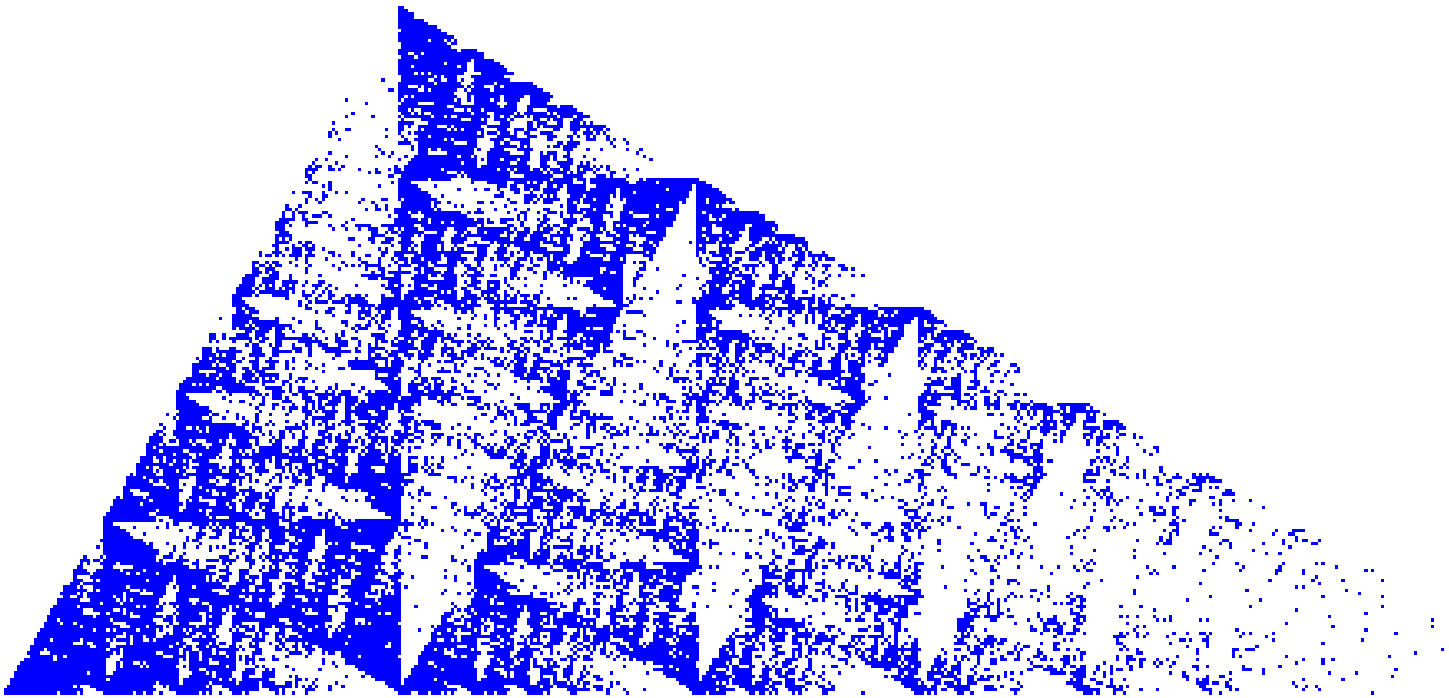
mappings $\mathbf{D_1} = \mathbf{f_1}(\mathbf{D}) \quad \mathbf{D_2} = \mathbf{f_2}(\mathbf{D})$



with prob. $\mathbf{p_1} = \mathbf{p_2} = \dfrac{\mathbf{1}}{\mathbf{2}}$

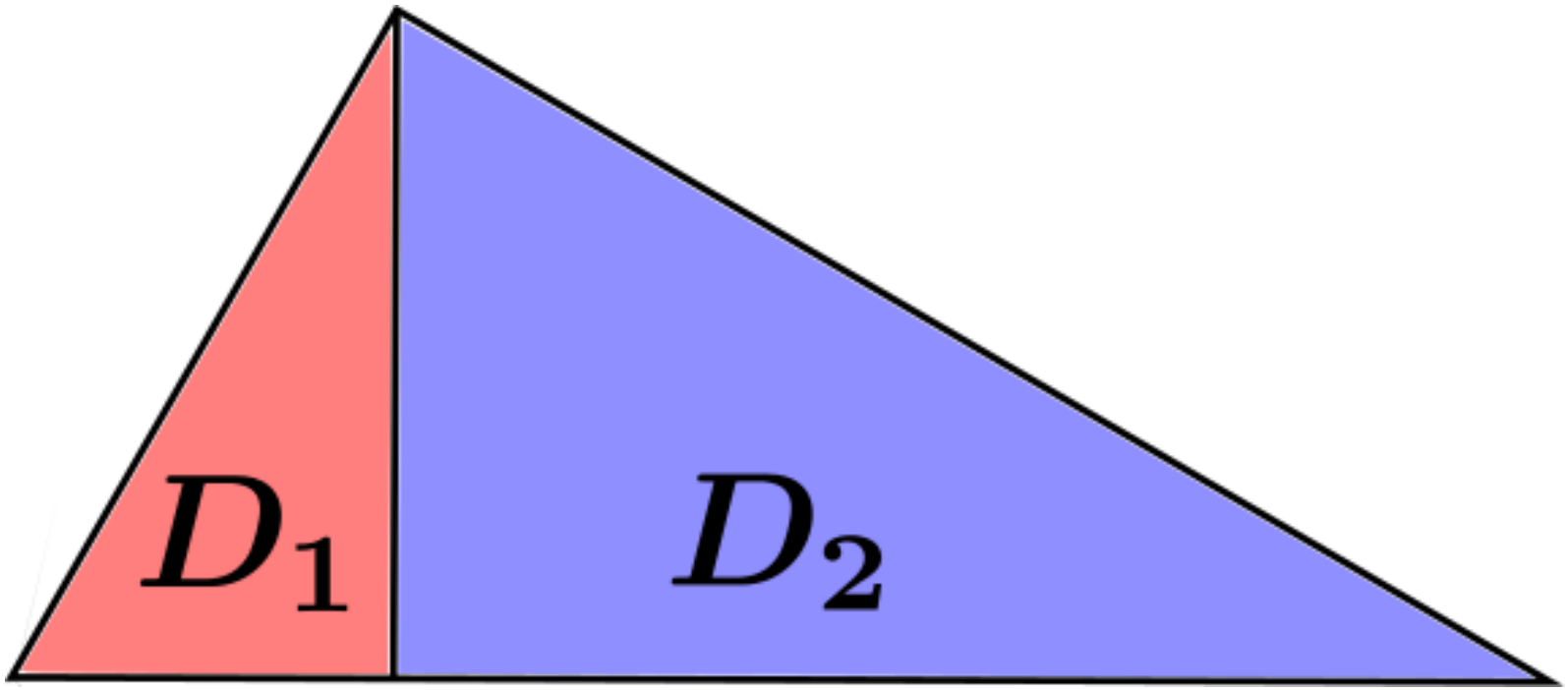
IFS – random trajectory

Points $\mathbf{x_0}, \mathbf{x_1}, \ldots, \mathbf{x_{100000}}$



IFS – right-angled triangle

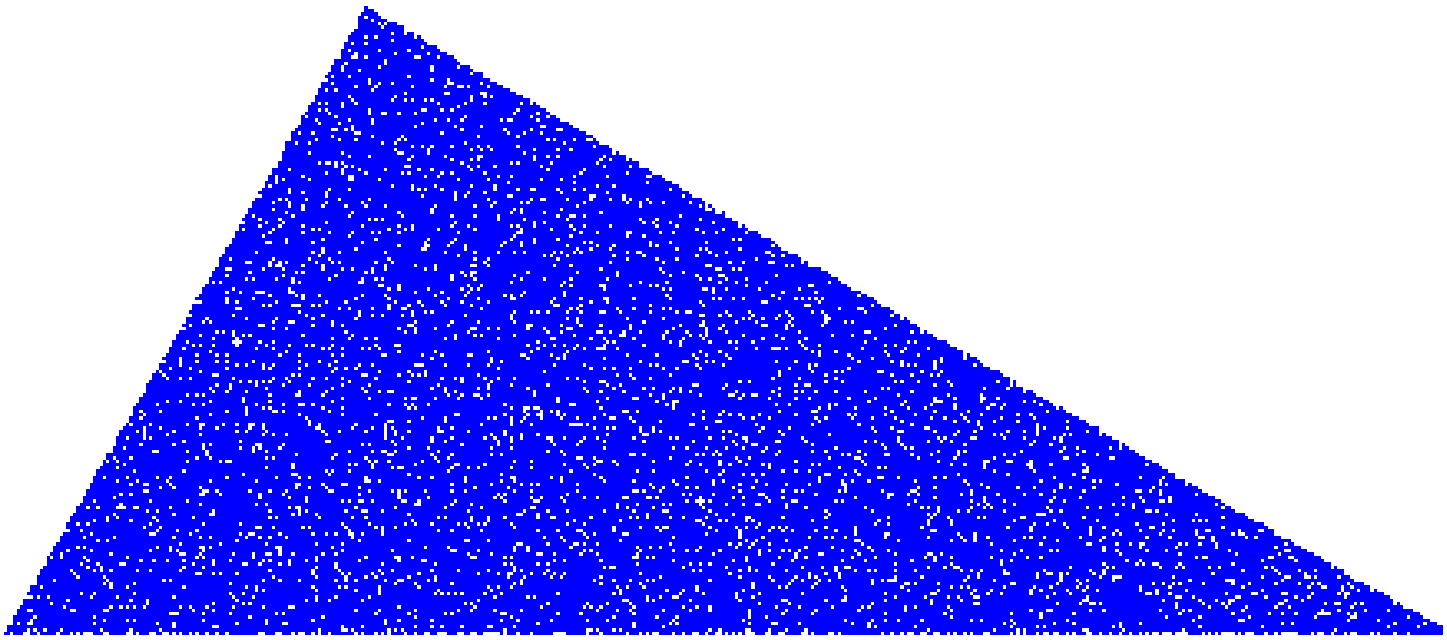
$\mathbf{D}$ right-angled triangle
mappings $\mathbf{D_1} = \mathbf{f_1}(\mathbf{D})$ and $\mathbf{D_2} = \mathbf{f_2}(\mathbf{D})$



with prob. $\mathbf{p_1} = \dfrac{|\mathbf{D_1}|}{|\mathbf{D}|} = \dfrac{1}{4}$ and $\mathbf{p_2} = \dfrac{|\mathbf{D_2}|}{|\mathbf{D}|} = \dfrac{3}{4}$

IFS – random trajectory

Points $\mathbf{x_0}, \mathbf{x_1}, \ldots, \mathbf{x_{100000}}$ of a random trajectory



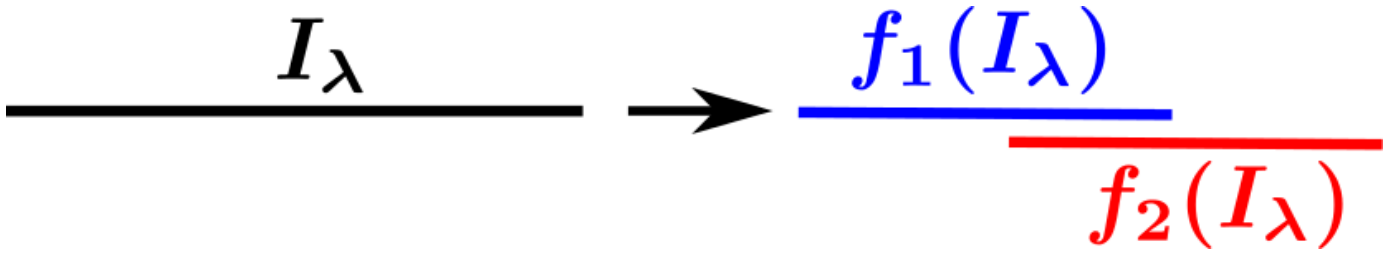
Bernoulli IFS

IFS on $\mathbb{R}$ with mappings

$$\mathbf{f_1(x) = \lambda x - 1} \text{ with prob. } \frac{\mathbf{1}}{\mathbf{2}}$$

$$\mathbf{f_2(x) = \lambda x + 1} \text{ with prob. } \frac{\mathbf{1}}{\mathbf{2}}$$

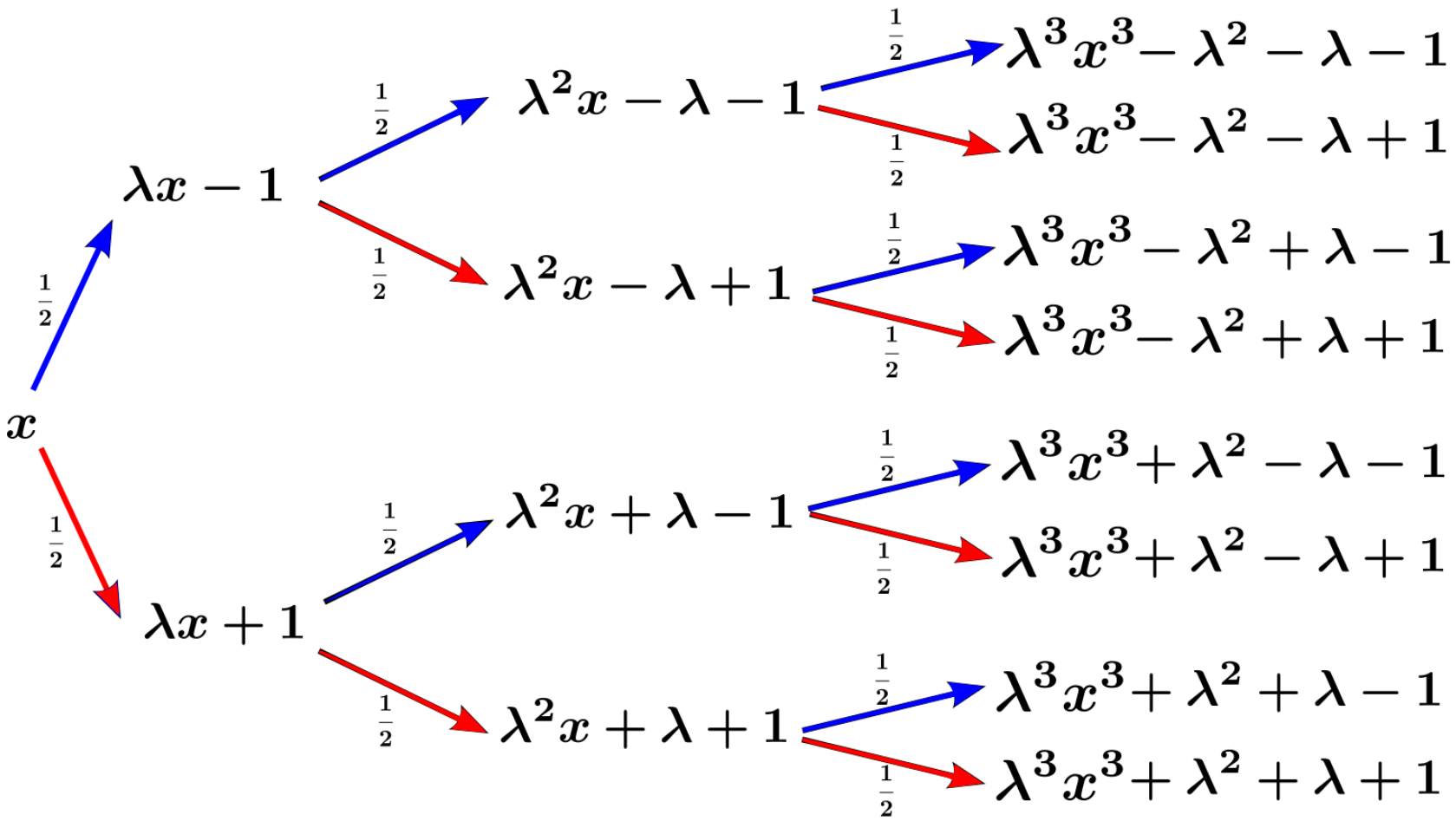
parameter $\boldsymbol{\lambda} \in [\frac{\mathbf{1}}{\mathbf{2}}, \mathbf{1})$



Interval $\mathbf{I}_\lambda = [-\frac{\mathbf{1}}{\mathbf{1}-\lambda}, \frac{\mathbf{1}}{\mathbf{1}-\lambda}]$ with $\mathbf{I}_\lambda = \mathbf{f_1}(\mathbf{I}_\lambda) \cup \mathbf{f_2}(\mathbf{I}_\lambda)$

Bernoulli IFS – trajectories

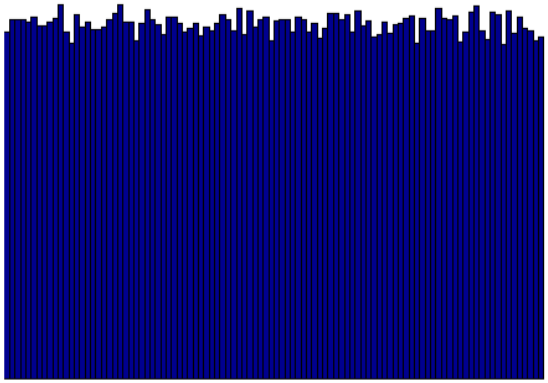
$f_1(x) = \lambda x - 1 \quad f_2(x) = \lambda x + 1$



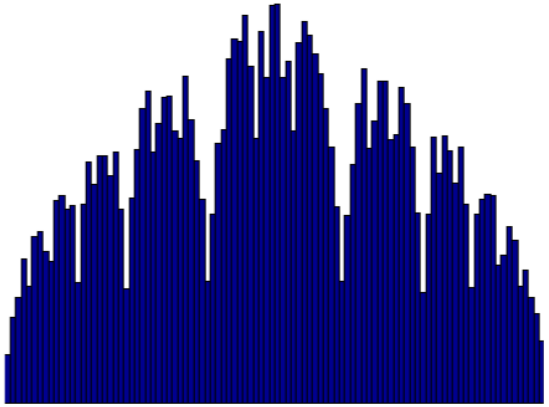
$\mathbf{n^{th}}$ iteration: $\pm 1 \pm \lambda \pm \lambda^2 \pm \lambda^3 \pm \ldots \pm \lambda^n x^n$

Bernoulli IFS - random trajectory

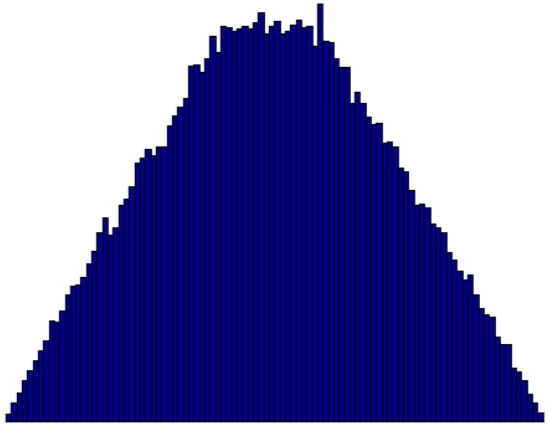
First 10^5 points (histogram)



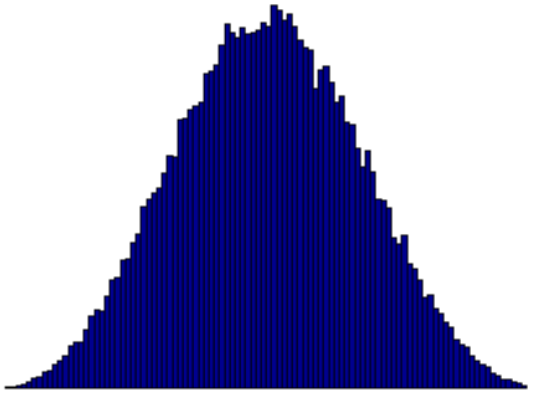
$\lambda = 0.5$



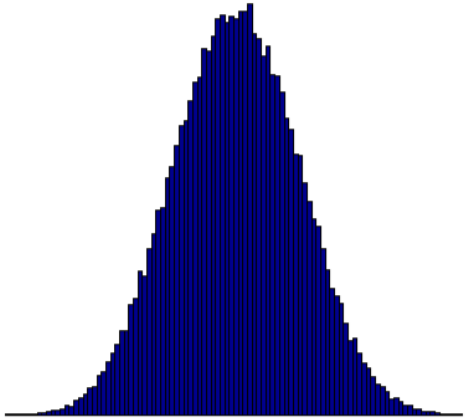
$\lambda = 0.6$



$\lambda = 0.7$



$\lambda = 0.8$



$\lambda = 0.9$

2. The transfer operator T for an IFS

The transfer operator T

$\mathcal{C}(\mathbf{K})$ space of continuous functions $\mathbf{K} \to \mathbb{K}$

$$\mathbf{T} : \mathcal{C}(\mathbf{K}) \to \mathcal{C}(\mathbf{K})$$

$$\mathbf{T}\mathbf{h}(\mathbf{x}) = \sum_{\mathbf{i}=1}^{\mathbf{N}} \mathbf{p}_{\mathbf{i}}\, \mathbf{h}(\mathbf{f}_{\mathbf{i}}(\mathbf{x}))$$

For a trajectory with starting point $\mathbf{x_0} \in \mathbf{K}$ we have

$$\mathbb{E}\, \mathbf{h}(\mathbf{x}_1) = \mathbf{T}\mathbf{h}(\mathbf{x}_0)$$
$$\mathbb{E}\, \mathbf{h}(\mathbf{x}_n) = \mathbf{T}^n\mathbf{h}(\mathbf{x}_0)$$

Dual operator of T

Let $\mathcal{M}(\mathbf{K})$ be the dual space of $\mathbf{C}(\mathbf{K})$ i.e. the space of Borel regular measures on $\mathbf{K}$.

The **Hutchinson operator**

$$\mathbf{H} : \mathcal{M}(\mathbf{K}) \to \mathcal{M}(\mathbf{K})$$

$$\mathbf{H}\mu = \sum_{\mathbf{i}=1}^{\mathbf{N}} \mathbf{p_i}\, \mu \circ \mathbf{f_i}^{-1}$$

is **dual to the transfer operator**, i.e.:

$$(\mathbf{H}\mu, \mathbf{h}) = (\mu, \mathbf{Th}) \quad \textbf{duality}$$

Hutchinson operator

$$\mathbf{H} : \mathcal{M}(\mathbf{K}) \to \mathcal{M}(\mathbf{K})$$

$$\mathbf{H}\mu(\mathbf{A}) = \sum_{\mathbf{i}=1}^{\mathbf{N}} \mathbf{p_i}\, \mu(\mathbf{f_i}^{-1}(\mathbf{A}))$$

Start with a distribution μ_0 auf $\mathbf{K}$. The mass in $\mathbf{A} \subset \mathbf{K}$ after one step comes with prob. $\mathbf{p_i}$ from the set $\mathbf{f_i}^{-1}(\mathbf{A})$, so that

$$\mu_1(\mathbf{A}) = \mathbf{H}\, \mu_0(\mathbf{A})$$

Distribution after $\mathbf{n}$ steps:

$$\mu_\mathbf{n} = \mathbf{H}^\mathbf{n} \mu_0$$

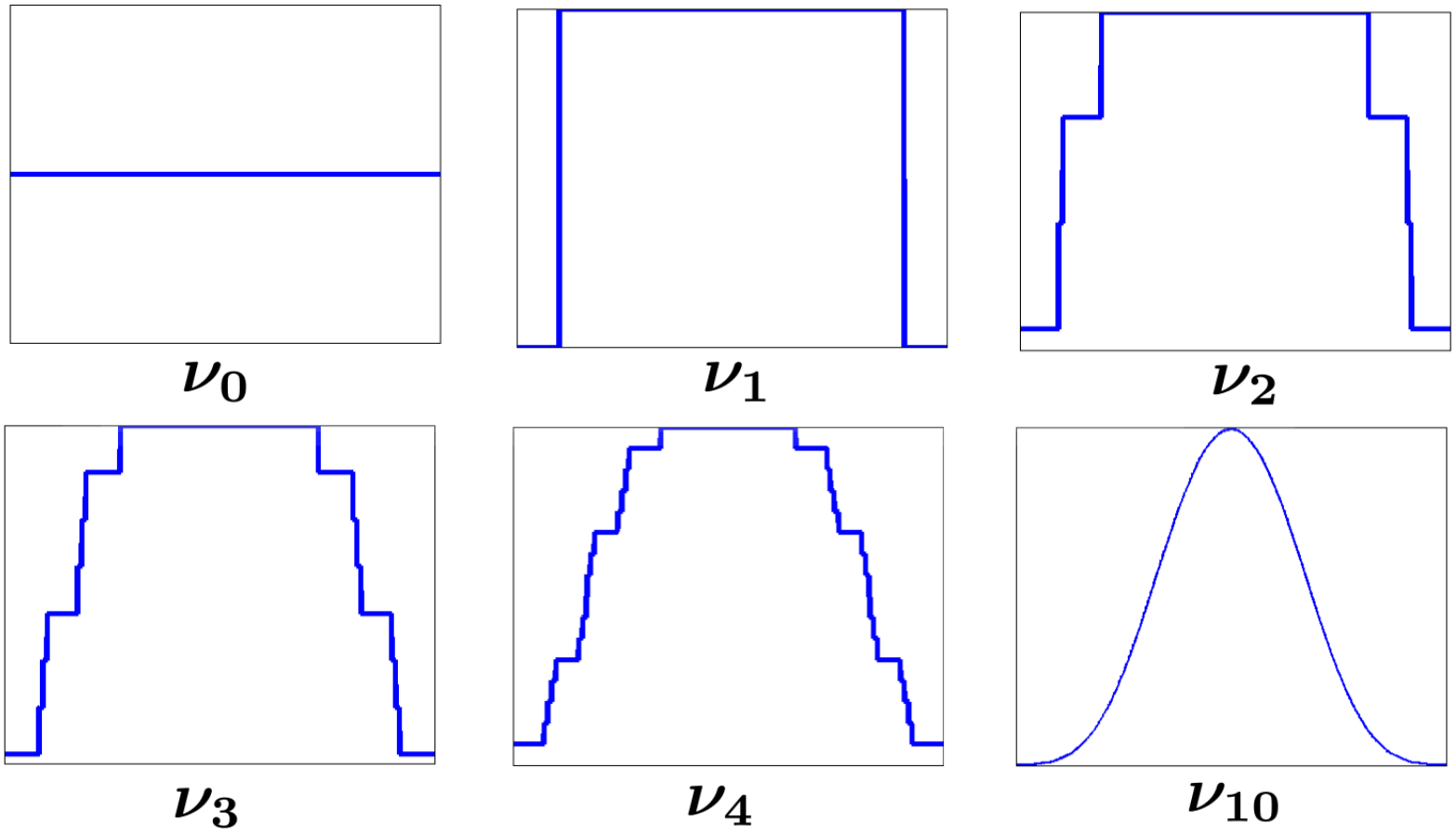
Implications of duality

Spectra:
The transfer operator $\mathbf{T}$ and the Hutchinson operator $\mathbf{H}$ have the **same spectra**.

Invariant measure of the IFS:
$\boldsymbol{\nu} = \mathbf{H}\boldsymbol{\nu}$ is **orthogonal to all eigenfunctions of $\mathbf{T}$** with eigenvalue $\neq \mathbf{1}$.

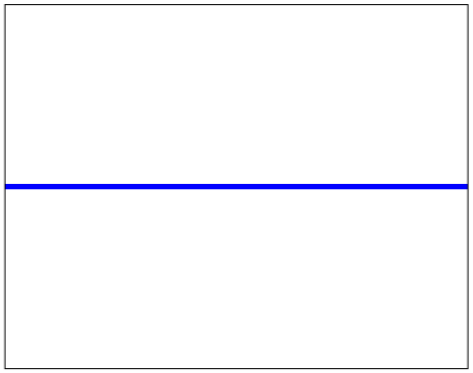
Bernoulli IFS - Hutchinson Operator

$\lambda = 0.9$

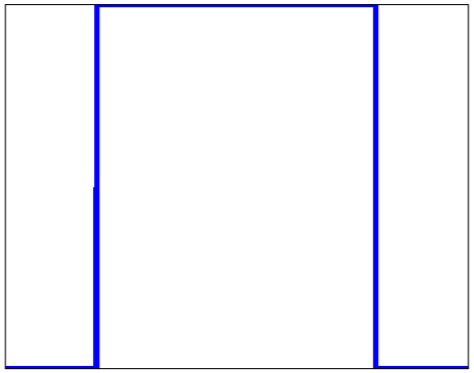


Bernoulli IFS - Hutchinson Operator

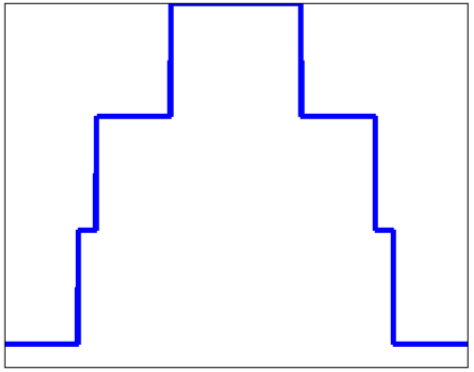
$\lambda = 0.8$



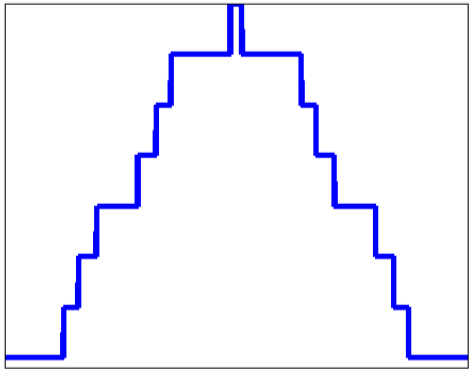
ν_0



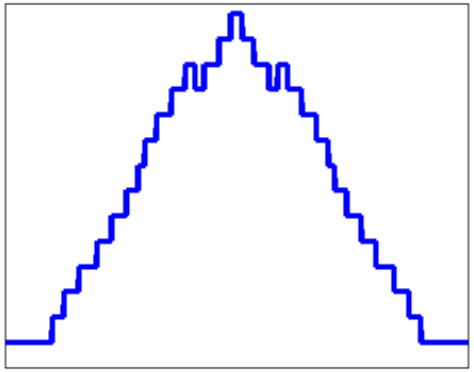
ν_1



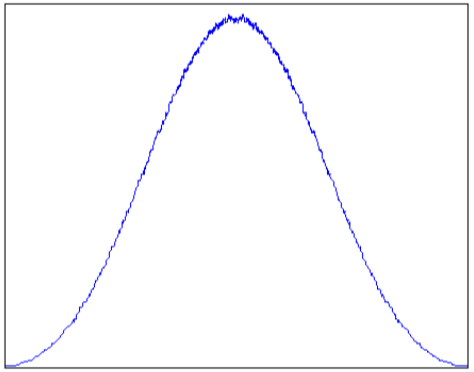
ν_2



ν_3



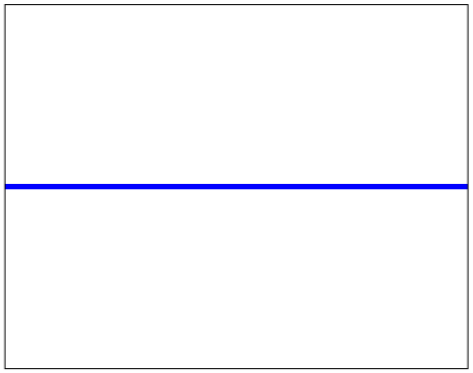
ν_4



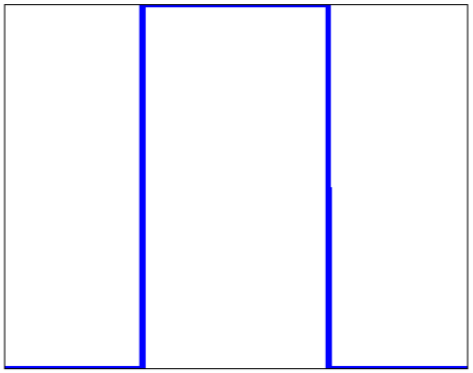
ν_{10}

Bernoulli IFS - Hutchinson Operator

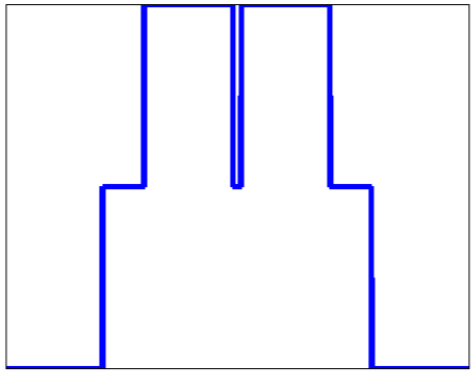
$\lambda = 0.7$



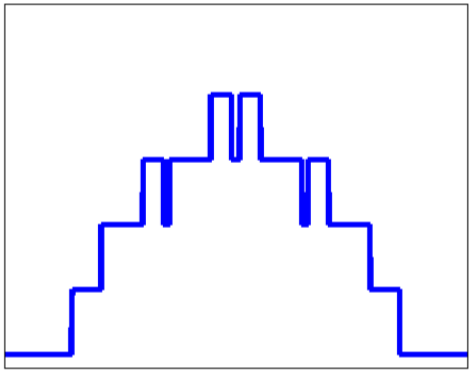
ν_0



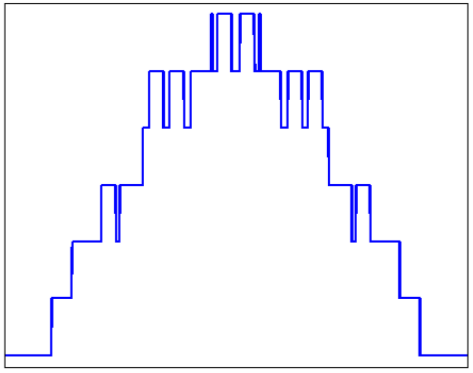
ν_1



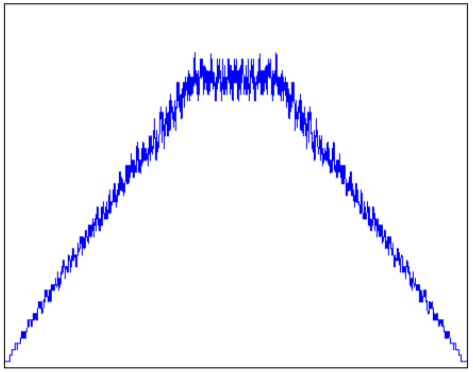
ν_2



ν_3



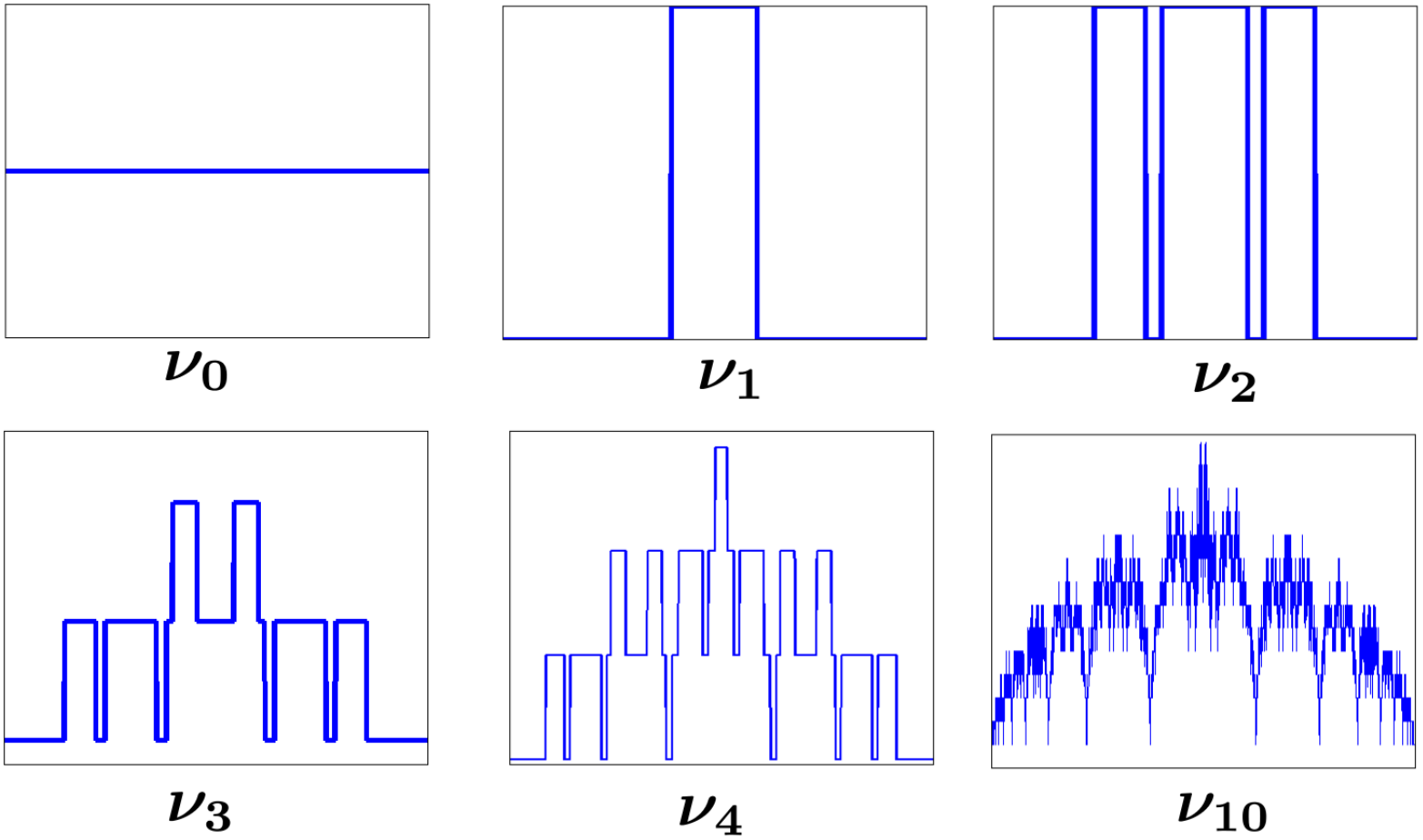
ν_4



ν_{10}

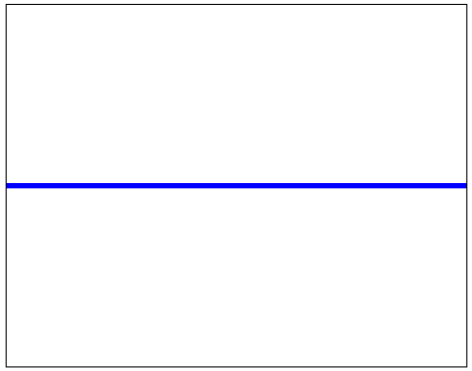
Bernoulli IFS - Hutchinson Operator

$\lambda = 0.6$

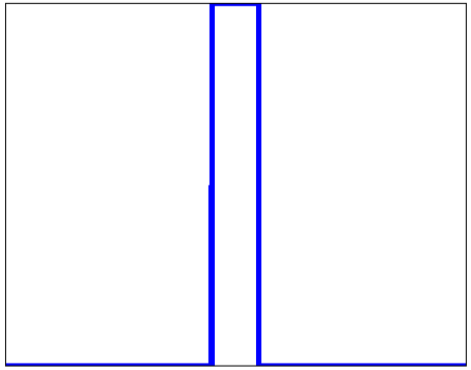


Bernoulli IFS - Hutchinson Operator

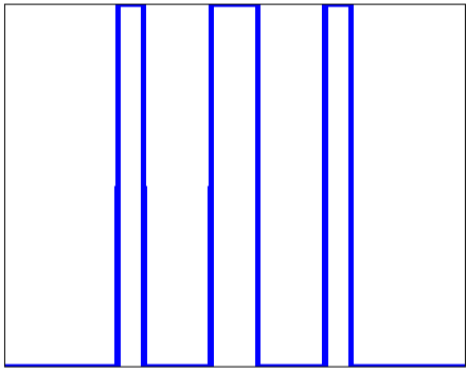
$\lambda = 0.55$



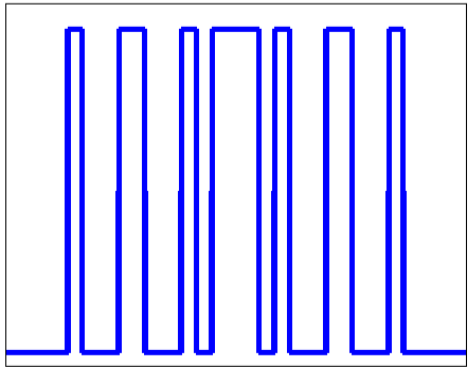
ν_0



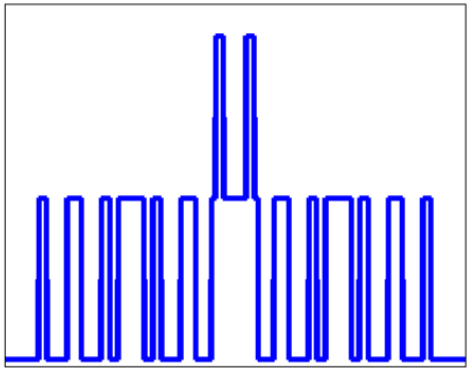
ν_1



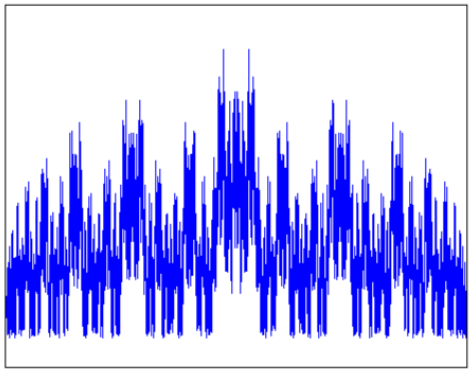
ν_2



ν_3



ν_4

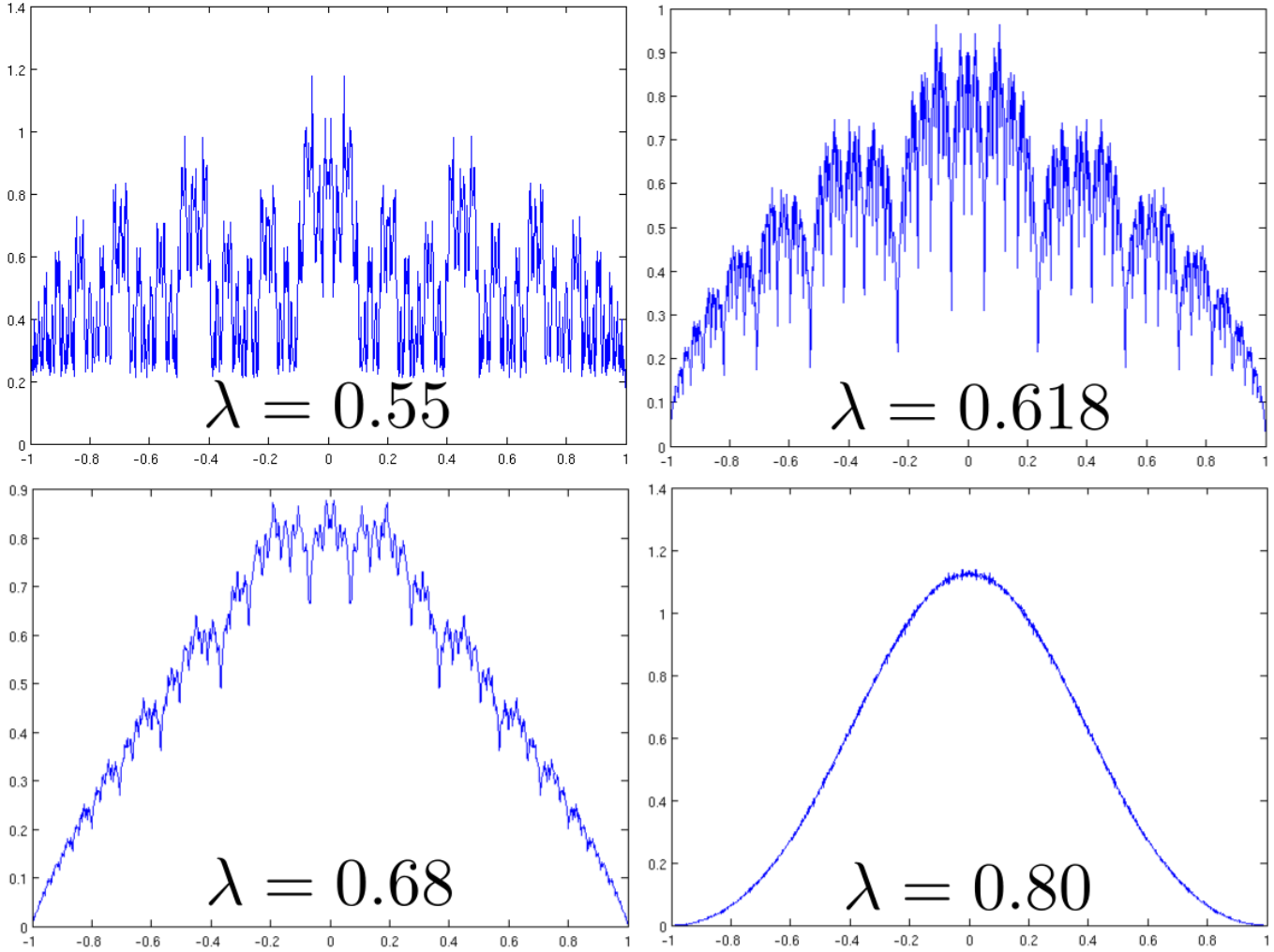


ν_{10}

Bernoulli convolution problem

ν distribution of the random sum $\sum_{n=0}^{\infty} \pm\lambda^n$

Does ν have a density?



Results on Bernoulli convolutions

ν distribution of the random sum $\sum_{n=0}^{\infty} \pm\lambda^n$

Jessen, Winter 1935: ν is either absolutely continuous or purely singular with respect to the Lebesgue measure.

Results on Bernoulli convolutions

$$\mathbf{S} = \{\lambda \in [\frac{1}{2}, 1) : \nu \text{ singular}\}$$

Erdös 1939: countably many examples in $\mathbf{S}$.

Garsia 1962: countably many examples in $\overline{\mathbf{S}} \cap [\frac{1}{2}, 1)$.

Solomyak 1995: $\mathbf{S}$ has Lebesgue measure 0.

Shmerkin 2013: $\mathbf{S}$ has Hausdorff dimension 0.

3. Eigenfunctions of T for affine IFS

T for affine IFS

Transfer operator for an IFS on $\mathbf{K} \subset \mathbb{K}^{\mathbf{d}}$ with affine contractions $\mathbf{f_i}(\mathbf{x}) = \mathbf{A_i x} + \mathbf{v_i}$

$$\mathbf{T} : \mathcal{C}(\mathbf{K}) \to \mathcal{C}(\mathbf{K})$$

$$\mathbf{Th}(\mathbf{x}) = \sum_{\mathbf{i=1}}^{\mathbf{N}} \mathbf{p_i}\, \mathbf{h}(\mathbf{A_i x} + \mathbf{v_i})$$

with matrix $\mathbf{A_i}$ and translation vector $\mathbf{v_i}$.

Invariant eigenspaces for affine IFS

Let $\mathcal{P}_{\mathbf{n}}(\mathbf{K})$ be the space of **polynomials**

$$\mathbf{p} : \mathbf{K} \to \mathbb{K}$$

of degree $\leq \mathbf{n}$. Then,

$$\mathbf{T} : \mathcal{P}_{\mathbf{n}}(\mathbf{K}) \to \mathcal{P}_{\mathbf{n}}(\mathbf{K})$$

is well-defined, i.e.

$\mathcal{P}_{\mathbf{n}}(\mathbf{K})$ is an invariant eigenspace of $\mathbf{T}$.

$\mathbf{T}$ for affine IFS on $\mathbb{K}$

Consider an IFS on $\mathbf{K} \subset \mathbb{K}$ with mappings

$$\mathbf{f_i(x) = \lambda_i x + v_i, \quad i = 1, \ldots, N}$$
$$\boldsymbol{\lambda_i,\ v_i} \in \mathbb{K}$$

The operator

$$\mathbf{T : \mathcal{P}_n(K) \to \mathcal{P}_n(K)}$$

is represented by a **matrix** $\mathbf{T_n} \in \mathbb{K}^{\mathbf{(n+1)\times(n+1)}}$.

Matrix for $T_n : \mathcal{P}_n(K) \to \mathcal{P}_n(K)$

$$T_n = \begin{pmatrix} 1 & * & * & * & \dots & * \\ 0 & \sum_i p_i \lambda_i & * & * & \dots & * \\ & 0 & \sum_i p_i \lambda_i^2 & * & \dots & * \\ \vdots & & & & & \\ & & \ddots & \ddots & & \vdots \\ 0 & & \dots & & 0 & \sum_i p_i \lambda_i^n \end{pmatrix}$$

with respect to the basis $1, x, x^2, \dots, x^n$.

Eigenvalues of T_n

$$\omega_0 = 1$$

$$\omega_1 = \sum_i p_i \lambda_i$$

$$\omega_2 = \sum_i p_i \lambda_i^2$$

$$\vdots$$

$$\omega_n = \sum_i p_i \lambda_i^n$$

$i = 1, \ldots, N$

Eigenvectors of $\mathbf{T_n}$

The **eigenvectors of $\mathbf{T_n}$** correspond to the **polynomial eigenfunctions** of

$$\mathbf{T} : \mathcal{P}_{\mathbf{n}}(\mathbf{K}) \to \mathcal{P}_{\mathbf{n}}(\mathbf{K}).$$

The constant function $\mathbf{1}$ is always an eigenfunction with eigenvalue 1, since

$$\mathbf{T1} = \sum_{\mathbf{i=1}}^{\mathbf{N}} \mathbf{p_i} = \mathbf{1}$$

Basis of eigenfunctions

Theorem. The transfer operator for an IFS with equal contraction factors $\boldsymbol{\lambda}$

$$\mathbf{T} : \mathcal{P}_{\mathbf{n}}(\mathbf{K}) \to \mathcal{P}_{\mathbf{n}}(\mathbf{K})$$

$$\mathbf{Th}(\mathbf{x}) = \sum_{\mathbf{i}=1}^{\mathbf{N}} \mathbf{p_i}\, \mathbf{h}(\boldsymbol{\lambda}\mathbf{x} + \mathbf{v_i})$$

has **eigenvalues $\lambda^{\mathbf{k}}$ for $0 \leq \mathbf{k} \leq \mathbf{n}$ and the eigenfunctions build a basis of $\mathcal{P}_{\mathbf{n}}(\mathbf{K})$**.

Basis of eigenfunctions

Theorem. The transfer operator

$$\mathbf{T} : \mathcal{C}(\mathbf{K}) \to \mathcal{C}(\mathbf{K})$$

$$\mathbf{Th}(\mathbf{x}) = \sum_{\mathbf{i}=1}^{\mathbf{N}} \mathbf{p_i}\, \mathbf{h}(\lambda \mathbf{x} + \mathbf{v_i})$$

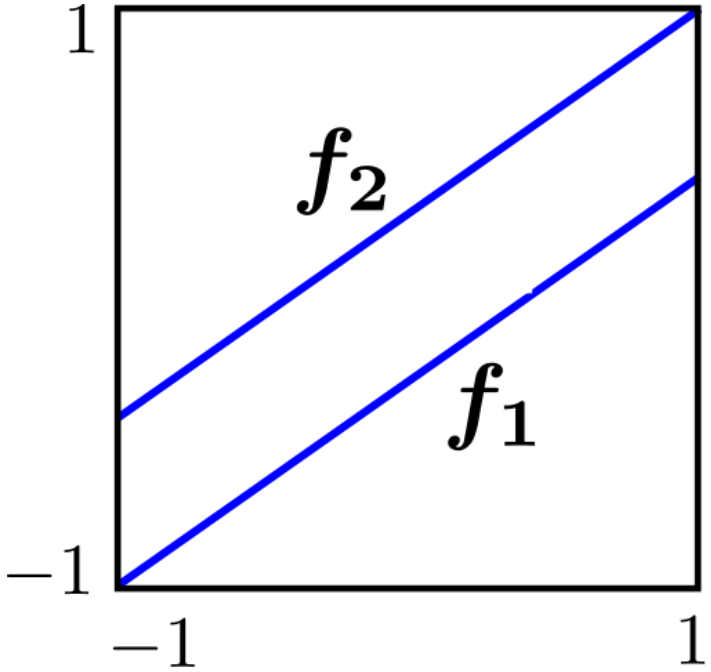
has **eigenvalues** $\{\lambda^{\mathbf{k}} : \mathbf{k} \in \mathbb{N}_0\}$ and their **eigenfunctions build a basis of** $\mathcal{P}(\mathbf{K})$**.**

Bernoulli IFS – Transfer operator

$$\mathsf{T} : \mathcal{C}([-1,1]) \to \mathcal{C}([-1,1])$$

$$\mathsf{T}h(x) = \frac{1}{2}h(\underbrace{\lambda x - 1 + \lambda}_{f_1(x)}) + \frac{1}{2}h(\underbrace{\lambda x + 1 - \lambda}_{f_2(x)})$$

with $\lambda \in [\frac{1}{2}, 1)$.



Bernoulli IFS – Transfer operator

We get the eigenpolynomials of

$$\mathbf{Th(x)} = \frac{\mathbf{1}}{\mathbf{2}}\mathbf{h}(\boldsymbol{\lambda}\mathbf{x} - \mathbf{1} + \boldsymbol{\lambda}) + \frac{\mathbf{1}}{\mathbf{2}}\mathbf{h}(\boldsymbol{\lambda}\mathbf{x} + \mathbf{1} - \boldsymbol{\lambda})$$

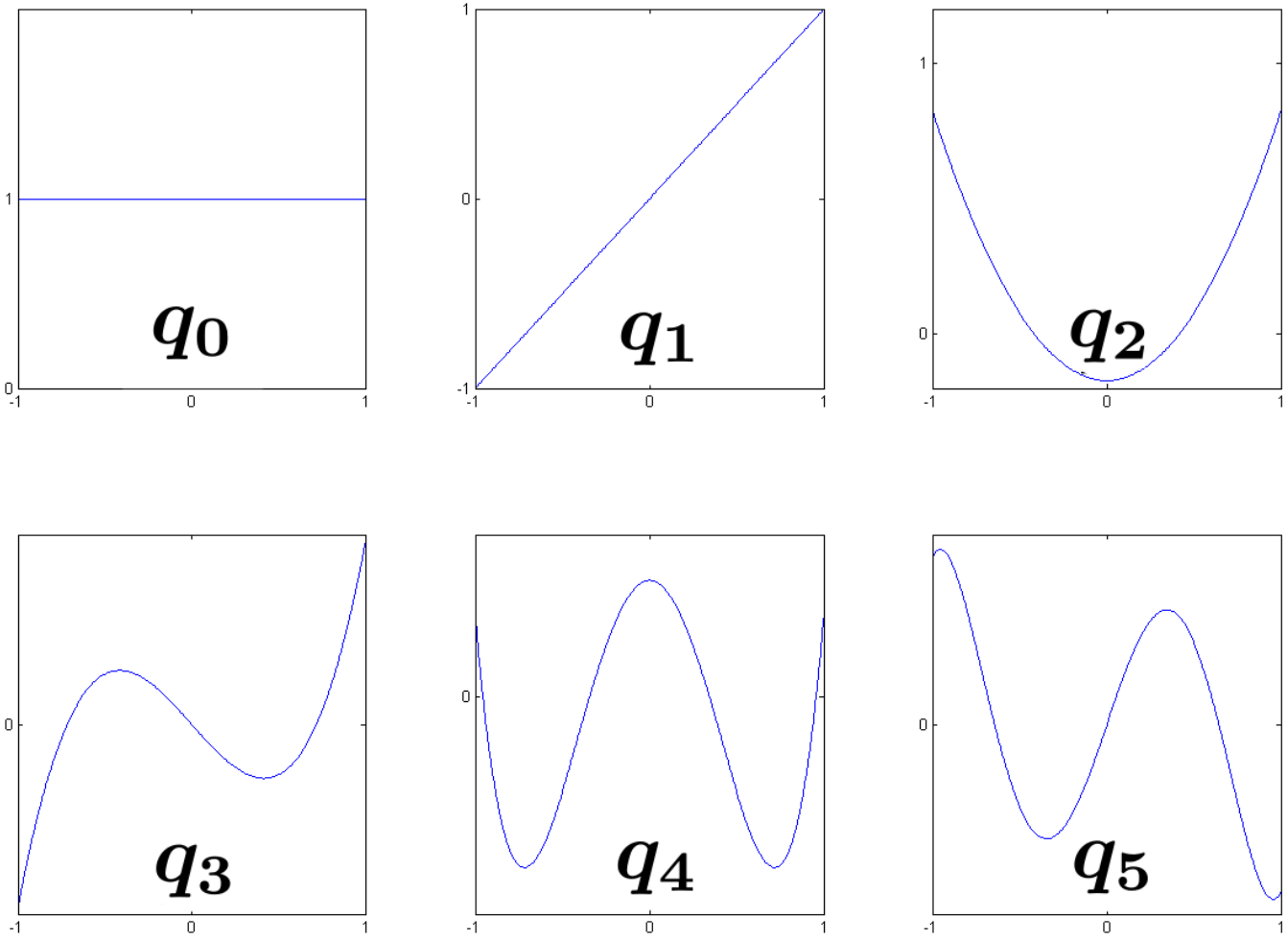
of degree ≤ 3 from the matrix

$$\mathbf{T_3} = \begin{pmatrix} 1 & 0 & (1-\lambda)^2 & 0 \\ 0 & \lambda & 0 & 3\lambda(1-\lambda)^2 \\ 0 & 0 & \lambda^2 & 0 \\ 0 & 0 & 0 & \lambda^3 \end{pmatrix}$$

$$\mathbf{q_0(x) = 1,\ q_1(x) = x,}$$

$$\mathbf{q_2(x) = x^2 + \frac{\lambda - 1}{\lambda + 1},\ \ q_3(x) = x^3 + 3\,\frac{\lambda - 1}{\lambda + 1}}$$

Bernoulli IFS – eigenfunctions of T



Polynomial eigenfunctions $\mathbf{q_0}, \ldots, \mathbf{q_5}$ for $\boldsymbol{\lambda} = \mathbf{0.7}$

Bernoulli IFS

Theorem. The transfer operator $\mathbf{T}$ has the **eigenfunctions**

$$\mathbf{q_n(x)} = \sum_{\mathbf{k=0}}^{\lfloor \frac{\mathbf{n}}{2} \rfloor} \mathbf{a_{n,k} x^{n-2k}} \qquad \mathbf{n} \in \mathbb{N}_0$$

with **eigenvalues** $\boldsymbol{\omega_n = \lambda^n}$. The coefficients are given recursively by

$$\mathbf{a_{n,k}} = \frac{\mathbf{1}}{\lambda^{2k} - \mathbf{1}} \sum_{\mathbf{j=0}}^{\mathbf{k-1}} \binom{\mathbf{n-2j}}{\mathbf{n-2k}} (\mathbf{1} - \lambda)^{\mathbf{2k-2j}} \mathbf{a_{n,j}}$$

for $\mathbf{k} \geq \mathbf{1}$ and $\mathbf{a_{n,0}} = \mathbf{1}$ else.

4. Approximation of the invariant measure

Setting

Consider an IFS on $\mathbf{K} \subset \mathbb{K}$ with invariant measure

$$\mathbf{H}\nu = \nu.$$

Assumption: the eigenfunctions of $\mathbf{T}$

$$\mathbf{q_k} \in \mathcal{P}_\mathbf{k}(\mathbf{K}), \quad \mathbf{k} \in \mathbb{N}_\mathbf{0}$$

build a **basis of** $\mathcal{P}(\mathbf{K})$ **(this is the case for the Bernoulli IFS)**

Approximating densities $\mathbf{v_n}$

Duality implies:

$$\nu \perp \mathbf{q_k} \quad \text{for} \quad \mathbf{k = 1, 2, \ldots}$$

We get a **sequence of polynomial probability densities** $\mathbf{v_n} \in \mathcal{P}_\mathbf{n}(\mathbf{K})$ by solving

$$\mathbf{v_n} \perp \mathbf{q_k} \quad \text{for} \quad \mathbf{1 \leq k \leq n}$$

or equivalently,

$$\langle \mathbf{v_n}, \mathbf{x^k} \rangle = \mathbf{m_k} \quad \text{for} \quad \mathbf{1 \leq k \leq n}$$

$\mathbf{m_k}$ is the $\mathbf{k^{th}}$ moment of the invariant measure ν

Linear system of equations for v_n

Theorem. The approximation $v_n(x) = \sum_{k=0}^{n} u_k x^k$ satisfies

$$G(u_0, u_1, \ldots, u_n)' = (m_0, m_1, \ldots, m_n)'$$

with the **Hilbert matrix** $G \in \mathbb{K}^{(n+1)\times(n+1)}$

$$G_{ij} = \int_K x^{i+j}\, dx.$$

$m_k = \int_K x^k\, d\nu$ are the moments of ν.

Approximating measures ν_n

Theorem. The approximating measures ν_n

$$\nu_n(A) = \int_A v_n(x)\, dx$$

converge

$$\nu_n \to \nu \quad \textbf{weakly}$$

to the invariant measure of the IFS.

Bernoulli IFS – Densities

The approximation $\mathbf{v_n(x)} = \sum_{\mathbf{k=0}}^{\mathbf{n}} \mathbf{u_k x^k}$ satisfies

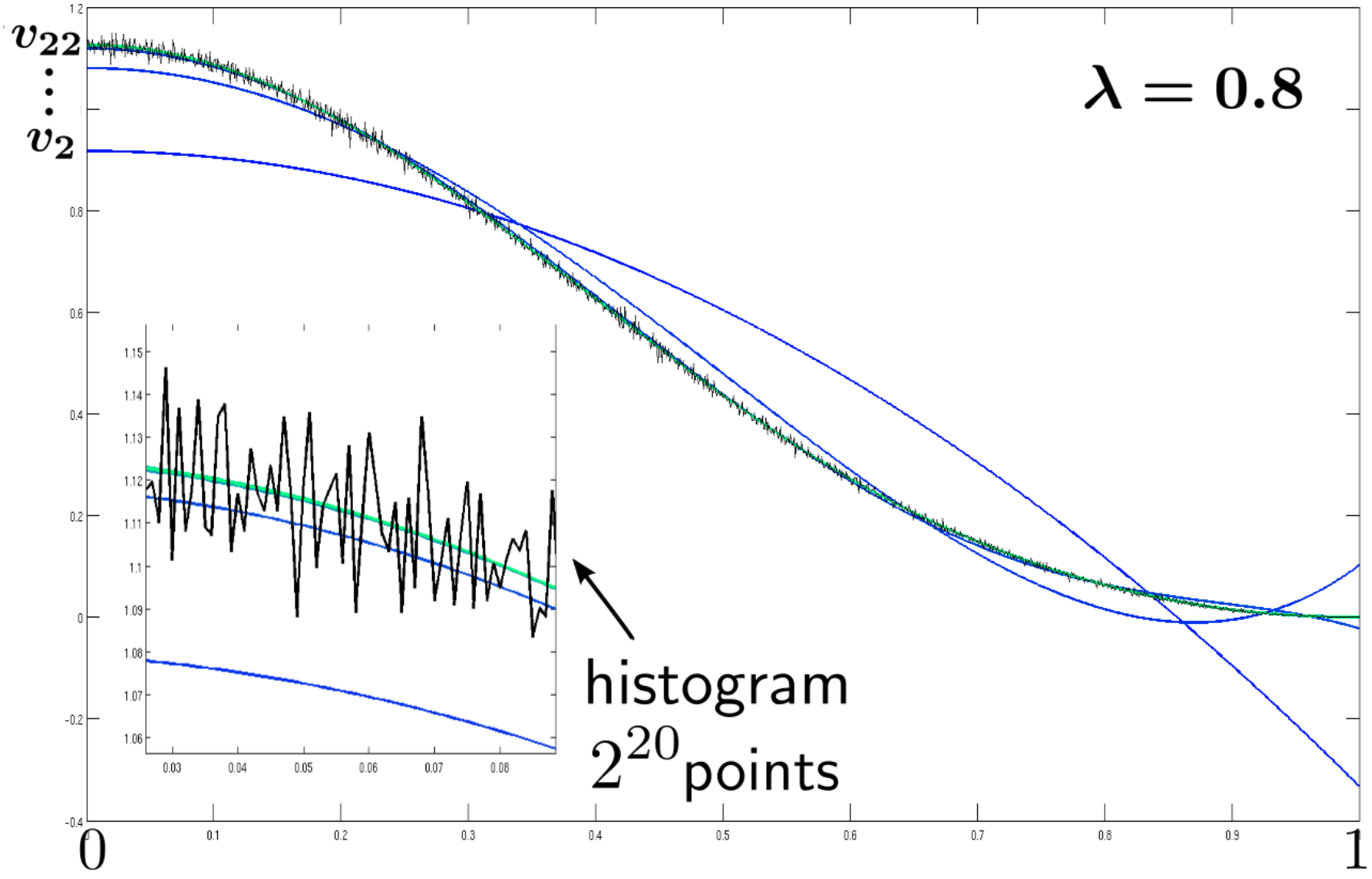
$$\mathbf{G(u_0, u_1, \ldots, u_n)' = (m_0, m_1, \ldots, m_n)'}$$

with the **Hilbert matrix** $\mathbf{G} \in \mathbb{K}^{\mathbf{(n+1)\times(n+1)}}$.

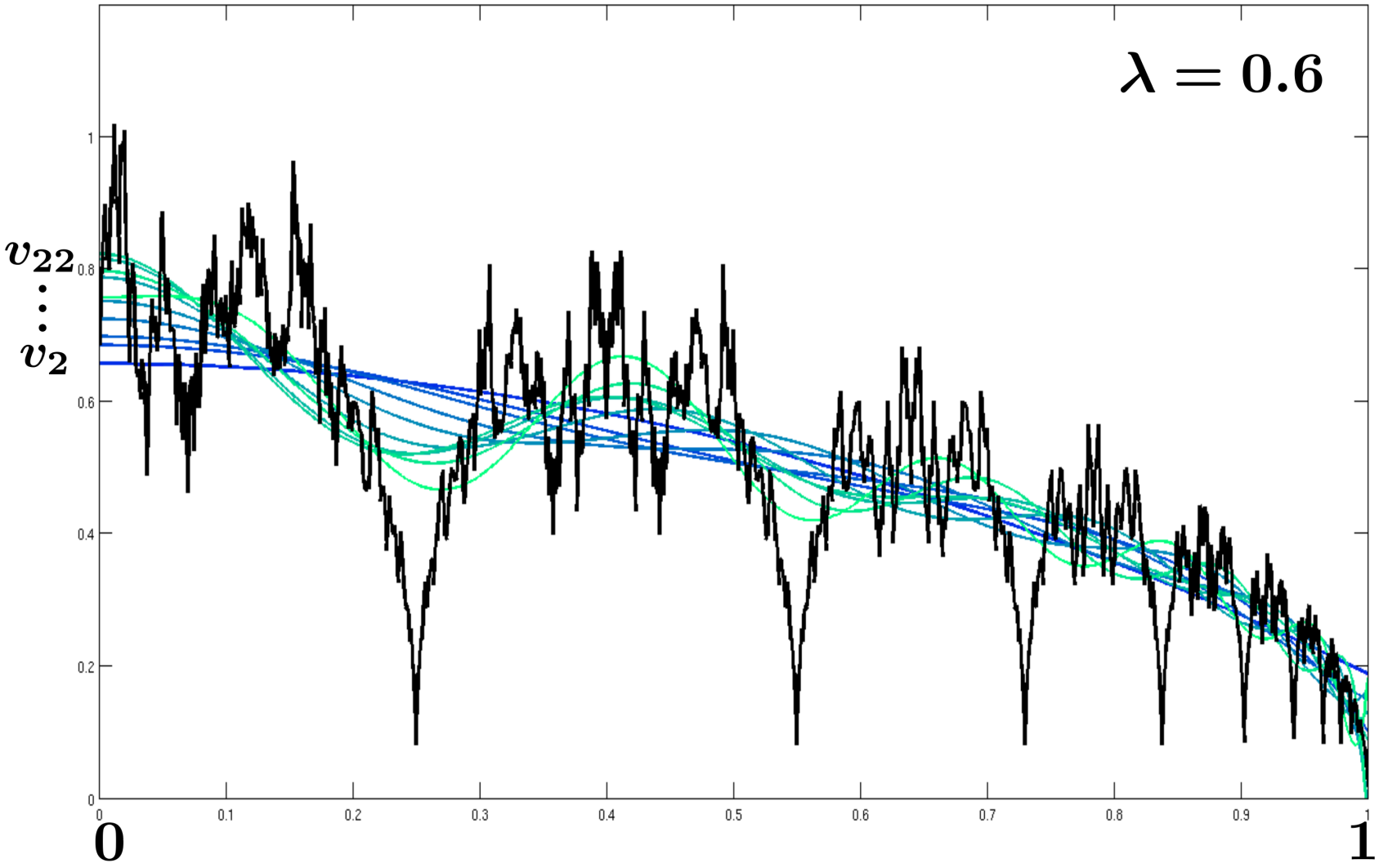
$$\mathbf{G} = 2 \begin{bmatrix} 1 & 0 & \frac{1}{3} & 0 & \frac{1}{5} & \ldots \\ 0 & \frac{1}{3} & 0 & \frac{1}{5} & 0 & \ldots \\ \frac{1}{3} & 0 & \frac{1}{5} & 0 & \frac{1}{7} & \ldots \\ 0 & \frac{1}{5} & 0 & \frac{1}{7} & 0 & \ldots \\ \vdots & . & . & . & . & \ldots \end{bmatrix}$$

$\mathbf{m_k}$ are the moments of ν

Bernoulli IFS – Densities



Bernoulli IFS – Densities



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Polynomial eigenfunctions of T

Let $\boldsymbol{\lambda} = \mathbf{0.5} - \mathbf{0.5i}$. We get the eigenpolynomials of

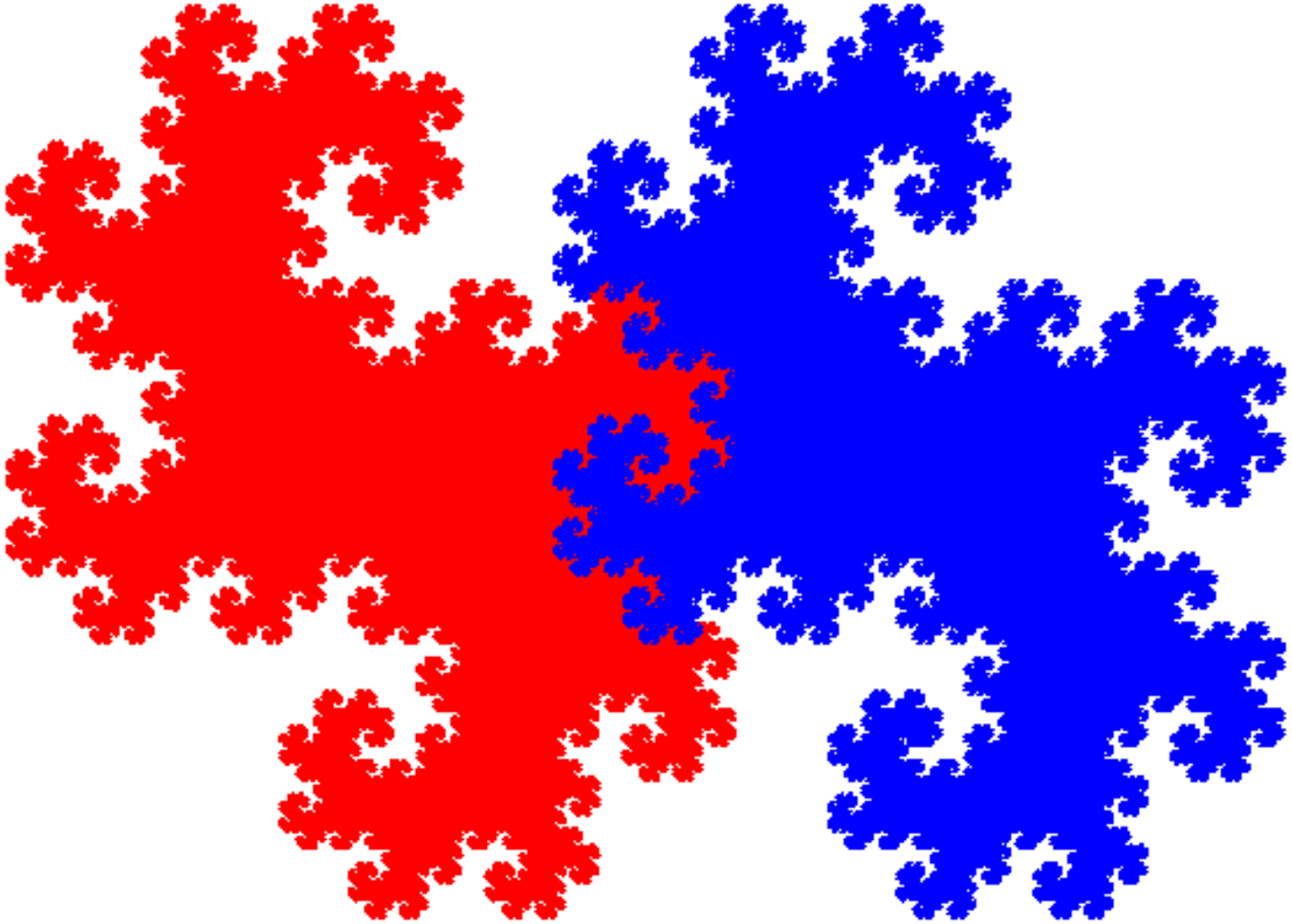
$$\mathbf{Th}(\mathbf{z}) = \frac{\mathbf{1}}{\mathbf{2}}\mathbf{h}(\boldsymbol{\lambda}\mathbf{z}) + \frac{\mathbf{1}}{\mathbf{2}}\mathbf{h}(\boldsymbol{\lambda}\mathbf{z} + \mathbf{1})$$

of degree ≤ 3 from the matrix

$$\mathbf{T_3} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0.5 - 0.5\,\mathrm{i} & 1 - \mathrm{i} & 1.5 - 1.5\,\mathrm{i} \\ 0 & 0 & -0.5\,\mathrm{i} & -1.5\,\mathrm{i} \\ 0 & 0 & 0 & -0.25 - 0.25\,\mathrm{i} \end{pmatrix}$$

Polynomial eigenfunctions of T

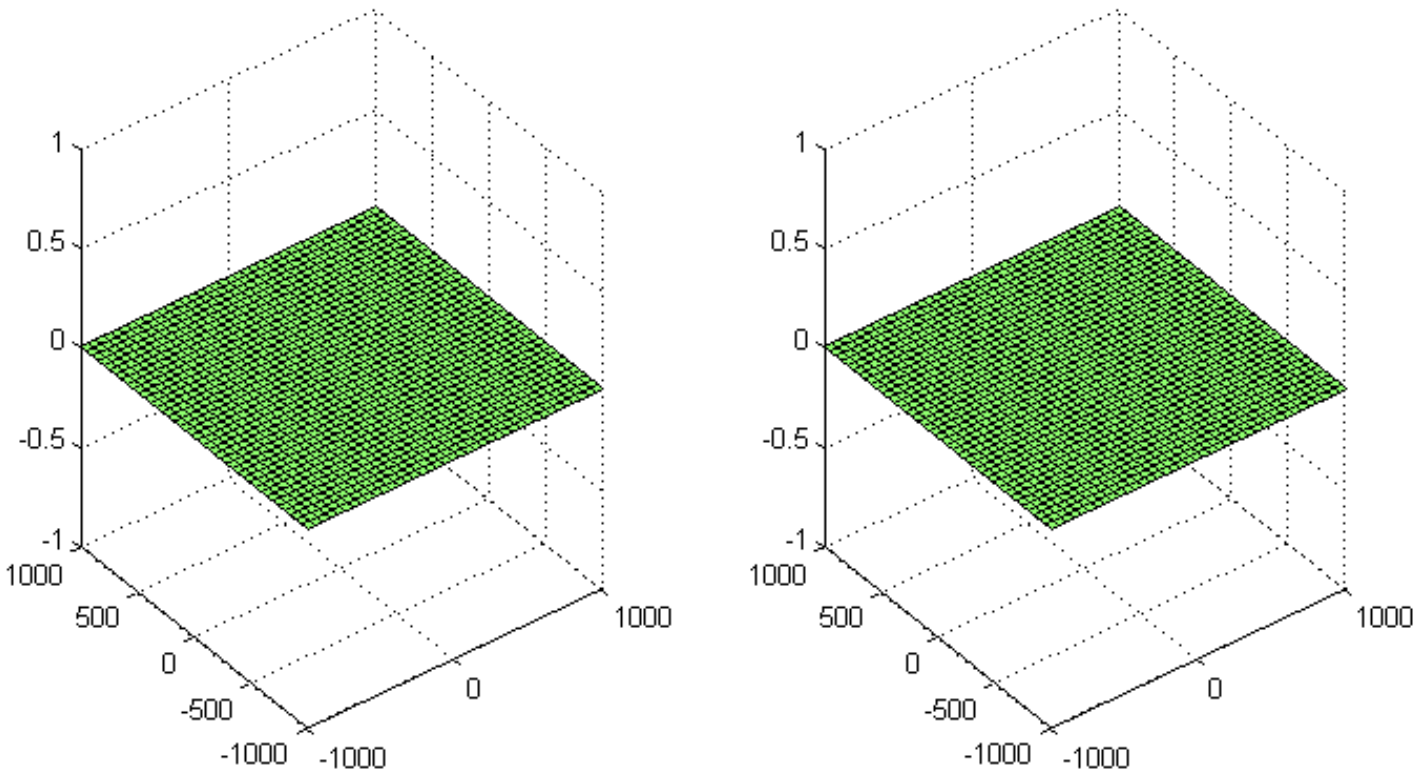
are orthogonal to a measure supported on the set



dragon curve

Polynomial eigenfunctions of T

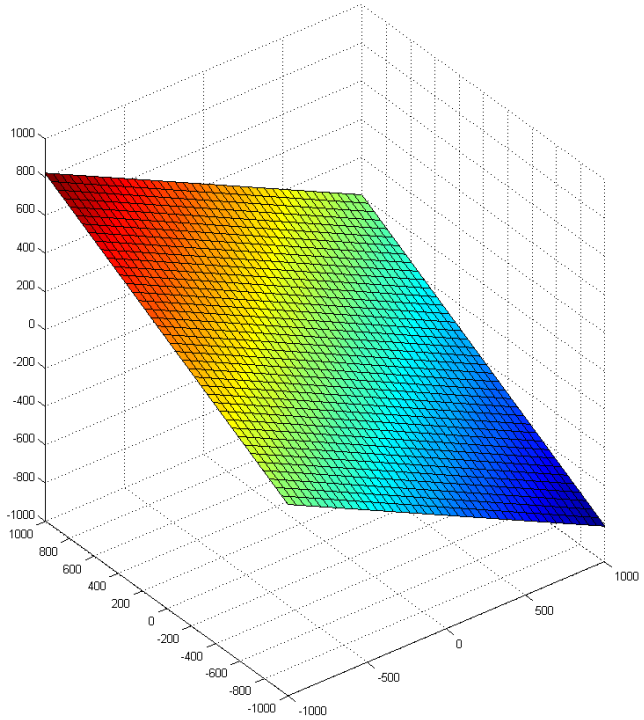
degree 0

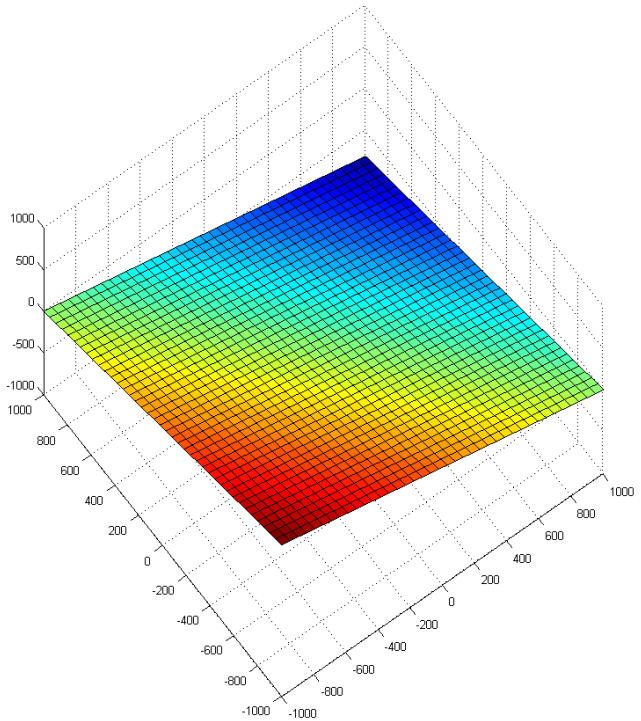


left: real part, right: imaginary part

Polynomial eigenfunctions of T

degree 1

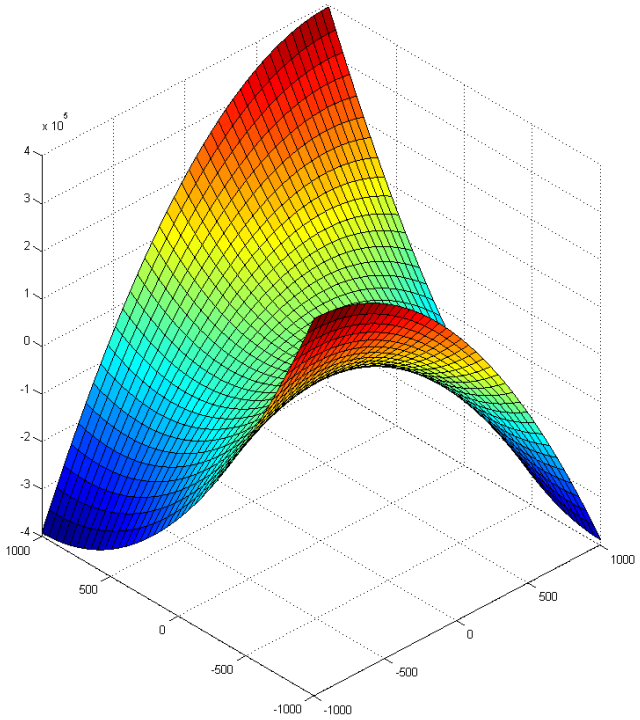


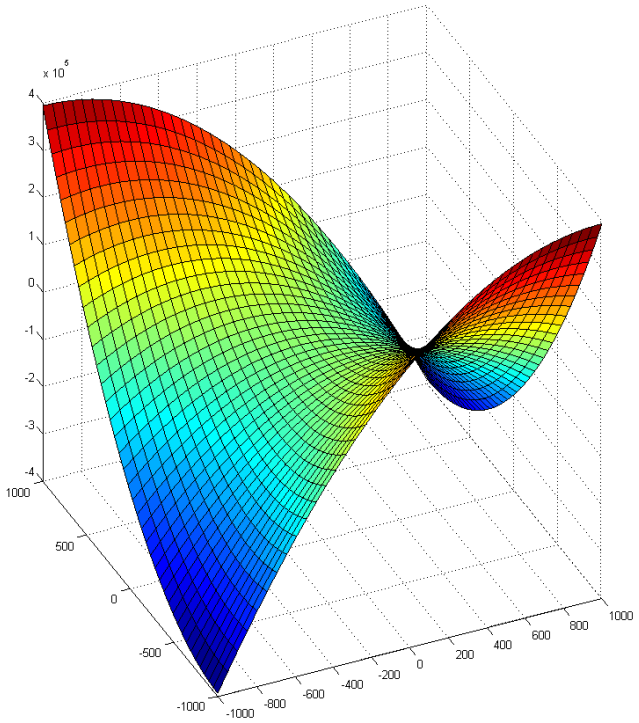


left: real part, right: imaginary part

Polynomial eigenfunctions of T

degree 2

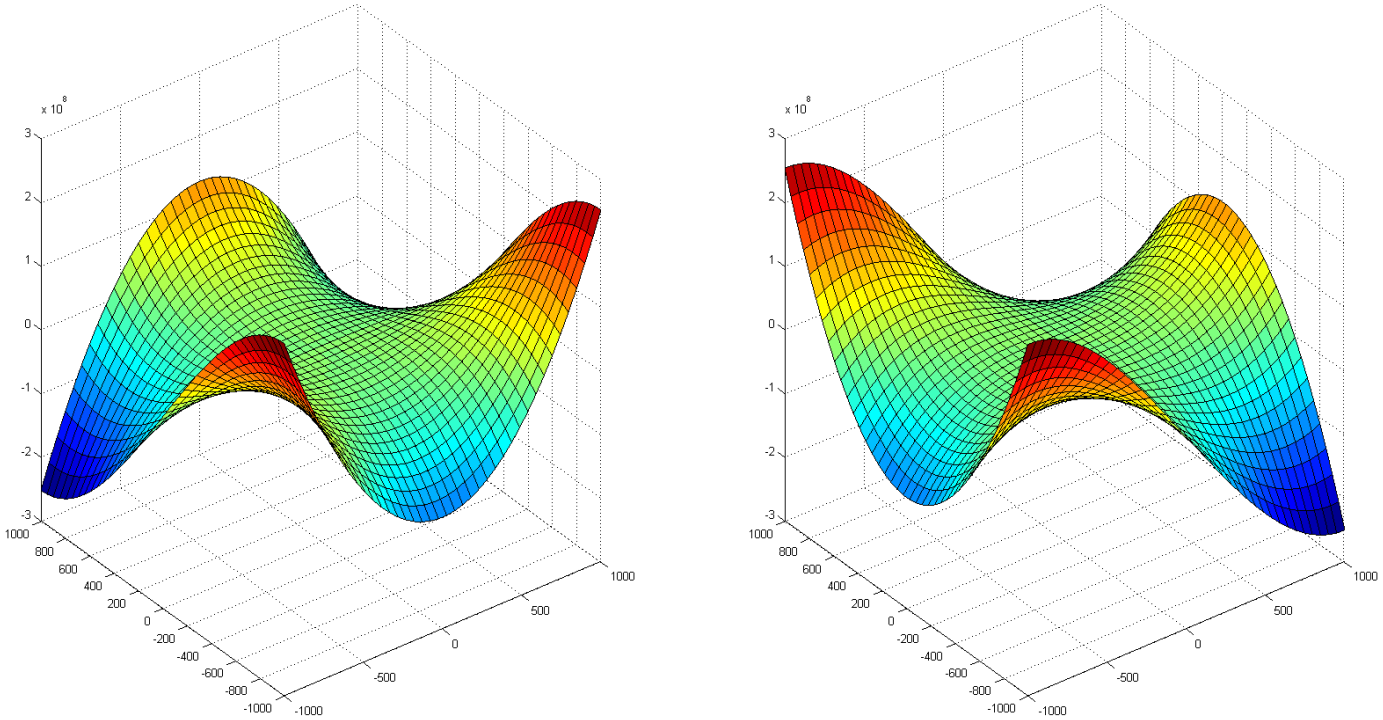




left: real part, right: imaginary part

Polynomial eigenfunctions of T

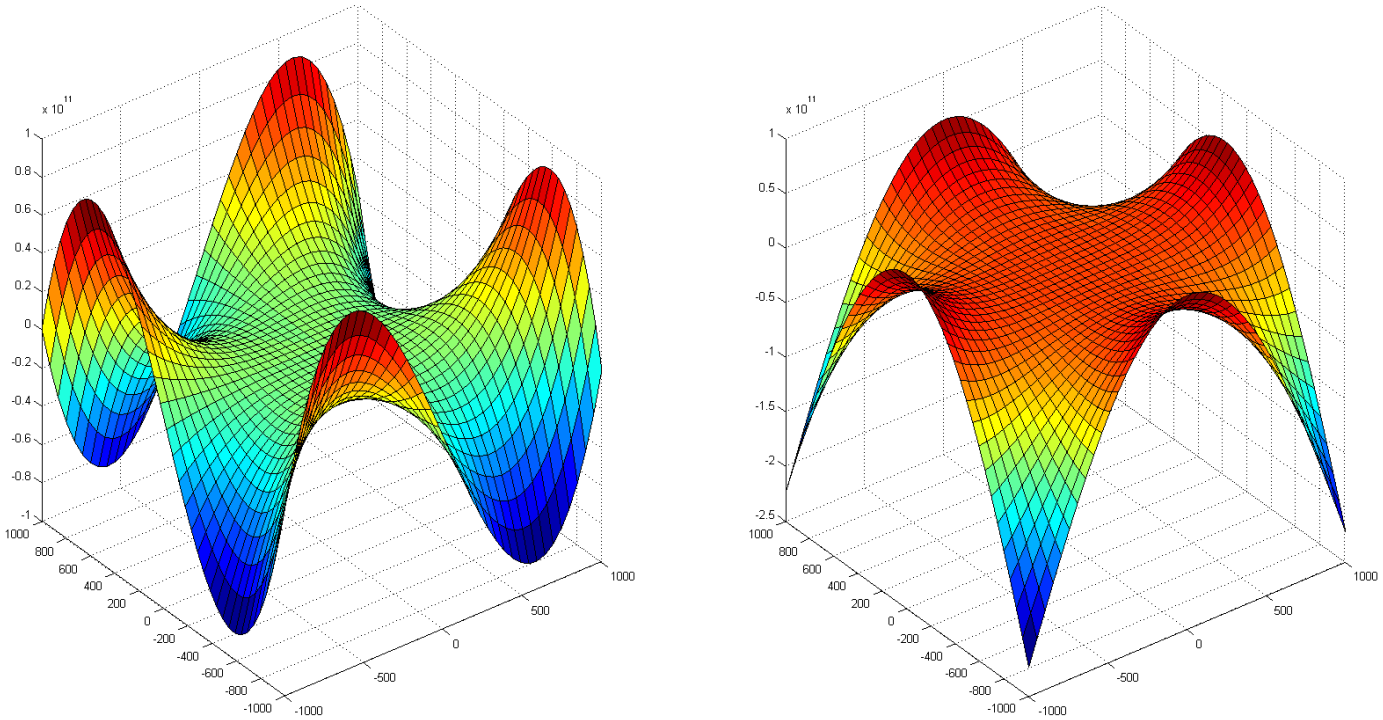
degree 3



left: real part, right: imaginary part

Polynomial eigenfunctions of T

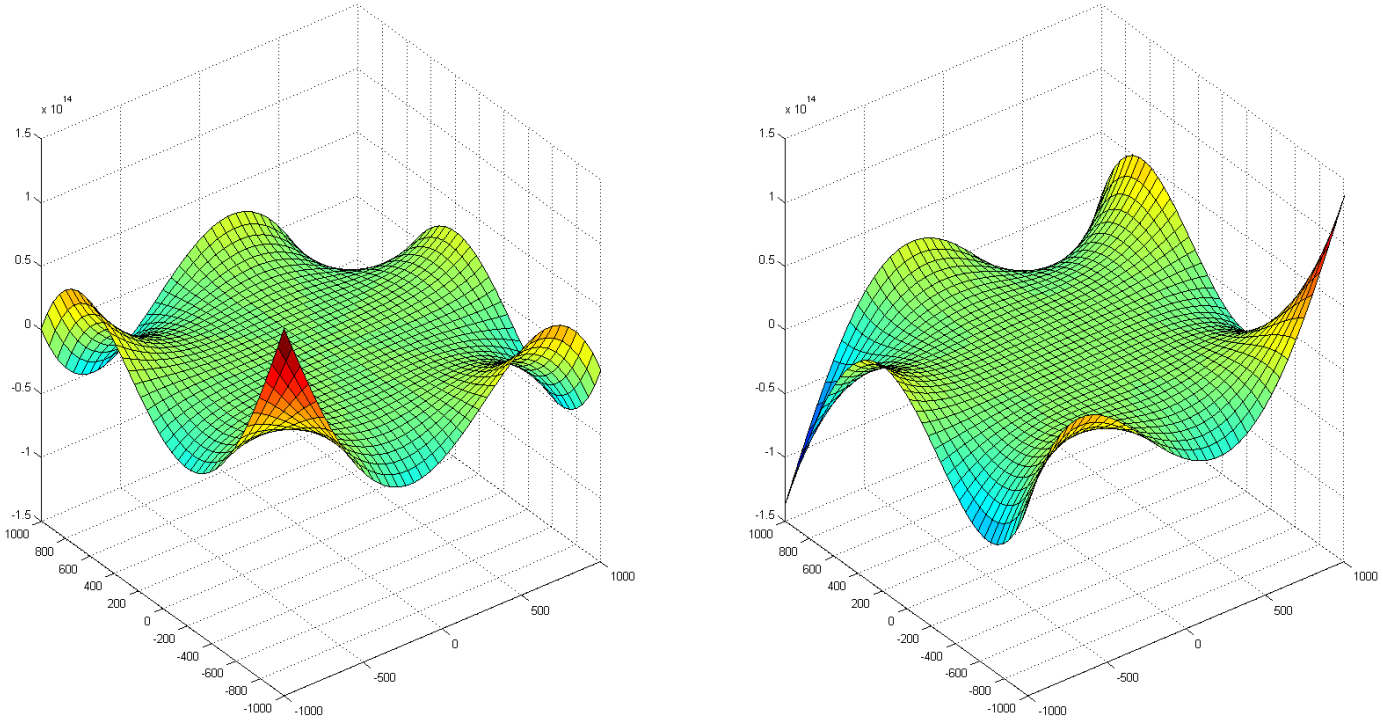
degree 4



left: real part, right: imaginary part

Polynomial eigenfunctions of T

degree 5



left: real part, right: imaginary part

Recursions

Moments of ν_λ

$$m_{2k} = -\sum_{i=1}^{k} a_{2k,i} m_{2k-2i}$$

with $m_0 = 1$ and

$$a_{n,i} = \frac{1}{\lambda^{2k-1}} \sum_{j=0}^{i-1} \binom{n-2j}{n-2i} (1-\lambda)^{2i-2j} a_{n,j}$$

the coefficient of x^i in the eigenpolynomial p_n of A with $a_{n,0} = 1$.

Recursions

Legendre moments of ν_λ

$$(\nu_\lambda, L_{2n}) = \sum_{k=0}^{n} (-1)^{n-k} 4^{-n} \binom{2n+2k}{2k} \binom{2n}{n+k} m_{2k}$$

where

$$m_{2k} = (\nu_\lambda, x^{2k})$$

are the moments of the convolution measure.

Recursions

Moments of ν_λ, another recursion:

$$m_{2k} = \frac{1}{1-\lambda^{2k}} \sum_{j=0}^{k-1} b_{2k,\lambda}(2j) m_{2j}$$

where $b_{n,\lambda}(\cdot)$ are the weights of the binomial distribution with parameters n and λ and $m_0 = 1$