Stein-Malliavin Approximations for Nonlinear Functionals of Random Eigenfunctions on $S^d$

Maurizia Rossi
Department of Mathematics, University of Rome “Tor Vergata”

(joint work with Domenico Marinucci)

Berlin – Padova Young Researchers Meeting
Stochastic Analysis and applications in Biology, Finance and Physics

WIAS Berlin - October 23, 2014

Research supported by ERC Grant 277742 Pascal
Introduction
- What are random eigenfunctions on $S^d$?
- Why are they so important?
- Aim of our work: quantitative CLT’s for nonlinear functionals of random eigenfunctions on $S^d$

2 Stein-Malliavin Normal approximations on $S^d$
- CLT’s via Wiener chaos decomposition
- Fourth moment theorems

3 Quantitative CLT’s for nonlinear functionals of random eigenfunctions on $S^d$
- Hermite transforms
- Arbitrary polynomial transforms
- General nonlinear functionals
  - Empirical measure of excursion sets
eigenfunctions on the $d$-dimensional sphere, $d \geq 2$

$\mathbb{S}^d \subset \mathbb{R}^{d+1} \rightarrow$ unit $d$-dim sphere
$\Delta_{\mathbb{S}^d} \rightarrow$ Laplacian operator on $\mathbb{S}^d$

- The eigenvalues of $\Delta_{\mathbb{S}^d}$ are $E_\ell := -\ell(\ell + d - 1), \ell \in \mathbb{N}$. 

eigenfunctions on the $d$-dimensional sphere, $d \geq 2$

$S^d \subset \mathbb{R}^{d+1} \longrightarrow \text{unit } d\text{-dim sphere}$

$\Delta_{S^d} \longrightarrow \text{Laplacian operator on } S^d$

- The eigenvalues of $\Delta_{S^d}$ are $E_\ell := -\ell(\ell + d - 1)$, $\ell \in \mathbb{N}$.
- The dimension of the eigenspace $H_\ell$ corresponding to $E_\ell$ is

$$n_{\ell,d} = \frac{2\ell + d - 1}{\ell} \left( \begin{array}{c} \ell + d - 2 \\ \ell - 1 \end{array} \right) \sim \frac{2}{(d-1)!} \ell^{d-1} \quad \text{as } \ell \to +\infty$$

(the number of l.i. homogeneous polynomials of degree $\ell$ in $d + 1$ variables).
eigenfunctions on the $d$-dimensional sphere, $d \geq 2$

$$S^d \subset \mathbb{R}^{d+1} \longrightarrow \text{unit } d\text{-dim sphere}$$
$$\Delta_{S^d} \longrightarrow \text{Laplacian operator on } S^d$$

- The eigenvalues of $\Delta_{S^d}$ are $E_\ell := -\ell(\ell + d - 1)$, $\ell \in \mathbb{N}$.
- The dimension of the eigenspace $H_\ell$ corresponding to $E_\ell$ is

$$n_{\ell,d} = \frac{2\ell + d - 1}{\ell} \binom{\ell + d - 2}{\ell - 1} \sim \frac{2}{(d - 1)!} \ell^{d-1} \text{ as } \ell \to +\infty$$

(the number of l.i. homogeneous polynomials of degree $\ell$ in $d+1$ variables).

- We consider the real orthonormal basis for $H_\ell$ given by the spherical harmonics in $d+1$-dimension $(Y_{\ell,m;d})_m$, $m = 1, 2, \ldots, n_{\ell,d}$,

$$\Delta_{S^d} Y_{\ell,m;d} = -\ell(\ell + d - 1) Y_{\ell,m;d}.$$
eigenfunctions on the \(d\)-dimensional sphere, \(d \geq 2\)

\[ S^d \subset \mathbb{R}^{d+1} \rightarrow \text{unit } d\text{-dim sphere} \]
\[ \Delta_{S^d} \rightarrow \text{Laplacian operator on } S^d \]

- The eigenvalues of \(\Delta_{S^d}\) are \(E_\ell := -\ell(\ell + d - 1), \ell \in \mathbb{N}\).
- The dimension of the eigenspace \(H_\ell\) corresponding to \(E_\ell\) is
  \[
  n_{\ell,d} = \frac{2\ell + d - 1}{\ell} \binom{\ell + d - 2}{\ell - 1} \sim \frac{2}{(d-1)!} \ell^{d-1} \text{ as } \ell \to +\infty
  \]
  (the number of l.i. homogeneous polynomials of degree \(\ell\) in \(d + 1\) variables).

- We consider the real orthonormal basis for \(H_\ell\) given by the spherical harmonics in \(d + 1\)-dimension \((Y_{\ell,m;d})_m, m = 1, 2, \ldots, n_{\ell,d}\),
  \[
  \Delta_{S^d} Y_{\ell,m;d} = -\ell(\ell + d - 1)Y_{\ell,m;d}.
  \]

- Each real-valued \(f \in L^2(S^d)\) admits the Fourier development
  \[
  f = \sum_{\ell \in \mathbb{N}} \sum_{m=1}^{n_{\ell,d}} \langle f, Y_{\ell,m;d} \rangle Y_{\ell,m;d}
  \]
random eigenfunctions on the $d$-dimensional sphere, $d \geq 2$

- What are random eigenfunctions on $S^d$?
  They are a linear combination of spherical harmonics of fixed degree with random coefficients.
random eigenfunctions on the $d$-dimensional sphere, $d \geq 2$

- What are random eigenfunctions on $S^d$? They are a linear combination of spherical harmonics of fixed degree with random coefficients.

- Precisely: for each $\ell \in \mathbb{N}$, we construct the random eigenfunction $T_\ell$ as

$$T_\ell = \sum_{m=1}^{n_{\ell,d}} a_{\ell,m;d} Y_{\ell,m;d},$$

where $(a_{\ell,m;d})_{m=1}^{n_{\ell,d}}$ are i.i.d. zero-mean Gaussian r.v.'s with $E[(a_{\ell,m;d})^2] = \mu_d n_{\ell,d}$, $\mu_d = \frac{\pi}{\Gamma(\frac{d+1}{2})^2} \text{denoting the measure of the surface of } S^d.$
random eigenfunctions on the \(d\)-dimensional sphere, \(d \geq 2\)

- What are random eigenfunctions on \(S^d\)?
  They are a linear combination of spherical harmonics of fixed degree with random coefficients.

- Precisely: for each \(\ell \in \mathbb{N}\), we construct the random eigenfunction \(T_\ell\) as

\[
T_\ell = \sum_{m=1}^{n_{\ell,d}} a_{\ell,m;d} Y_{\ell,m;d},
\]

- where \((a_{\ell,m;d})_{m=1,...,n_{\ell,d}}\) are i.i.d. zero-mean Gaussian r.v.’s with

\[
\mathbb{E}[(a_{\ell,m;d})^2] = \frac{\mu_d}{n_{\ell,d}},
\]

\[
\mu_d = \frac{2\pi^{\frac{d+1}{2}}}{\Gamma\left(\frac{d+1}{2}\right)}
\]
denoting the measure of the surface of \(S^d\).
Main properties

- \( T_\ell \) is a **Gaussian** and **isotropic** field, i.e. its **law** is **invariant** under the action of \( SO(d+1) \), that is \( \forall g \in SO(d+1) \),

\[
T_\ell(\cdot) \overset{\text{law}}{=} T_\ell(g \cdot) .
\]

- \( T_\ell \) is centered and \( E[T_\ell(x) T_\ell(y)] = G_\ell; d(\cos d(x,y)) \), \( d(x,y) \) is the **spherical geodesic distance** between \( x \) and \( y \).

- \( G_\ell; d \) is the \( \ell \)-th **Gegenbauer polynomial**, normalized in such a way that \( G_\ell; d(1) = 1 \).

- Precisely, \( G_\ell; d \) is defined as \( P_\ell(d^2-1, d^2-1) \), where \( P_\alpha,\beta \) are **Jacobi polynomials**.

- Gegenbauer polynomials \( (G_\ell; d) \) are orthogonal polynomials on the interval \([-1, 1]\) with respect to the weight \( w(t) = (1-t^2)^{d/2-1} \).

For instance, if \( d = 2 \), then \( G_\ell; 2 = P_\ell \) the Legendre polynomials.
Main properties

- $T_\ell$ is a Gaussian and isotropic field, i.e. its law is invariant under the action of $SO(d + 1)$, that is $\forall g \in SO(d + 1),\$

$$T_\ell(\cdot) \overset{\text{law}}{=} T_\ell(g \cdot).$$

- $T_\ell$ is centered and

$$\mathbb{E}[T_\ell(x)T_\ell(y)] = G_{\ell;d}(\cos d(x, y)),$$

where $d(x, y) =$ spherical geodesic distance between $x$ and $y$

$G_{\ell;d} = \ell$-th Gegenbauer polynomial, normalized in such a way that $G_{\ell;d}(1) = 1.$
Main properties

- \( T_\ell \) is a Gaussian and isotropic field, i.e. its law is invariant under the action of \( SO(d + 1) \), that is \( \forall g \in SO(d + 1), \)

\[
T_\ell(\cdot) \overset{\text{law}}{=} T_\ell(g\cdot).
\]

- \( T_\ell \) is centered and

\[
\mathbb{E}[T_\ell(x)T_\ell(y)] = G_{\ell,d}(\cos d(x, y)),
\]

\( d(x, y) = \) spherical geodesic distance between \( x \) and \( y \)

\( G_{\ell,d} = \ell\)-th Gegenbauer polynomial, normalized in such a way that \( G_{\ell,d}(1) = 1. \)

- Precisely

\[
G_{\ell,d} = \frac{P_\ell^{\left(\frac{d}{2} - 1, \frac{d}{2} - 1\right)}}{(\ell + \frac{d}{2} - 1)},
\]

where \( P_\ell^{(\alpha, \beta)} \) are Jacobi polynomials.
Main properties

• $T_\ell$ is a Gaussian and isotropic field, i.e. its law is invariant under the action of $SO(d + 1)$, that is $\forall g \in SO(d + 1)$,

\[ T_\ell(\cdot) \overset{\text{law}}{=} T_\ell(g \cdot) . \]

• $T_\ell$ is centered and

\[ \mathbb{E}[T_\ell(x)T_\ell(y)] = G_{\ell;d}(\cos d(x, y)) , \]

$d(x, y) =$ spherical geodesic distance between $x$ and $y$

$G_{\ell;d} =$ $\ell$-th Gegenbauer polynomial, normalized in such a way that

$G_{\ell;d}(1) = 1$.

• Precisely

\[ G_{\ell;d} = \frac{P_{\ell}^{\left( \frac{d}{2} - 1, \frac{d}{2} - 1 \right)}}{\left( \ell + \frac{d}{2} - 1 \right)} , \]

where $P_{\ell}^{(\alpha, \beta)}$ are Jacobi polynomials.

• Gegenbauer polynomials $(G_{\ell;d})_\ell$ are orthogonal polynomials on the interval $[-1, 1]$ with respect to the weight $w(t) = (1 - t^2)^{d/2 - 1}$.

For instance, if $d = 2$, then $G_{\ell;2} = P_{\ell}$ the Legendre polynomials.
why do we study spherical random eigenfunctions?

There are many reasons.

- Every Gaussian and isotropic random field \( T \) on \( S^d \) is mean-square continuous (Marinucci and Peccati, 2013) and satisfy in \( L^2 \) the spectral representation

\[
T(x) = \sum_{\ell=1}^{\infty} c_\ell T_\ell(x), \quad \mathbb{E} [T(x)^2] = \sum_{\ell=1}^{\infty} c_\ell^2 < \infty,
\]

where the deterministic sequence \((c_\ell)\) is the power spectrum of \( T \).
why do we study spherical random eigenfunctions?

There are many reasons.

• Every Gaussian and isotropic random field $T$ on $S^d$ is mean-square continuous (Marinucci and Peccati, 2013) and satisfy in $L^2$ the spectral representation

\[
T(x) = \sum_{\ell=1}^{\infty} c_\ell T_\ell(x), \quad \mathbb{E} [T(x)^2] = \sum_{\ell=1}^{\infty} c_\ell^2 < \infty,
\]

where the deterministic sequence $(c_\ell)_\ell$ is the power spectrum of $T$.

• $\Rightarrow (T_\ell)_{\ell \in \mathbb{N}}$ can be viewed as the Fourier components of $T$. 

M. Rossi (Rome Tor Vergata) 
Stein-Malliavin approximations on $S^d$ 

WIAS Berlin - October 23, 2014 6 / 27
why do we study spherical random eigenfunctions?

There are many reasons.

• Every Gaussian and isotropic random field $T$ on $S^d$ is mean-square continuous (Marinucci and Peccati, 2013) and satisfy in $L^2$ the spectral representation

$$T(x) = \sum_{\ell=1}^{\infty} c_\ell T_\ell(x), \quad \mathbb{E} [T(x)^2] = \sum_{\ell=1}^{\infty} c_\ell^2 < \infty,$$

where the deterministic sequence $(c_\ell)_{\ell}$ is the power spectrum of $T$.

• $\Rightarrow (T_\ell)_{\ell \in \mathbb{N}}$ can be viewed as the Fourier components of $T$.

• Moreover they are important in physical contexts, mainly related to the analysis of isotropic spherical random fields on $S^2$, (in connection with the analysis of Cosmic Microwave Background).
why do we study spherical random eigenfunctions?

There are many reasons.

• Every Gaussian and isotropic random field $T$ on $S^d$ is mean-square continuous ([Marinucci and Peccati, 2013]) and satisfy in $L^2$ the spectral representation

$$T(x) = \sum_{\ell=1}^{\infty} c_{\ell} T_{\ell}(x), \quad \mathbb{E} [T(x)^2] = \sum_{\ell=1}^{\infty} c_{\ell}^2 < \infty,$$

where the deterministic sequence $(c_{\ell})_{\ell}$ is the power spectrum of $T$.

• $\Rightarrow (T_{\ell})_{\ell \in \mathbb{N}}$ can be viewed as the Fourier components of $T$.

• Moreover they are important in physical contexts, mainly related to the analysis of isotropic spherical random fields on $S^2$, (in connection with the analysis of Cosmic Microwave Background).

• According to Berry’s Universality conjecture ([Berry, 1977]) random Gaussian monochromatic waves (similar to e.g. random Gaussian spherical harmonics) could model deterministic eigenfunctions on a “generic” manifold with or without boundary.
our work

⇒ growing interest for geometric functionals of spherical random eigenfunctions
  • more generally nonlinear functionals of random eigenfunctions on compact manifolds (e.g. Nazarov and Sodin, 2009 - Granville and Wigman, 2011 - Krishnapur, Kurlberg and Wigman, 2013 - Marinucci and Wigman, 2014).

AIM: investigate quantitative CLT’s for nonlinear functionals of $T_\ell$, as $\ell \to +\infty$.

We consider functionals of the form $S_\ell(M) = \int S_d M(T_\ell(x)) \, dx$,

where $M: \mathbb{R} \to \mathbb{R}$ is any square integrable measurable function.

MAIN RESULT: Under a mild assumption on $M$, the Normal approximation $S_\ell(M) - E[S_\ell(M)] \sqrt{\text{Var}[S_\ell(M)]} \xrightarrow{L} N(0, 1)$.

The rate of convergence w.r.t. the Wasserstein distance = $O(\ell^{-1/2})$.

Important application (Excursion area) If $M = 1(\cdot \leq z)$, for $z \in \mathbb{R}$, then $S_\ell(z) = \int S_d 1(T_\ell(x) \leq z) \, dx$. 

M. Rossi (Rome Tor Vergata)}
our work

growing interest for geometric functionals of spherical random eigenfunctions

more generally nonlinear functionals of random eigenfunctions on compact manifolds (e.g. Nazarov and Sodin, 2009 - Granville and Wigman, 2011 - Krishnapur, Kurlberg and Wigman, 2013 - Marinucci and Wigman, 2014).

AIM: investigate quantitative CLT’s for nonlinear functionals of \( T_\ell \), as \( \ell \to +\infty \).

We consider functionals of the form

\[
S_\ell(M) = \int_{S^d} M(T_\ell(x)) \, dx ,
\]

where \( M : \mathbb{R} \to \mathbb{R} \) is any square integrable measurable function.
our work

⇒ growing interest for geometric functionals of spherical random eigenfunctions
• more generally nonlinear functionals of random eigenfunctions on compact
manifolds (e.g. Nazarov and Sodin, 2009 - Granville and Wigman, 2011
- Krishnapur, Kurlberg and Wigman, 2013 - Marinucci and Wigman,
2014).
• AIM : investigate quantitative CLT’s for nonlinear functionals of $T_\ell$, as
$\ell \to +\infty$.

We consider functionals of the form

$$S_\ell(M) = \int_{S^d} M(T_\ell(x)) \, dx ,$$

where $M : \mathbb{R} \to \mathbb{R}$ is any square integrable measurable function.
• MAIN RESULT : Under a mild assumption on $M$, the Normal approximation

$$\frac{S_\ell(M) - \mathbb{E}[S_\ell(M)]}{\sqrt{\text{Var}[S_\ell(M)]}} \xrightarrow{L} \mathcal{N}(0, 1) .$$

The rate of convergence w.r.t. the Wasserstein distance $= O(\ell^{-1/2})$. 
our work

⇒ growing interest for geometric functionals of spherical random eigenfunctions
• more generally nonlinear functionals of random eigenfunctions on compact manifolds (e.g. Nazarov and Sodin, 2009 - Granville and Wigman, 2011 - Krishnapur, Kurlberg and Wigman, 2013 - Marinucci and Wigman, 2014).

• AIM : investigate quantitative CLT’s for nonlinear functionals of $T_\ell$, as $\ell \to +\infty$.

We consider functionals of the form

$$S_\ell(M) = \int_{S^d} M(T_\ell(x)) \, dx,$$

where $M : \mathbb{R} \to \mathbb{R}$ is any square integrable measurable function.

• MAIN RESULT : Under a mild assumption on $M$, the Normal approximation

$$\frac{S_\ell(M) - \mathbb{E}[S_\ell(M)]}{\sqrt{\text{Var}[S_\ell(M)]}} \xrightarrow{\mathcal{L}} \mathcal{N}(0, 1).$$

The rate of convergence w.r.t. the Wasserstein distance $= O(\ell^{-1/2})$.

• Important application (Excursion area) If $M = 1(\cdot \leq z)$, for $z \in \mathbb{R}$, then

$$S_\ell(z) = \int_{S^d} 1(T_\ell(x) \leq z) \, dx.$$
• Isonormal Gaussian field \( X \) on \( L^2(S^d) \)
  
  Centered Gaussian family \( \{X(f), \ f \in L^2(S^d)\} \) with covariance structure
  
  \[
  \text{Cov} (X(f), X(f')) = \langle f, f' \rangle_2
  \]
**background: wiener chaos**

- **Isonormal Gaussian field** $X$ on $L^2(S^d)$
  
  Centered Gaussian family $\{X(f), f \in L^2(S^d)\}$ with covariance structure
  
  $\text{Cov}(X(f), X(f')) = \langle f, f' \rangle_2$

- For each $q \geq 0$, the $q$-th Wiener chaos $\mathcal{H}_q$ of $X$ is the closure in $L^2(\mathbb{P})$ of the subspace generated by r.v.’s of the form

  $$H_q(X(f)), \quad f \in L^2(S^d), \quad \|f\|_2 = 1,$$

  where $H_q$ is the $q$-th Hermite polynomial.
background: wiener chaoses

- Isonormal Gaussian field $X$ on $L^2(\mathbb{S}^d)$
  
  Centered Gaussian family $\{X(f), f \in L^2(\mathbb{S}^d)\}$ with covariance structure
  
  $\text{Cov}(X(f), X(f')) = \langle f, f' \rangle_2$

- For each $q \geq 0$, the $q$-th Wiener chaos $\mathcal{H}_q$ of $X$ is the closure in $L^2(\mathbb{P})$ of the subspace generated by r.v.’s of the form

  $$H_q(X(f)), \quad f \in L^2(\mathbb{S}^d), \quad \|f\|_2 = 1,$$

  where $H_q$ is the $q$-th Hermite polynomial.

- Wiener-Itô chaos decomposition into orthogonal subspaces

  $$L^2(\mathbb{P}) = \bigoplus_{q \geq 0} \mathcal{H}_q$$

  $$\updownarrow$$

  $$F \in L^2(\mathbb{P}) \Rightarrow F = \mathbb{E}[F] + \sum_{q \geq 1} F_q$$

  $F_q$ being the projection of $F$ on $\mathcal{H}_q$. 

**background: wiener chaoses**

- **Isonormal Gaussian field** $X$ on $L^2(\mathbb{S}^d)$
  Centered Gaussian family $\{X(f), \ f \in L^2(\mathbb{S}^d)\}$ with covariance structure
  \[ \text{Cov}(X(f), X(f')) = \langle f, f' \rangle_2 \]

- For each $q \geq 0$, the $q$-th Wiener chaos $\mathcal{H}_q$ of $X$ is the closure in $L^2(\mathbb{P})$ of the subspace generated by r.v.'s of the form
  \[ H_q(X(f)), \ f \in L^2(\mathbb{S}^d), \ |f|_2 = 1, \]
  where $H_q$ is the $q$-th Hermite polynomial.

- **Wiener-Itô chaos decomposition** into orthogonal subspaces
  \[ L^2(\mathbb{P}) = \bigoplus_{q \geq 0} \mathcal{H}_q \]
  \[ F \in L^2(\mathbb{P}) \Rightarrow F = \mathbb{E}[F] + \sum_{q \geq 1} F_q \]
  $F_q$ being the projection of $F$ on $\mathcal{H}_q$.

- **Borel function** $\varphi : \mathbb{R} \rightarrow \mathbb{R} : F = \varphi(X(f)) \in L^2(\mathbb{P})$
  \[ \varphi(X(f)) = \sum_{q \geq 0} \frac{J_q(\varphi)}{q!} H_q(X(f)), \quad J_q(\varphi) = \mathbb{E}[\varphi(X(f))H_q(X(f))] \]
clt’s via chaos decomposition

- Isonormal representation $T_\ell(x) = X(f_x)$, $f_x(\cdot) = \sqrt{\frac{n\ell;d}{\mu_d}} G_\ell(\cos d(x, \cdot))$
**clt’s via chaos decomposition**

- Isonormal representation \( T_\ell(x) = X(f_x) \), \( f_x(\cdot) = \sqrt{\frac{n_\ell;d}{\mu_d}} G_\ell(\cos d(x, \cdot)) \)

- \( S_\ell(M) = \int_{S^d} M(X(f_x)) \, dx \in L^2(\mathbb{P}) \Rightarrow \) Wiener-Itô chaos expansion

\[
S_\ell(M) = \sum_{q=0}^{\infty} \frac{J_q(M)}{q!} \int_{S^d} H_q(T_\ell(x)) \, dx 
\]  
(2.1)
clt’s via chaos decomposition

- Isonormal representation \( T_\ell(x) = X(f_x) \), \( f_x(\cdot) = \sqrt{\frac{n_\ell}{\mu_d}} G_\ell(\cos d(x, \cdot)) \)

- \( S_\ell(M) = \int_{S^d} M(X(f_x)) \, dx \in L^2(P) \Rightarrow \) Wiener-Itô chaos expansion

\[
S_\ell(M) = \sum_{q=0}^{\infty} \frac{J_q(M)}{q!} \int_{S^d} H_q(T_\ell(x)) \, dx
\] (2.1)

- In order to prove a quantitative CLT for \( S_\ell(M) \), as \( \ell \to +\infty \), we would
clt’s via chaos decomposition

- Isonormal representation: \( T_\ell(x) = X(f_x) \), \( f_x(\cdot) = \sqrt{\frac{n_\ell; d}{\mu_d}} G_\ell(\cos d(x, \cdot)) \)
- \( S_\ell(M) = \int_{S^d} M(X(f_x)) \, dx \in L^2(\mathbb{P}) \Rightarrow \) Wiener-Itô chaos expansion

\[
S_\ell(M) = \sum_{q=0}^{\infty} \frac{J_q(M)}{q!} \int_{S^d} H_q(T_\ell(x)) \, dx
\] (2.1)

- In order to prove a quantitative CLT for \( S_\ell(M) \), as \( \ell \to +\infty \), we would
  - 1) prove a quantitative CLT for \( h_{\ell; q, d} := \int_{S^d} H_q(T_\ell(x)) \, dx \), that are the basic building blocks for (2.1);
**clt’s via chaos decomposition**

- Isonormal representation: \( T_\ell(x) = X(f_x) \), \( f_x(\cdot) = \sqrt{\frac{n_\ell}{\mu_d}} G_\ell(\cos d(x, \cdot)) \)
- \( S_\ell(M) = \int_{S^d} M(X(f_x)) \, dx \in L^2(\mathbb{P}) \Rightarrow \text{Wiener-Itô chaos expansion} \)

\[
S_\ell(M) = \sum_{q=0}^{\infty} \frac{J_q(M)}{q!} \int_{S^d} H_q(T_\ell(x)) \, dx \quad (2.1)
\]

- In order to prove a quantitative CLT for \( S_\ell(M) \), as \( \ell \to +\infty \), we would
  - **1)** prove a quantitative CLT for \( h_{\ell; q, d} := \int_{S^d} H_q(T_\ell(x)) \, dx \), that are the basic building blocks for (2.1);
  - \( \Rightarrow \) then the CLT for *finite* linear combinations of \( h_{\ell; q, d} \) *immediately* follows, **Peccati and Taqqu, 2010** (but not the rate of convergence, which will require some work);
clt’s via chaos decomposition

• Isonormal representation
  \[ T_\ell(x) = X(f_x), \quad f_x(\cdot) = \sqrt{\frac{n_\ell \cdot d}{\mu_d}} G_\ell(\cos d(x, \cdot)) \]

• \( S_\ell(M) = \int_{S_d} M(X(f_x)) \, dx \in L^2(\mathbb{P}) \Rightarrow \) Wiener-Itô chaos expansion
  \[ S_\ell(M) = \sum_{q=0}^{\infty} \frac{J_q(M)}{q!} \int_{S_d} H_q(T_\ell(x)) \, dx \]  
  \hspace{1cm} (2.1)

• In order to prove a quantitative CLT for \( S_\ell(M) \), as \( \ell \to +\infty \), we would
  \begin{enumerate}
  \item prove a quantitative CLT for \( h_{\ell; q, d} := \int_{S_d} H_q(T_\ell(x)) \, dx \), that are the basic building blocks for (2.1);
  \item then the CLT for finite linear combinations of \( h_{\ell; q, d} \) immediately follows, \textit{Peccati and Taqqu, 2010} (but not the rate of convergence, which will require some work);
  \item show that, under a mild assumption on \( M \), the asymptotic behaviour of the r.v. in (2.1) is dominated by a single Wiener chaos or at most a finite linear combination of \( h_{\ell; q, d} \).
  \end{enumerate}
fourth moment theorems

- \( Z, N \) r.v.'s: Kolmogorov \( d_K \), Total Variation \( d_{TV} \) and Wasserstein \( d_W \) distances

\[
d_K(Z, N) = \sup_{z \in \mathbb{R}} |\mathbb{P}(Z \leq z) - \mathbb{P}(N \leq z)|,
\]
\[
d_{TV}(Z, N) = \sup_{A \in \mathcal{B}(\mathbb{R})} |\mathbb{P}(Z \in A) - \mathbb{P}(N \in A)|,
\]
\[
d_W(Z, N) = \sup_{h \in \text{Lip}(1)} |\mathbb{E}[h(Z)] - \mathbb{E}[h(N)]|,
\]

- to prove a quantitative CLT for \( h^{\ell}; q, d \); 
- \( S^d \) Stein-Malliavin approximations
fourth moment theorems

- \( Z, N \) r.v.’s: Kolmogorov \( d_K \), Total Variation \( d_{TV} \) and Wasserstein \( d_W \) distances

\[
d_K(Z, N) = \sup_{z \in \mathbb{R}} |\mathbb{P}(Z \leq z) - \mathbb{P}(N \leq z)| ,
\]

\[
d_{TV}(Z, N) = \sup_{A \in \mathcal{B}(\mathbb{R})} |\mathbb{P}(Z \in A) - \mathbb{P}(N \in A)| ,
\]

\[
d_W(Z, N) = \sup_{h \in \text{Lip}(1)} |\mathbb{E}[h(Z)] - \mathbb{E}[h(N)]| ,
\]

- to prove a quantitative CLT for \( h_{\ell; q, d} = \int_{S^d} H_q(T_{\ell}(x)) \, dx \Rightarrow \) fourth moment theorem by Nourdin and Peccati: for \( d_D = d_{TV}, d_W, d_K \)

\[
d_D \left( \frac{h_{\ell; q, d}}{\sqrt{\text{Var}[h_{\ell; q, d}]}, N(0, 1)} \right) \leq \sqrt{\frac{q - 1}{3q} \frac{\text{cum}_4[h_{\ell; q, d}]}{\left(\text{Var}[h_{\ell; q, d}]\right)^2}} \quad (2.2)
\]
fourth moment theorems

- $Z, N$ r.v.'s: Kolmogorov $d_K$, TotalVariation $d_{TV}$ and Wasserstein $d_W$ distances

\[
d_K(Z, N) = \sup_{z \in \mathbb{R}} |\mathbb{P}(Z \leq z) - \mathbb{P}(N \leq z)|,
\]

\[
d_{TV}(Z, N) = \sup_{A \in \mathcal{B}(\mathbb{R})} |\mathbb{P}(Z \in A) - \mathbb{P}(N \in A)|,
\]

\[
d_W(Z, N) = \sup_{h \in \text{Lip}(1)} |\mathbb{E}[h(Z)] - \mathbb{E}[h(N)]|,
\]

- to prove a quantitative CLT for $h_{\ell;q,d} = \int_{S^d} H_q(T_\ell(x)) \, dx \Rightarrow$ fourth moment theorem by Nourdin and Peccati: for $d_D = d_{TV}, d_W, d_K$

\[
d_D \left( \frac{h_{\ell;q,d}}{\sqrt{\text{Var}[h_{\ell;q,d}]}} , \mathcal{N}(0,1) \right) \leq \sqrt{\frac{q - 1}{3q}} \frac{\text{cum}_4[h_{\ell;q,d}]}{(\text{Var}[h_{\ell;q,d}])^2} \quad (2.2)
\]

- Therefore we have to study the variance of $h_{\ell;q,d}$ and the fourth cumulant of $h_{\ell;q,d}$ and show that the r.h.s. in (2.2) goes to 0, as $\ell \to +\infty$
the variance of $h_{\ell;q,d}$

- $\text{Var}[h_{\ell;q,d}] = \mathbb{E} \left[ \left( \int_{S^d} H_q(T_\ell(x)) \, dx \right)^2 \right] = q! \int_{(S^d)^2} G_{\ell;d}(\cos d(x,y))^q \, dx \, dy = q! \mu_d \mu_{d-1} \int_0^\pi G_{\ell;d}(\cos \vartheta)^q (\sin \vartheta)^{d-1} \, d\vartheta$. 

• Our first result: upper bound for these variances, asymptotic as $\ell \to +\infty$ (for "notational" simplicity: just even $\ell$).

• Method: investigate the asymptotic behaviour as $\ell \to +\infty$ of all order moments of Gegenbauer polynomials $\int_{S^2} G_{\ell;d}(\cos \vartheta)^q (\sin \vartheta)^{d-1} \, d\vartheta$.

• We are inspired by the proof for moments of Legendre polynomials ($d = 2$) in Marinucci and Wigman, 2011.
the variance of $h_{\ell;q,d}$

- $\text{Var}[h_{\ell;q,d}] = \mathbb{E} \left[ \left( \int_{S^d} H_q(T_{\ell}(x)) \, dx \right)^2 \right] = q! \int_{(S^d)^2} G_{\ell;d}(\cos d(x, y))^q \, dxdy = q! \mu_d \mu_{d-1} \int_0^\pi G_{\ell;d}(\cos \vartheta)^q (\sin \vartheta)^{d-1} \, d\vartheta$.

- $G_{\ell;d}(t) = (-1)^{\ell} G_{\ell;d}(-t)$, so $h_{\ell;q,d} = 0 \ a.s.$ when both $\ell$ and $q$ are odd, otherwise

$$\text{Var}[h_{\ell;q,d}] = 2q! \mu_d \mu_{d-1} \int_0^{\pi/2} G_{\ell;d}(\cos \vartheta)^q (\sin \vartheta)^{d-1} \, d\vartheta.$$
the variance of $h_{\ell; q, d}$

- $\text{Var}[h_{\ell; q, d}] = \mathbb{E} \left[ (\int_{S^d} H_q(T_\ell(x)) \, dx)^2 \right] = q! \int_{(S^d)^2} G_{\ell; d}(\cos d(x, y))^q \, dx \, dy = q! \mu_d \mu_{d-1} \int_0^\pi G_{\ell; d}(\cos \vartheta)^q (\sin \vartheta)^{d-1} \, d\vartheta$.

- $G_{\ell; d}(t) = (-1)^\ell G_{\ell; d}(-t)$, so $h_{\ell; q, d} = 0$ a.s. when both $\ell$ and $q$ are odd, otherwise

$$\text{Var}[h_{\ell; q, d}] = 2q! \mu_d \mu_{d-1} \int_0^{\pi/2} G_{\ell; d}(\cos \vartheta)^q (\sin \vartheta)^{d-1} \, d\vartheta.$$ 

- **Our first result**: upper bound for these variances, asymptotic as $\ell \to +\infty$ (for "notational" simplicity: just even $\ell$).
the variance of $h_{\ell;q,d}$

- $\text{Var}[h_{\ell;q,d}] = \mathbb{E} \left[ (\int_{S^d} H_q(T_{\ell}(x)) \, dx)^2 \right] = q! \int_{(S^d)^2} G_{\ell;d}(\cos d(x, y))^q \, dxdy = q! \mu_d \mu_{d-1} \int_0^\pi G_{\ell;d}(\cos \vartheta)^q (\sin \vartheta)^{d-1} \, d\vartheta$.

- $G_{\ell;d}(t) = (-1)^\ell G_{\ell;d}(-t)$, so $h_{\ell;q,d} = 0$ a.s. when both $\ell$ and $q$ are odd, otherwise

$$\text{Var}[h_{\ell;q,d}] = 2q! \mu_d \mu_{d-1} \int_0^{\frac{\pi}{2}} G_{\ell;d}(\cos \vartheta)^q (\sin \vartheta)^{d-1} \, d\vartheta$$.

- **Our first result**: upper bound for these variances, asymptotic as $\ell \to +\infty$ (for "notational" simplicity: just even $\ell$).

- **Method**: investigate the asymptotic behaviour as $\ell \to +\infty$ of all order moments of Gegenbauer polynomials

$$\int_0^{\frac{\pi}{2}} G_{\ell;d}(\cos \vartheta)^q (\sin \vartheta)^{d-1} \, d\vartheta$$. 
the variance of $h_{\ell; q, d}$

- $\text{Var}[h_{\ell; q, d}] = \mathbb{E} \left[ (\int_{S^d} H_q(T_\ell(x)) \, dx)^2 \right] = q! \int_{(S^d)^2} G_{\ell; d}(\cos d(x, y))^q \, dx \, dy =$
  
  $= q! \mu_d \mu_{d-1} \int_0^\pi G_{\ell; d}(\cos \theta)^q (\sin \theta)^{d-1} \, d\theta$.

- $G_{\ell; d}(t) = (-1)^\ell G_{\ell; d}(-t)$, so $h_{\ell; q, d} = 0$ a.s. when both $\ell$ and $q$ are odd, otherwise

  $$\text{Var}[h_{\ell; q, d}] = 2q! \mu_d \mu_{d-1} \int_0^{\pi/2} G_{\ell; d}(\cos \theta)^q (\sin \theta)^{d-1} \, d\theta.$$  

- **Our first result**: upper bound for these variances, asymptotic as $\ell \to +\infty$ (for ”notational” simplicity: just even $\ell$).

- **Method**: investigate the asymptotic behaviour as $\ell \to +\infty$ of all order moments of Gegenbauer polynomials

  $$\int_0^{\pi/2} G_{\ell; d}(\cos \theta)^q (\sin \theta)^{d-1} \, d\theta.$$  

- We are inspired by the proof for moments of Legendre polynomials ($d = 2$) in *Marinucci and Wigman, 2011*. 
main tool for investigating these variances

Recall the definition of the Bessel functions

\[ J_\nu(\psi) = \sum_{k=0}^{\infty} (-1)^k \frac{\psi^{2k+n}}{2^{2k+n} k! \Gamma(k+n+1)} , \quad \nu \geq 0 , \quad \psi \geq 0 ; \]

- Hilb's asymptotic formula for Jacobi polynomials (Szegö, 1975):

\[
\frac{(\sin \vartheta)^{d-1}}{(\ell + d-1)} G_{\ell;d}(\cos \vartheta) = \frac{2^{d-1}}{(\ell + \frac{d}{2} - 1)} \left[ \frac{\Gamma(\ell + \frac{d}{2})}{(\ell + d-1)^{\frac{d}{2} - 1} \ell!} \right] \sqrt{\vartheta} \sin \vartheta \left( (\ell + \frac{d-1}{2}) \vartheta \right) + \delta(\vartheta)
\]

\( \sim 1 \) as \( \ell \to +\infty \)

The remainder is

\[
\delta(\vartheta) = \begin{cases} 
\sqrt{\vartheta} \ell^{-\frac{3}{2}} & \text{if } \ell^{-1} < \vartheta < \frac{\pi}{2}, \\
\vartheta \left( \frac{d}{2} - 1 \right)^2 + \frac{d}{2} - 1 & \text{if } 0 < \vartheta < \ell^{-1}.
\end{cases}
\]
sketch of the proof \((d, q \geq 3)\)

By Hilb’s asymptotic

\[
\int_{0}^{\pi/2} G_{\ell,d}(\cos \vartheta)^q (\sin \vartheta)^{d-1} d\vartheta =
\]

\[
= \left(\frac{2^{d/2-1}}{(\ell + \frac{d}{2} - 1)}\right)^q \int_{0}^{\pi/2} (\sin \vartheta)^{-q(\frac{d}{2} - 1)} \left(\frac{\vartheta}{\sin \vartheta}\right)^{\frac{q}{2}} J_{\frac{d-1}{2}}(\vartheta) (\sin \vartheta)^{d-1} d\vartheta + \text{error}
\]

\*error = \(o\left(\frac{1}{L^d}\right)\)! By the change of variable \(\psi = (\ell + \frac{d-1}{2})\vartheta \equiv L\vartheta\) in the integral

\[
= \left(\frac{2^{d/2-1}}{(\ell + \frac{d}{2} - 1)}\right)^q \frac{1}{L} \int_{0}^{L\pi/2} (\sin \frac{\psi}{L})^{-q(\frac{d}{2} - 1)} \left(\frac{\psi/L}{\sin \frac{\psi}{L}}\right)^{\frac{q}{2}} J_{\frac{d-1}{2}}(\psi) (\sin \frac{\psi}{L})^{d-1} d\psi
\]

and since \(\psi/L = o(1)\),

\[
\sim \frac{1}{L^d} \left(\frac{2^{d/2-1} L (\frac{d}{2} - 1)}{(\ell + \frac{d}{2} - 1)}\right)^q \int_{0}^{L\pi/2} \left(\frac{\psi/L}{\sin \frac{\psi}{L}}\right)^{q(\frac{d}{2} - \frac{1}{2}) - d + 1} J_{\frac{d-1}{2}}(\psi) \psi^{-q(\frac{d}{2} - 1) + d - 1} d\psi,
\]

\[
\rightarrow c_q
\]

\[
c_q := \left(2^{d/2-1} \left(\frac{d}{2} - 1\right)\right)^q \int_{0}^{+\infty} J_{\frac{d-1}{2}}(\psi)^q \psi^{-q(\frac{d}{2} - 1) + d - 1} d\psi.
\]
summary on the variance of \( h_{\ell; q, d} \) for \( d \geq 2 \)

- **Our result:** as \( \ell \to \infty \), for \( d, q \geq 3 \),

\[
\text{Var}[h_{\ell; q, d}] = 2q! \mu_d \mu_{d-1} c_q \frac{1}{\ell^d} (1 + o(1)) .
\]
summary on the variance of $h_{\ell;q,d}$ for $d \geq 2$

• **Our result**: as $\ell \to \infty$, for $d, q \geq 3$,

$$\text{Var}[h_{\ell;q,d}] = 2q!\mu_d\mu_{d-1}c_q \frac{1}{\ell^d} (1 + o(1)).$$

• **Second moment**: for $d \geq 2$, there exists an exact formula when $q = 2$:

$$\int_0^\pi G_{\ell;d}(\cos \vartheta)^2 (\sin \vartheta)^{d-1} \, d\vartheta = \frac{\mu_d}{\mu_{d-1} n_{\ell,d}} \sim c_2 \frac{1}{\ell^{d-1}}$$
Summary on the variance of $h_{\ell; q, d}$ for $d \geq 2$

- **Our result:** as $\ell \to \infty$, for $d, q \geq 3$,

$$\text{Var}[h_{\ell; q, d}] = 2q! \mu_d \mu_{d-1} c_q \frac{1}{\ell^d} (1 + o(1)) .$$

- **Second moment:** for $d \geq 2$, there exists an exact formula when $q = 2$:

$$\int_0^\pi G_{\ell; d}(\cos \vartheta)^2 (\sin \vartheta)^{d-1} \, d\vartheta = \frac{\mu_d}{\mu_{d-1} n_{\ell, d}} \sim c_2 \frac{1}{\ell^{d-1}}$$

- **Case $d = 2$, $q \geq 3$:** in Marinucci and Wigman, 2011.
  - $q = 3, 5, \ldots$

$$\text{Var}[h_{\ell; q, 2}] = (4\pi)^2 q! c_q \frac{1}{\ell^2} (1 + o(1)) ,$$

$$c_q = \int_0^\infty J_0(\psi)^q \psi \, d\psi ,$$

- $q = 4$, as $\ell \to +\infty$,

$$\text{Var}[h_{\ell; 4, 2}] \sim 24^2 \frac{\log \ell}{\ell^2} .$$
the fourth cumulant of \( h_{\ell; q, d} \)

- circulant diagrams \( \rightarrow \) give the major contribution for \( \text{cum}_4[h_{\ell; q, d}] \) (Nourdin and Peccati)

\( \downarrow \)
corresponds to multiple integrals of Gegenbauer polynomials as follows:

\[
K_{\ell}(q; r) := \int_{(\mathbb{S}^d)^4} G_{\ell; d}^{q-r}(\cos d(x_1, x_2)) G_{\ell; d}^r(\cos d(x_2, x_3)) \times \\
G_{\ell; d}^{q-r}(\cos d(x_3, x_4)) G_{\ell; d}^r(\cos d(x_4, x_1)) \, dx_1 \, dx_2 \, dx_3 \, dx_4,
\]

for \( q = 2, 3, 4, ..., \) \( r = 1, 2, ... \lfloor \frac{q}{2} \rfloor \).

**How do we estimate them?** : We bound \( |G_{\ell; d}^r(\cos d(x_4, x_1))| \) in \( |K_{\ell}(q; r)| \) by 1 and we use previous results for all order moments of \( G_{\ell; d} \).

**Example:** \( q = 5 \) and \( d = 2 \) \( \rightarrow \) \( G_{\ell; 2} \equiv P_{\ell} \) Legendre polynomials

\[
|K_{\ell}(5; 1)| \leq \int_{(\mathbb{S}^2)^4} |P_{\ell}(\langle x_1, x_2 \rangle)|^4 \left| \int_{\mathbb{S}^2} \right| P_{\ell}(\langle x_2, x_3 \rangle) \, dx_3 \right|^4 \left| P_{\ell}(\langle x_4, x_1 \rangle) \right| \, dx_1 \, dx_2 \, dx_3 \, dx_4 \leq 1
\]

\[
= O \left( (\log \ell) \ell^{-2} \right) \times \int_{(\mathbb{S}^2)^2} |P_{\ell}(\langle x_1, x_2 \rangle)|^4 \left( \int_{\mathbb{S}^2} |P_{\ell}(\langle x_2, x_3 \rangle)\, dx_3 \right) \, dx_1 \, dx_2
\]

\[
= O \left( (\log \ell) \ell^{-2} \right) \times O \left( \ell^{-1/2} \right) \times \int_{(\mathbb{S}^2)^2} |P_{\ell}(\langle x_1, x_2 \rangle)|^4 \, dx_1 \, dx_2 = O \left( (\log^2 \ell) \ell^{-9/2} \right).
\]
**results for** $\text{cum}_4[h_{\ell;q_2}]$ ($d = 2$)

- For $r = 1, 2, ..., \left[\frac{q}{2}\right]$ we have

  \[ |K_\ell(q; r)| = O\left(\ell^{-5}\right) \text{ for } q = 3 , \]  
  \[ |K_\ell(q; r)| = O\left(\ell^{-4}\right) \text{ for } q = 4 , \]  
  \[ |K_\ell(q; r)| = O\left((\log \ell)\ell^{-9/2}\right) \text{ for } q = 5, 6 , \]  
  \[ |K_\ell(q; r)| = O\left(\ell^{-9/2}\right) \text{ for } q \geq 7 . \]
**results for** $\text{cum}_4[h_\ell;q,2]$ **($d = 2$)**

- For $r = 1, 2, ..., \left[ \frac{q}{2} \right]$ we have

\[
|K_\ell(q; r)| = O(\ell^{-5}) \text{ for } q = 3, \quad (3.1)
\]
\[
|K_\ell(q; r)| = O(\ell^{-4}) \text{ for } q = 4, \quad (3.2)
\]
\[
|K_\ell(q; r)| = O\left((\log \ell)\ell^{-9/2}\right) \text{ for } q = 5, 6, \quad (3.3)
\]
\[
|K_\ell(q; r)| = O\left(\ell^{-9/2}\right) \text{ for } q \geq 7. \quad (3.4)
\]

- The bounds (3.1), (3.2) are known and indeed the corresponding integrals can be evaluated explicitly in terms of Wigner’s 3j and 6j coefficients (Marinucci, 2008, Marinucci and Peccati, 2011, Marinucci and Wigman, 2014).
For \( r = 1, 2, \ldots, \left[ \frac{q}{2} \right] \) we have

\[
|K_\ell(q; r)| = O\left( \ell^{-5} \right) \text{ for } q = 3, \quad (3.1)
\]

\[
|K_\ell(q; r)| = O\left( \ell^{-4} \right) \text{ for } q = 4, \quad (3.2)
\]

\[
|K_\ell(q; r)| = O\left( (\log \ell) \ell^{-9/2} \right) \text{ for } q = 5, 6, \quad (3.3)
\]

\[
|K_\ell(q; r)| = O\left( \ell^{-9/2} \right) \text{ for } q \geq 7. \quad (3.4)
\]

The bounds (3.1), (3.2) are known and indeed the corresponding integrals can be evaluated explicitly in terms of Wigner’s 3j and 6j coefficients (Marinucci, 2008, Marinucci and Peccati, 2011, Marinucci and Wigman, 2014).

The bounds in (3.3),(3.4) improve the existing bounds in Marinucci and Wigman, 2014, where they use a different method to bound \( \text{cum}_4[h_\ell; q, 2] \).
results for $\text{cum}_4[h; q, d]$ when $d \geq 3$

- For all $r = 1, 2, \ldots, \left[\frac{q}{2}\right]$ we have

\[
|\mathcal{K}_\ell(q; r)| = O\left(\frac{1}{\ell^{2d+\frac{d-5}{2}}}\right) \quad \text{for } q = 3 ,
\]

\[
|\mathcal{K}_\ell(q; r)| = O\left(\frac{1}{\ell^{2d+\frac{d-3}{2}}}\right) \quad \text{for } q = 4 ,
\]

\[
|\mathcal{K}_\ell(q; r)| = O\left(\frac{1}{\ell^{2d+\frac{d-1}{2}}}\right) \quad \text{for } q \geq 5 .
\]
results for $\text{cum}_4[h_{\ell;q,d}]$ when $d \geq 3$

- For all $r = 1, 2, \ldots, \left[ \frac{q}{2} \right]$ we have

\[
|K_{\ell}(q;r)| = O \left( \frac{1}{\ell^{2d+\frac{d-5}{2}}} \right) \quad \text{for } q = 3 ,
\]

\[
|K_{\ell}(q;r)| = O \left( \frac{1}{\ell^{2d+\frac{d-3}{2}}} \right) \quad \text{for } q = 4 ,
\]

\[
|K_{\ell}(q;r)| = O \left( \frac{1}{\ell^{2d+\frac{d-1}{2}}} \right) \quad \text{for } q \geq 5 .
\]

- Recall that for $q \geq 3$, $\text{Var}[h_{\ell;q,d}] \sim \frac{1}{\ell^d}$, therefore we obtain a useful upper bound for $\text{cum}_4[h_{\ell;q,d}]$, i.e.

\[
\frac{\text{cum}_4[h_{\ell;q,d}]}{\text{Var}[h_{\ell;q,d}]^2} \to 0 , \quad \ell \to +\infty
\]

but for the cases

(1) $q = 3$ and $d = 3, 4, 5$;
(2) $q = 4$ and $d = 3$.

We leave these cases for future research as we need estimates on Clebsch Gordan coefficients for $SO(d + 1)$, $d = 3, 4, 5$. 
quantitative clt’s for $h_{\ell;q,d}$

∀q s.t. $(d, q) \neq (3, 4), (3, 3), (4, 3), (5, 3)$ and $c_q > 0$, we have

\[
\frac{h_{\ell;q,d}}{\sqrt{\text{Var}[h_{\ell;q,d}]}} \xrightarrow{\text{law}} \mathcal{N}(0, 1)
\]

and the rate of convergence $R(\ell; q, d)$ for the metrics $d_D = d_{TV}, d_W, d_K$ is

\[
d = 2 \quad R(\ell; q, 2) = \begin{cases} 
\ell^{-\frac{1}{2}} & \text{if } q = 2, 3, \\
(\log \ell)^{-1} & \text{if } q = 4, \\
(\log \ell)\ell^{-\frac{1}{4}} & \text{if } q = 5, 6, \\
\ell^{-\frac{1}{4}} & \text{if } q = 7, 8, \ldots ;
\end{cases}
\]

\[
d \geq 3 \quad R(\ell; q, d) = \begin{cases} 
\ell^{-\left(\frac{d-1}{2}\right)} & \text{if } q = 2, 3, \\
\ell^{-\left(\frac{d-3}{4}\right)} & \text{if } q = 4, \\
\ell^{-\left(\frac{d-1}{4}\right)} & \text{if } q = 5, 6, \ldots .
\end{cases}
\]

• The results for $d = 2$ improve the quantitative CLT in Marinucci and Wigman, 2014, actually they show that the total variation rate satisfies (up to logarithmic terms)

\[d_{TV} = O(\ell^{-\delta_q}), \text{ where } \delta_4 = \frac{1}{10}, \delta_5 = \frac{1}{7}, \text{ and } \delta_q = \frac{q-6}{4q-6} < \frac{1}{4} \text{ for } q = 7, 8, \ldots .\]
arbitrary polynomial transforms

• The transforms of $T_\ell$ by a finite order polynomial $p$ are of the form

$$Z_\ell = \int_{S^d} p(T_\ell(x)) \, dx = \sum_{q=2}^{Q} \beta_q h_\ell;q,d , \quad Q \in \mathbb{N}, \beta_q \in \mathbb{R} .$$
arbitrary polynomial transforms

• The transforms of $T_\ell$ by a finite order polynomial $p$ are of the form

\[ Z_\ell = \int_{S^d} p(T_\ell(x)) \, dx = \sum_{q=2}^Q \beta_q h_{\ell;q,d} , \quad Q \in \mathbb{N}, \, \beta_q \in \mathbb{R} . \]

• Assume that for $d = 3$, $\beta_3 = 0$, $\beta_4 = 0$ and for $d = 4, 5$, $\beta_3 = 0$. 
The transforms of $T_\ell$ by a finite order polynomial $p$ are of the form

$$Z_\ell = \int_{S^d} p(T_\ell(x)) \, dx = \sum_{q=2}^{Q} \beta_q h_{\ell;q,d} , \quad Q \in \mathbb{N}, \beta_q \in \mathbb{R} .$$

Assume that for $d = 3$, $\beta_3 = 0$, $\beta_4 = 0$ and for $d = 4, 5$, $\beta_3 = 0$.

CLT's for $h_{\ell;q,d} \Rightarrow$ CLT for $Z_\ell$ (Peccati and Tudor, 2010).
**arbitrary polynomial transforms**

- The transforms of $T_\ell$ by a finite order polynomial $p$ are of the form

$$Z_\ell = \int_{S^d} p(T_\ell(x)) \, dx = \sum_{q=2}^{Q} \beta_q h_{\ell;q,d} , \quad Q \in \mathbb{N}, \, \beta_q \in \mathbb{R} .$$

- Assume that for $d = 3$, $\beta_3 = 0, \beta_4 = 0$ and for $d = 4, 5$, $\beta_3 = 0$.
- CLT's for $h_{\ell;q,d} \Rightarrow$ CLT for $Z_\ell$ ([Peccati and Tudor, 2010]).
- **Our result:** The rate of convergence for $Z_\ell$ is the same as what we obtained for the Hermite polynomial case. If there exists $q$ such that $\beta_q \neq 0$ and $c_q > 0$

$$d_D \left( \frac{Z_\ell - \mathbb{E}[Z_\ell]}{\sqrt{\text{Var}[Z_\ell]}}, \mathcal{N}(0,1) \right) = O(R(Z_\ell; d)) ,$$

where $d_D = d_{TV}, d_W, d_K$ and for $d \geq 2$

$$R(Z_\ell; d) = \begin{cases} \ell^{-\frac{d-1}{2}} & \text{if } \beta_2 \neq 0 , \\ \max_{q=3,\ldots,Q} : \beta_q \neq 0 \quad R(\ell; q, d) & \text{if } \beta_2 = 0 . \end{cases}$$
why ? (sketch)

- Following Nourdin and Peccati, the CLT rate for random variables with finite chaotic expansion is

\[
d_D \left( \frac{Z_\ell - \mathbb{E}[Z_\ell]}{\sqrt{\text{Var}[Z_\ell]}}, \mathcal{N}(0, 1) \right) \leq (A) + (B),
\]

where in (A) and (B) there are quantities of two different form as follows:

(A) \[\int_{(\mathbb{S}^d)_4} G_{\ell,d}^{q-r}(\cos d(x_1, x_2)) G_{\ell,d}^{r}(\cos d(x_2, x_3)) \times \]
\[\times G_{\ell,d}^{q-r}(\cos d(x_3, x_4)) G_{\ell,d}^{r}(\cos d(x_4, x_1)) \, dx_1 dx_2 dx_3 dx_4 ,\]

(B) \[\int_{(\mathbb{S}^d)_4} G_{\ell,d}^{q_1}(\cos d(x_1, x_2)) G_{\ell,d}^{q_2-q_1}(\cos d(x_2, x_3)) G_{\ell,d}^{q_1}(\cos d(x_3, x_4)) \, dx_1 dx_2 dx_3 dx_4 .\]
why ? (sketch)

- Following Nourdin and Peccati, the CLT rate for random variables with finite chaotic expansion is

$$d_D \left( \frac{Z_\ell - \mathbb{E}[Z_\ell]}{\sqrt{\text{Var}[Z_\ell]}}, \mathcal{N}(0,1) \right) \leq (A) + (B),$$

where in (A) and (B) there are quantities of two different form as follows:

(A) $$\int_{(S^d)^4} G^{q-r}_{\ell;d} (\cos d(x_1, x_2)) G^r_{\ell;d}(\cos d(x_2, x_3)) \times$$
$$\times G^{q-r}_{\ell;d} (\cos d(x_3, x_4)) G^r_{\ell;d}(\cos d(x_4, x_1)) \, dx_1 dx_2 dx_3 dx_4,$$

(B) $$\int_{(S^d)^4} G^{q_1}_{\ell;d}(\cos d(x_1, x_2)) G^{q_2-q_1}_{\ell;d}(\cos d(x_2, x_3)) G^{q_1}_{\ell;d}(\cos d(x_3, x_4)) \, dx_1 dx_2 dx_3 dx_4$$

- (A) directly follows from the analysis of $\text{cum}_4[h_{\ell;q,d}]$. 
why? (sketch)

• Following Nourdin and Peccati, the CLT rate for random variables with finite chaotic expansion is

\[ d_D \left( \frac{Z_\ell - \mathbb{E}[Z_\ell]}{\sqrt{\text{Var}[Z_\ell]}}, \mathcal{N}(0, 1) \right) \leq (A) + (B), \]

where in (A) and (B) there are quantities of two different form as follows:

(A) \[ \int_{(S^d)^4} G^{q-r}_{\ell; d}(\cos d(x_1, x_2))G^r_{\ell; d}(\cos d(x_2, x_3)) \times \]
\[ \times G^{q-r}_{\ell; d}(\cos d(x_3, x_4))G^r_{\ell; d}(\cos d(x_4, x_1)) \, dx_1 \, dx_2 \, dx_3 \, dx_4, \]

(B) \[ \int_{(S^d)^4} G^{q_1}_{\ell; d}(\cos d(x_1, x_2))G^{q_2-q_1}_{\ell; d}(\cos d(x_2, x_3))G^{q_1}_{\ell; d}(\cos d(x_3, x_4)) \, dx_1 \, dx_2 \, dx_3 \, dx_4 \]

• (A) directly follows from the analysis of \( \text{cum}_4[h_\ell; q; d] \).
• (B) corresponds to diagrams with no proper loop. Usually Cauchy-Schwartz inequality is used to estimate \( |(B)| \).
why ? (sketch)

• Following Nourdin and Peccati, the CLT rate for random variables with finite chaotic expansion is

\[
d_D \left( \frac{Z_\ell - \mathbb{E}[Z_\ell]}{\sqrt{\text{Var}[Z_\ell]}}, \mathcal{N}(0, 1) \right) \leq (A) + (B),
\]

where in (A) and (B) there are quantities of two different form as follows:

\[(A) \int_{(\mathbb{S}^d)^4} G_{q-r}^{q-r}(\cos d(x_1, x_2)) G_{r}^{r}(\cos d(x_2, x_3)) \times\]
\[\times G_{q-r}^{q-r}(\cos d(x_3, x_4)) G_{r}^{r}(\cos d(x_4, x_1)) \, dx_1 \, dx_2 \, dx_3 \, dx_4,
\]

\[(B) \int_{(\mathbb{S}^d)^4} G_{q_1}^{q_1}(\cos d(x_1, x_2)) G_{q_2-q_1}^{q_2-q_1}(\cos d(x_2, x_3)) G_{q_1}^{q_1}(\cos d(x_3, x_4)) \, dx_1 \, dx_2 \, dx_3 \, dx_4.
\]

• (A) directly follows from the analysis of \(\text{cum}_4[h_{\ell;q,d}]\).

• (B) corresponds to diagrams with no proper loop. Usually Cauchy-Schwartz inequality is used to estimate \(|(B)|\).

• We estimate (B) in a different way.
estimates for (b)

\[(B) \int_{(S^d)^4} G_{\ell;d}^{q_1} (\cos d(x_1, x_2)) G_{\ell;d}^{q_2-q_1} (\cos d(x_2, x_3)) G_{\ell;d}^{q_1} (\cos d(x_3, x_4)) \, dx_1 \, dx_2 \, dx_3 \, dx_4 =

\int_{(S^d)^2} \left( G_{\ell;d}^{q_2-q_1} (\cos d(x_2, x_3)) \int_{S^d} G_{\ell;d}^{q_1} (\cos d(x_1, x_2)) \, dx_1 \int_{S^d} G_{\ell;d}^{q_1} (\cos d(x_3, x_4)) \, dx_4 \right) \, dx_2 \, dx_3

= \left( \frac{1}{q_1!} \text{Var}[h_{\ell;q_1,d}] \right)^2 \int_{(S^d)^2} G_{\ell;d}^{q_2-q_1} (\cos d(x_2, x_3)) \, dx_2 \, dx_3

= \begin{cases} 
0 & \text{if } q_2 - q_1 = 1 ; \\
\leq \left( \frac{1}{q_1!} \text{Var}[h_{\ell;q_1,d}] \right)^2 \int_{(S^d)^2} G_{\ell;d}^{2} (\cos d(x_2, x_3)) \, dx_2 \, dx_3 & \text{if } q_2 - q_1 \geq 2 .
\end{cases}

Therefore

\[(B) = O \left( \left( \frac{1}{q_1!} \text{Var}[h_{\ell;q_1,d}] \right)^2 \times \ell^{-(d-1)} \right) .\]
• It is easy to analyze the variance of $Z_\ell$, indeed

\[
\begin{aligned}
d = 2 \quad \text{Var}[Z_\ell] &= \sum_{q=2}^{Q} \beta_q^2 \text{Var}[h_\ell;q,d] \sim \begin{cases} 
\ell^{-1}, & \text{for } \beta_2 \neq 0 \\
\ell^{-2} \log \ell, & \text{for } \beta_2 = 0, \beta_4 \neq 0 \\
\ell^{-2}, & \text{otherwise.}
\end{cases} \\

\ell \geq 3 \quad \text{Var}[Z_\ell] &= \sum_{q=2}^{Q} \beta_q^2 \text{Var}[h_\ell;q,d] \sim \begin{cases} 
\ell^{-d+1}, & \text{for } \beta_2 \neq 0 \\
\ell^{-d}, & \text{otherwise.}
\end{cases}
\end{aligned}
\]
• It is easy to analyze the variance of $Z_\ell$, indeed

$$d = 2 \quad \text{Var}[Z_\ell] = \sum_{q=2}^{Q} \beta_q^2 \text{Var}[h_\ell; q, d] \sim \begin{cases} \ell^{-1}, & \text{for } \beta_2 \neq 0 \\ \ell^{-2} \log \ell, & \text{for } \beta_2 = 0, \beta_4 \neq 0 \\ \ell^{-2}, & \text{otherwise.} \end{cases}$$

$$d \geq 3 \quad \text{Var}[Z_\ell] = \sum_{q=2}^{Q} \beta_q^2 \text{Var}[h_\ell; q, d] \sim \begin{cases} \ell^{-d+1}, & \text{for } \beta_2 \neq 0 \\ \ell^{-d}, & \text{otherwise.} \end{cases}$$

• In the special case of polynomials of Hermite rank 2, i.e. $\beta_2 \neq 0$,

the asymptotic behaviour of $Z_\ell$ is dominated by the term $h_\ell; 2, d$, that is the projection on the 2nd Wiener chaos.

The variance of $h_\ell; 2, d$ is of order $\ell^{-(d-1)}$, rather than $O(\ell^{-d})$ as for the other terms. Therefore we find the simple behaviour

$$d_D \left( \frac{Z_\ell - \mathbb{E}[Z_\ell]}{\sqrt{\text{Var}[Z_\ell]}}, \mathcal{N}(0, 1) \right) \sim d_D \left( \frac{h_\ell; 2, d}{\sqrt{\text{Var}[h_\ell; 2, d]}}, \mathcal{N}(0, 1) \right) = O \left( \ell^{-\left(\frac{d-1}{2}\right)} \right).$$
This simple behaviour holds also for the general case (under a mild assumption). Precisely, consider

\[ S_\ell(M) = \int_{\mathbb{S}^d} M(T_\ell(x)) dx; \]
This simple behaviour holds also for the general case (under a mild assumption). Precisely, consider

\[ S_\ell(M) = \int_{\mathbb{S}^d} M(T_\ell(x)) \, dx ; \]

- \( M : \mathbb{R} \to \mathbb{R} \) is a measurable function such that

\[ E\left[ M(T_\ell(x))^2 \right] < \infty \]

\[ J_2(M) \neq 0, \]

that is, the projection on the 2nd Wiener chaos is nonzero.

\( \rightarrow J_0(M) = E[M(T_\ell(x))] = 0, \) assuming we work with centred variables.

\[ h_\ell;1 = \int S_d T_\ell(x) \, dx = 0, \]

whence the value of \( J_1(M) \) can be taken \( = 0 \).

\( \Rightarrow M \) has Hermite rank \( = 2 \) \( \Rightarrow \) its asymptotic behaviour is dominated by the projection on the 2nd Wiener chaos.
general nonlinear transforms

- This simple behaviour holds also for the general case (under a mild assumption).
  Precisely, consider

\[ S_\ell(M) = \int_{S^d} M(T_\ell(x)) \, dx ; \]

- \( M : \mathbb{R} \to \mathbb{R} \) is a measurable function such that
- \( \mathbb{E}[M(T_\ell)^2] < \infty \)
This simple behaviour holds also for the general case (under a mild assumption).
Precisely, consider

\[ S_\ell(M) = \int_{\mathbb{S}^d} M(T_\ell(x)) \, dx ; \]

- \( M : \mathbb{R} \to \mathbb{R} \) is a measurable function such that
- \( \mathbb{E}[M(T_\ell)^2] < \infty \)
- \( J_2(M) \neq 0 \), where

\[ J_q(M) = \mathbb{E}[M(T_\ell)H_q(T_\ell)] , \]

that is, the projection on the 2nd Wiener chaos is nonzero.
This simple behaviour holds also for the general case (under a mild assumption). Precisely, consider

\[ S_\ell(M) = \int_{S^d} M(T_\ell(x)) \, dx ; \]

- \( M : \mathbb{R} \to \mathbb{R} \) is a measurable function such that
- \( \mathbb{E}[M(T_\ell)^2] < \infty \)
- \( J_2(M) \neq 0 \), where
  \[ J_q(M) = \mathbb{E}[M(T_\ell)H_q(T_\ell)] , \]
  that is, the projection on the 2nd Wiener chaos is nonzero.
- w.l.o.g, the first two coefficients of \( S_\ell(M) \), i.e. \( J_0(M), J_1(M) \) can always be taken \( = 0 \) in the present framework.
  \[ \rightarrow J_0(M) = \mathbb{E}[M(T_\ell)] = 0, \text{ assuming we work with centred variables.} \]
  \[ \rightarrow h_{\ell,1,d} = \int_{S^d} T_\ell(x) \, dx = 0, \text{ whence the value of } J_1(M) \text{ can be taken } = 0. \]
  \[ \Rightarrow M \text{ has Hermite rank } = 2 \Rightarrow \text{ its asymptotic behaviour is dominated by the projection on the 2nd Wiener chaos} \]
quantitative clt’s for general nonlinear functionals

- Under previous hypothesis, the 2nd Wiener chaos dominates and

\[ \text{Var}[S_\ell(M)] \sim \frac{1}{\ell^{d-1}} \]
quantitative clt’s for general nonlinear functionals

- Under previous hypothesis, the 2nd Wiener chaos dominates and

\[ \text{Var}[S_\ell(M)] \sim \frac{1}{\ell^{d-1}} \]

- As \( \ell \to \infty \), we have the CLT and the rate

\[ d_W \left( \frac{S_\ell(M)}{\sqrt{\text{Var}(S_\ell(M))}}, \mathcal{N}(0, 1) \right) = O \left( \frac{1}{\sqrt{\ell}} \right), \]

where \( d_W \) is the Wasserstein distance.
quantitative clt’s for general nonlinear functionals

• Under previous hypothesis, the 2nd Wiener chaos dominates and

\[ \text{Var}[S_\ell(M)] \sim \frac{1}{\ell^{d-1}} \]

• As \( \ell \to \infty \), we have the CLT and the rate

\[ d_W \left( \frac{S_\ell(M)}{\sqrt{\text{Var}(S_\ell(M))}}, \mathcal{N}(0, 1) \right) = O \left( \frac{1}{\sqrt{\ell}} \right), \]

where \( d_W \) is the Wasserstein distance.

• Our hypothesis \( J_2(M) \neq 0 \) covers the case of Hermite rank \( = 2 \) but also Hermite rank \( = 1 \) for \( M \).
quantitative clt’s for general nonlinear functionals

• Under previous hypothesis, the 2nd Wiener chaos dominates and

\[ \text{Var}[S_\ell(M)] \sim \frac{1}{\ell^{d-1}} \]

• As \( \ell \to \infty \), we have the CLT and the rate

\[ d_W \left( \frac{S_\ell(M)}{\sqrt{\text{Var}(S_\ell(M))}}, \mathcal{N}(0, 1) \right) = O \left( \frac{1}{\sqrt{\ell}} \right), \]

where \( d_W \) is the Wasserstein distance.

• Our hypothesis \( J_2(M) \neq 0 \) covers the case of Hermite rank = 2 but also Hermite rank = 1 for \( M \).

• \( M \) does not need to be smooth in any meaningful sense \( \Rightarrow \) analysis of the asymptotic behaviour of functionals of geometric interest, e.g. empirical measure of excursion sets.
empirical measure of excursion sets

• excursion set: fix $z \in \mathbb{R}$,

$$\{ x \in S^d : T_\ell(x) \leq z \}$$

empirical measure: nonlinear functional of $T_\ell$

$$S_\ell(z) := S_\ell(1(\cdot \leq z)) = \int_{S^d} 1(T_\ell(x) \leq z) \, dx,$$

where $1(\cdot \leq z)$ is the indicator function of the interval $(-\infty, z]$. 
**empirical measure of excursion sets**

- **excursion set**: fix $z \in \mathbb{R}$,

$$\{x \in S^d : T_\ell(x) \leq z\}$$

- **empirical measure**: nonlinear functional of $T_\ell$

$$S_\ell(z) := S_\ell(1(\cdot \leq z)) = \int_{S^d} 1(T_\ell(x) \leq z) \, dx ,$$

where $1(\cdot \leq z)$ is the indicator function of the interval $(-\infty, z]$.

- $J_2(1(\cdot \leq z)) = z\phi(z)$, $\phi$ being the Gaussian density $\Rightarrow$
empirical measure of excursion sets

- **excursion set**: fix $z \in \mathbb{R}$,

$$\{ x \in S^d : T_\ell(x) \leq z \}$$

- **empirical measure**: nonlinear functional of $T_\ell$

$$S_\ell(z) := S_\ell(1(\cdot \leq z)) = \int_{S^d} 1(T_\ell(x) \leq z) \, dx,$$

where $1(\cdot \leq z)$ is the indicator function of the interval $(-\infty, z]$.

- $J_2(1(\cdot \leq z)) = z\phi(z)$, $\phi$ being the Gaussian density $\Rightarrow$

- As $\ell \to \infty$, for $z \neq 0$, we have that

$$d_W \left( \frac{S_\ell(z) - \mu_d \Phi(z)}{\sqrt{\text{Var}[S_\ell(z)]}}, \mathcal{N}(0, 1) \right) = O \left( \frac{1}{\sqrt{\ell}} \right),$$

where $\Phi$ is the Gaussian cdf.
Thank you for your attention!


