A stochastic model for speciation by mating preferences

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Ecology

Definition

Species (Ernst Mayr, 1942): largest group of organisms in which two individuals are capable of reproducing fertile offspring.

Definition

Speciation: evolutionary process by which reproductively isolated biological populations evolve to become distinct species.

$\Rightarrow$ How does reproductive isolation appear?

Motivation

Pundamilia pundamilia

Pundamilia nyererei

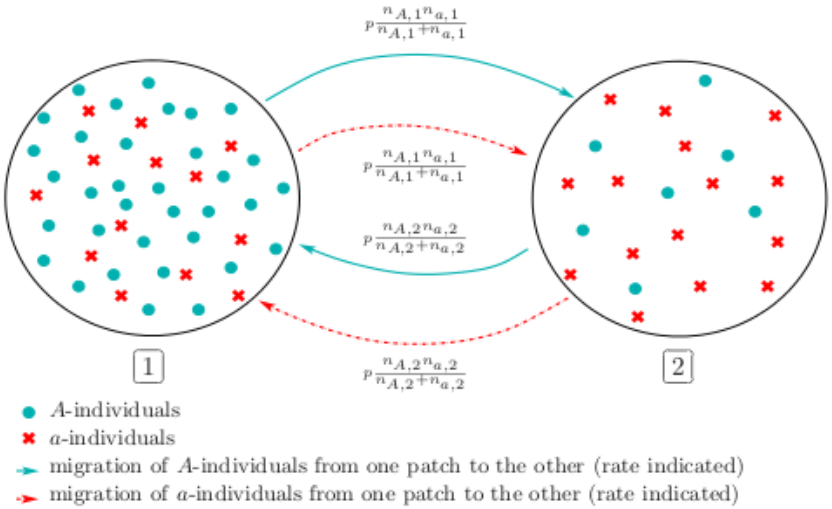
Definition

Homogamy: individuals with similar phenotypes have a higher reproductive success

$\Rightarrow$ Can homogamy entail reproductive isolation?

General description

- 2 patches
- Haploid individuals
- 1 gene, 2 alleles: A and a



$$(\mathbf{N}(t), t \geq 0) = (N_{\alpha,i}(t), t \geq 0)_{\alpha \in \{A,a\}, i \in \{1,2\}}$$

$$N_{\alpha,i}(t) = \sharp\{\text{individuals of type } \alpha \text{ in the patch } i\}$$

$(\mathbf{N}(t), t \geq 0)$ is a birth-and-death process with migration.

Birth

- Uniform individual reproduction rate
- Uniform intrademe mating
- Homogamy
- Mendelian reproduction

$$b_{\alpha,i}(\mathbf{n}) = b n_{\alpha,i} \frac{\beta n_{\alpha,i} + n_{\bar{\alpha},i}}{n_{\alpha,i} + n_{\bar{\alpha},i}}$$

Ecological relevance:

- Simultaneous mating of both male and female organs in flowering plants
- Bodysize dependent mating probability

Migration and death

- Death: natural and due to competition (logistic)

$$d_{\alpha,i}(\mathbf{n}) = n_{\alpha,i}(d + c(n_{\alpha,i} + n_{\bar{\alpha},i}))$$

- Migration:

$$m_{\alpha,i}(\mathbf{n}) = p n_{\alpha,i} \frac{n_{\bar{\alpha},i}}{n_{\alpha,i} + n_{\bar{\alpha},i}}$$

Ecological relevance: mating success dependent dispersal

Rescaling

Scaling assumption : Initial number of individuals of order $K \to +\infty$

$$(\mathbf{Z}^K(t), t \geq 0) = (Z^K_{\alpha,i}(t), t \geq 0)_{\alpha \in \{A,a\}, i \in \{1,2\}}$$
$$= \left(\frac{N_{\alpha,i}(t)}{K}, t \geq 0 \right)_{\alpha \in \{A,a\}, i \in \{1,2\}}$$

$$c \text{ becomes } \frac{c}{K}$$

Motivations:

- Simpler limiting model
- Models populations with different individual sizes

Limiting dynamical system

Lemma (Ethier & Kurtz 1986, Theorem 2.1 p. 456)

If $(\mathbf{N}^K(0)/K)_{K\geq 1} \to \mathbf{z}^0$ *when* $K \to \infty$, *then*

$$\lim_{K\to\infty} \sup_{s\leq T} \|\mathbf{N}^K(s)/K - \mathbf{z}^{(\mathbf{z}^0)}(s)\| = 0 \quad \textit{in probability,}$$

where $\mathbf{z}^{(\mathbf{z}^0)}$ *is the unique solution starting at* $\mathbf{z}^0$, *of*

$$\begin{cases}
\frac{d}{dt} z_{A,1}(t) = z_{A,1}\left[b\frac{\beta z_{A,1}+z_{a,1}}{z_{A,1}+z_{a,1}} - d - c(z_{A,1}+z_{a,1}) - p\frac{z_{a,1}}{z_{A,1}+z_{a,1}}\right] + p\frac{z_{A,2}z_{a,2}}{z_{A,2}+z_{a,2}} \\
\frac{d}{dt} z_{a,1}(t) = z_{a,1}\left[b\frac{\beta z_{a,1}+z_{A,1}}{z_{A,1}+z_{a,1}} - d - c(z_{A,1}+z_{a,1}) - p\frac{z_{A,1}}{z_{A,1}+z_{a,1}}\right] + p\frac{z_{A,2}z_{a,2}}{z_{A,2}+z_{a,2}} \\
\frac{d}{dt} z_{A,2}(t) = z_{A,2}\left[b\frac{\beta z_{A,2}+z_{a,2}}{z_{A,2}+z_{a,2}} - d - c(z_{A,2}+z_{a,2}) - p\frac{z_{a,2}}{z_{A,2}+z_{a,2}}\right] + p\frac{z_{A,1}z_{a,1}}{z_{A,1}+z_{a,1}} \\
\frac{d}{dt} z_{a,2}(t) = z_{a,2}\left[b\frac{\beta z_{a,2}+z_{A,2}}{z_{A,2}+z_{a,2}} - d - c(z_{A,2}+z_{a,2}) - p\frac{z_{A,2}}{z_{A,2}+z_{a,2}}\right] + p\frac{z_{A,1}z_{a,1}}{z_{A,1}+z_{a,1}}.
\end{cases}$$

Equilibria and stability

Theorem

4 stable stationary states:

- *2 equilibria for which each type remains present*

$$(\zeta, 0, 0, \zeta), \quad (0, \zeta, \zeta, 0)$$

- *2 equilibria for which only one type remains present*

$$(\zeta, 0, \zeta, 0), \quad (0, \zeta, 0, \zeta)$$

where $\zeta = \frac{b\beta - d}{c}$.

Equilibria and stability

Theorem

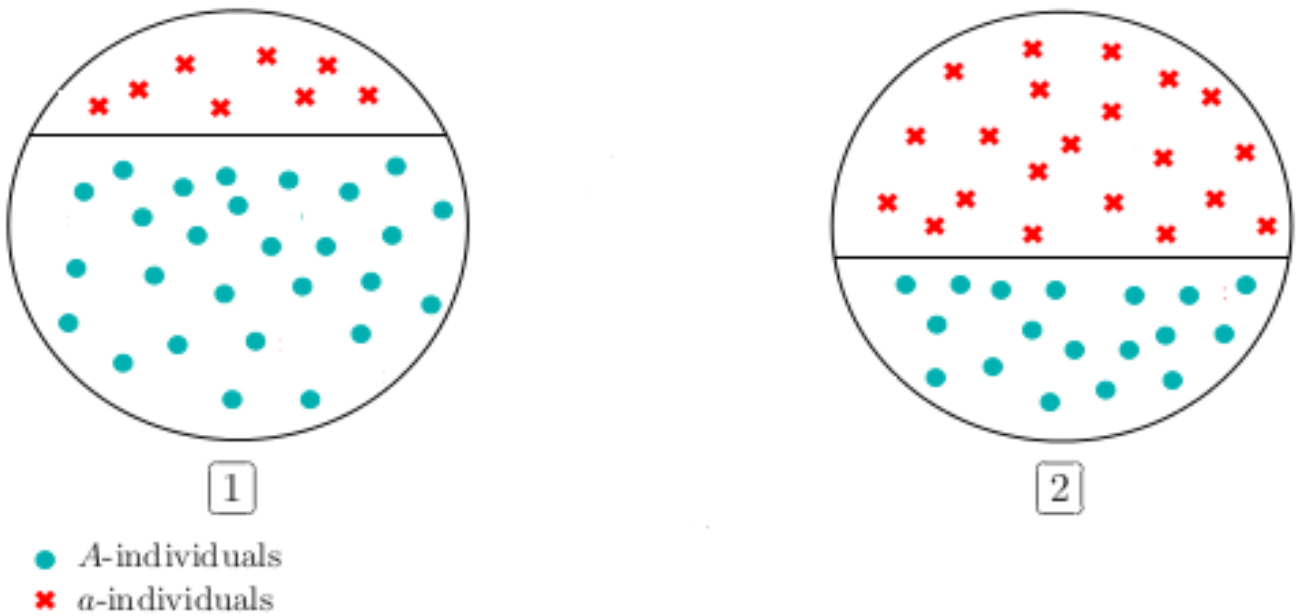
10 unstable stationary states:

- *the extinction state:* $(0, 0, 0, 0)$,
- *4 equilibria with only one type in only one patch,*
- *the equilibrium with both types remaining in both patches with the same quantity of each type,*
- *4 equilibria with both types remaining in both patches with different quantities of each type.*

Convergence of the dynamical system

Let

$$\mathcal{D} := \{\mathbf{z} \in \mathbb{R}^4_+, z_{A,1} - z_{a,1} > 0, z_{a,2} - z_{A,2} > 0\}.$$



Convergence of the dynamical system

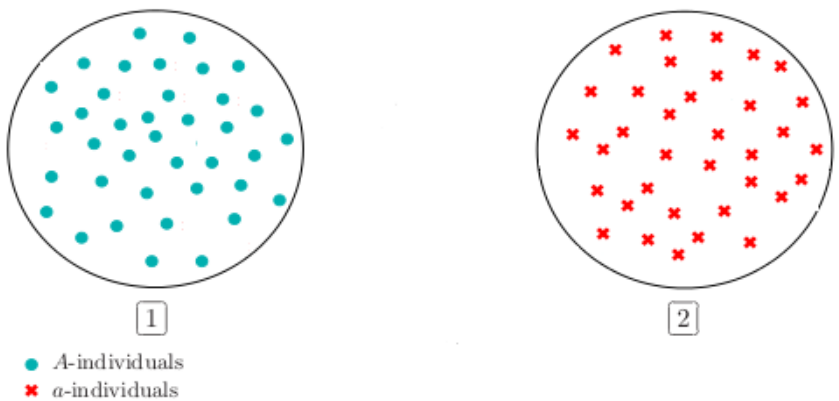
Let

$$\mathcal{D} := \{\mathbf{z} \in \mathbb{R}^4_+, z_{A,1} - z_{a,1} > 0, z_{a,2} - z_{A,2} > 0\}.$$

$$p_0 = \frac{\sqrt{b(\beta-1)[b(3\beta+1)-4d]} - b(\beta-1)}{2}.$$

Theorem

Let $p < p_0$. Then any solution which starts from $\mathcal{D}$ converges to the equilibrium $(\zeta, 0, 0, \zeta)$.



Convergence of the dynamical system

Let

$$\mathcal{D} := \{\mathbf{z} \in \mathbb{R}^4_+, z_{A,1} - z_{a,1} > 0, z_{a,2} - z_{A,2} > 0\}.$$

Theorem

Let $p < p_0$. Then if the initial condition lies in

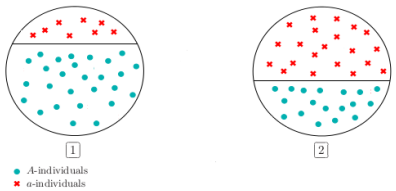
$$\mathcal{K} = \left\{\mathbf{z} \in \mathcal{D},\ \{z_{A,1} + z_{a,1},\ z_{A,2} + z_{a,2}\} \in \left[\frac{b(\beta+1) - 2d - p}{2c}, \frac{2b\beta - 2d + p}{2c}\right]\right\},$$

there exist two positive constants k_1 and k_2 such that for every $t \geq 0$,

$$\|\mathbf{z}(t) - (\zeta, 0, 0, \zeta)\| \leq k_1 e^{-k_2 t}.$$

Stochastic process long time behavior

Hyp: $(\mathbf{Z}(0)/K)_{K\geq 1}$ converges to $\mathbf{z}^0 \in \mathcal{D}$



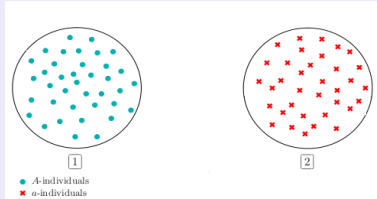
Theorem

For $\epsilon > 0$, let

$$T^K_{\mathcal{B}_\epsilon} = \inf\{t \geq 0, \mathbf{Z}^K(t) \in [\zeta - \epsilon, \zeta + \epsilon] \times \{0\} \times \{0\} \times [\zeta - \epsilon, \zeta + \epsilon]\,.\}$$

Then there exist ε_0, C_0 and m, such that if $\epsilon \leq \epsilon_0$, and $p < p_0$,

$$\lim_{K\to\infty} \mathbb{P}\left(\left|\frac{T^K_{\mathcal{B}_\epsilon}}{\log K} - \frac{1}{b(\beta-1)}\right| \leq C_0\epsilon\right) = 1$$

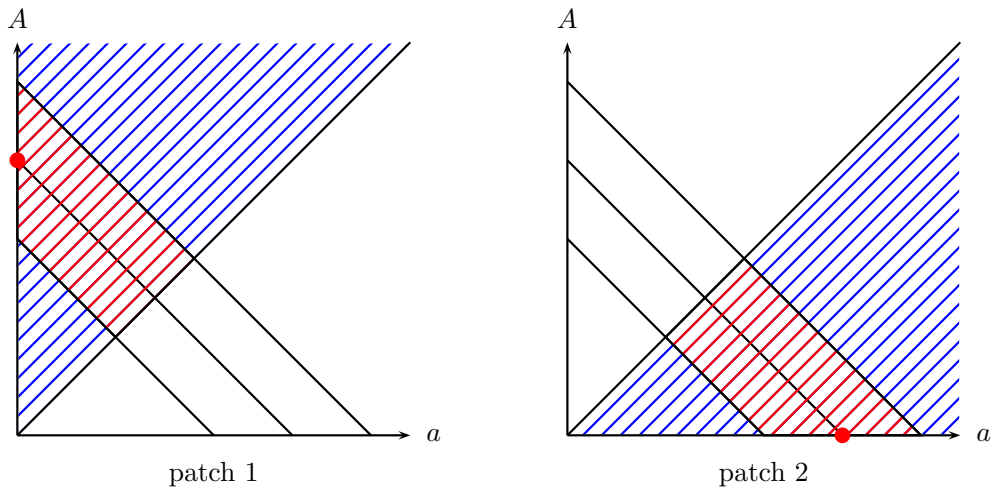


Mating preferences are enough to entail **reproductive isolation**.

Dynamical system convergence - Containment

Let

$$\mathcal{K} = \{\mathbf{z} \in \mathcal{D},\ \{z_{A,1} + z_{a,1},\ z_{A,2} + z_{a,2}\} \in [z_{min}, z_{max}]\}\,.$$



- $\mathcal{K}$ is invariant under the dynamical system
- The dynamical system reaches $\mathcal{K}$ in finite time from any point of $\mathcal{D}$.

Dynamical system convergence - Lyapunov

On $\mathcal{K}$, the function

$$V(\mathbf{z}) = \ln\left(\frac{z_{A,1}+z_{a,1}}{z_{A,1}-z_{a,1}}\right) + \ln\left(\frac{z_{a,2}+z_{A,2}}{z_{a,2}-z_{A,2}}\right).$$

satisfies

$$\frac{d}{dt}V(\mathbf{z}(t)) \leq -C_1(z_{a,1}(t)+z_{A,2}(t)), \quad t \geq 0,$$

for $C_1 > 0$.

Stochastic process long time behavior

- Let

$$T_1^{K,M} = \inf\{t \geq 0 : |Z_{A,1}^K(t) - \zeta| \geq M\epsilon, \quad or \quad |Z_{a,2}^K(t) - \zeta| \geq M\epsilon\}$$

and

$$T_2^K = \inf\{t \geq 0 : Z_{a,1}^K(t) \geq \epsilon, \quad or \quad Z_{a,2}^K(t) \geq \epsilon\}.$$

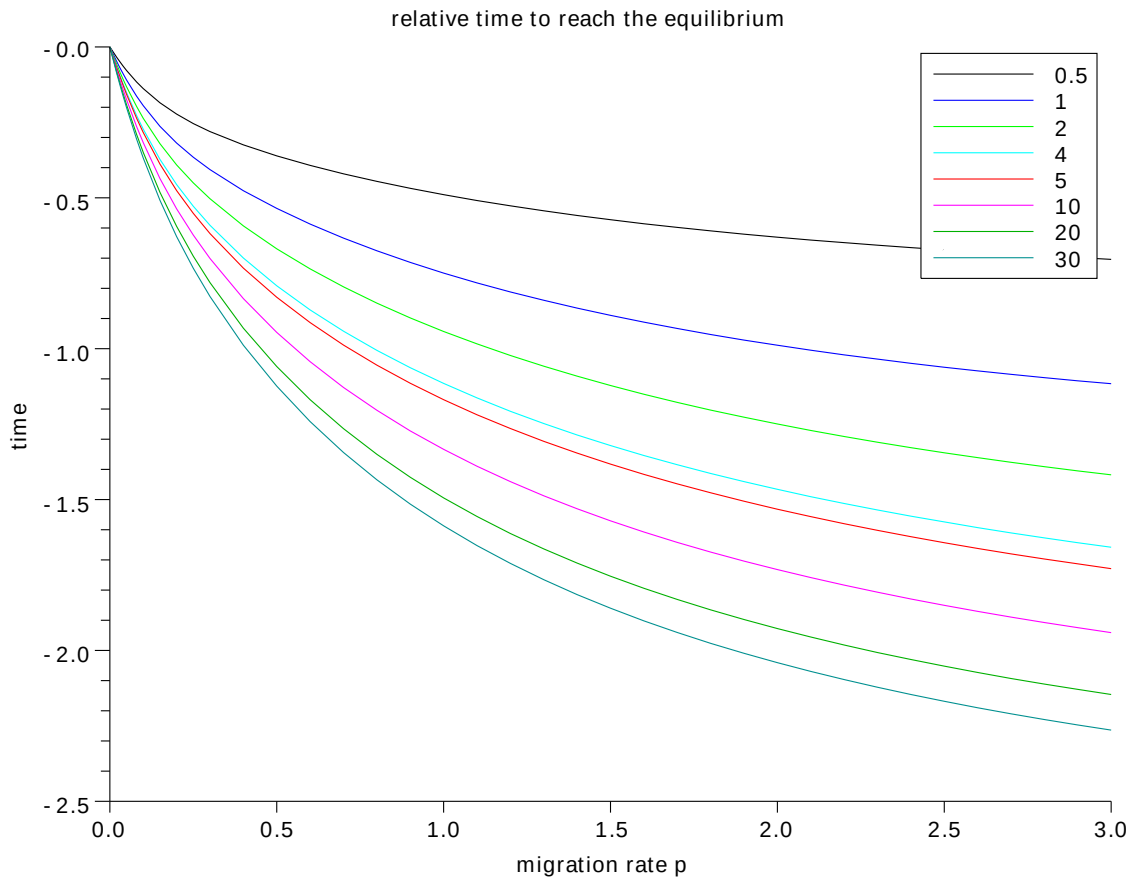
There exist $M, V > 0$ such that

$$\mathbb{P}(T_2^K \wedge e^{KV} > T_1^K) \underset{K \to +\infty}{\longrightarrow} 0$$

- $(Z_{A,2}(t) + Z_{a,1}(t), t \geq 0)$: stochastically bounded by two linear birth-and-death processes.

Influence of the migration parameter p

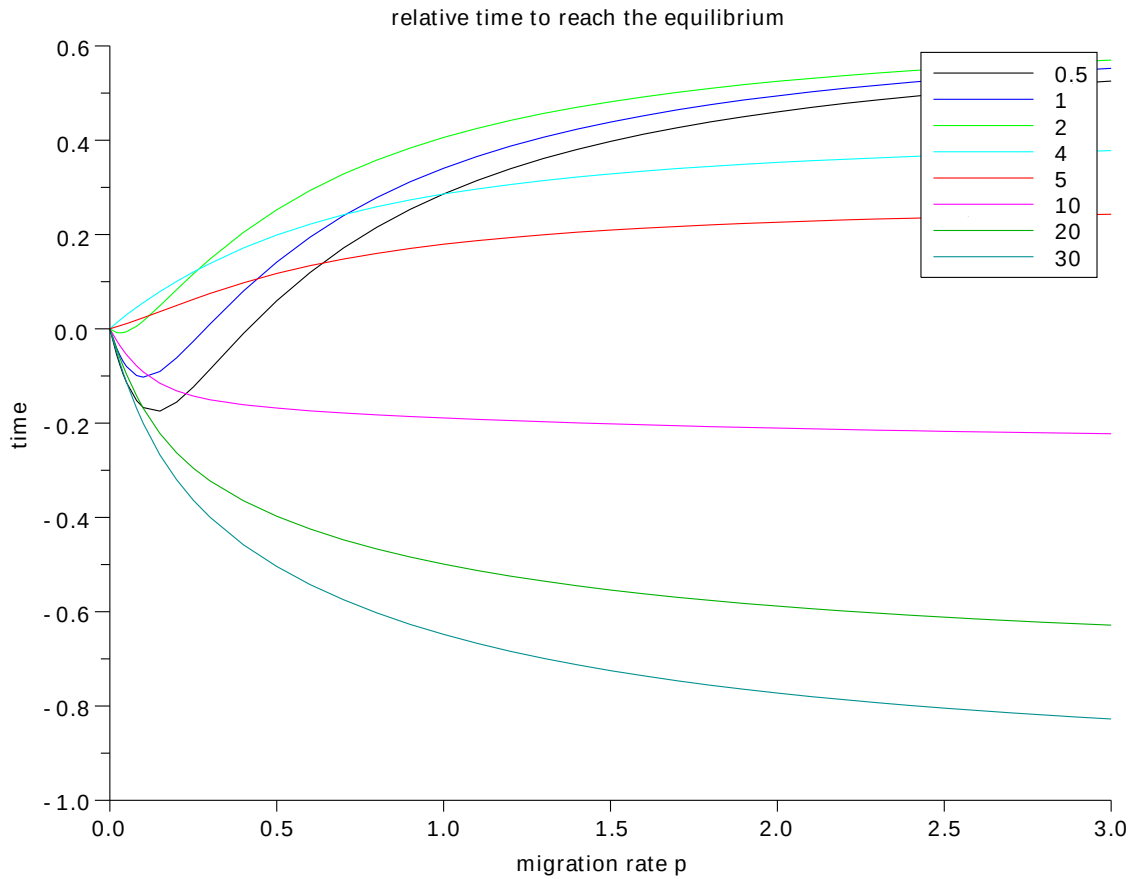
$$p \mapsto T_\varepsilon(p) - T_\varepsilon(0)$$



$$z_{a,1}(0) = z_{A,1}(0) - 0.1, \quad z_{A,2}(0) = 1, \quad z_{a,2}(0) = 30.$$

Influence of the migration parameter p

$$p \mapsto T_\varepsilon(p) - T_\varepsilon(0)$$



$$z_{a,1}(0) = z_{A,1}(0) - 0.1, \quad z_{A,2}(0) = 15, \quad z_{a,2}(0) = 16.$$