Continuous-state branching processes, extremal processes and super-individuals

Clément Foucart
Université Paris 13

with

Chunhua Ma
Nankai University

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Introduction

Consider a continuous-state branching process $(X_t(x), t \geq 0)$ starting from x.

Questions

- *Can we characterize the growth rate of a supercritical CSBP with infinite mean (no Malthusian parameter)?*
- *Can we characterise the decay rate of a subcritical CSBP with infinite variation ?*

Consider a *continuous branching population* encoded by *a flow of CSBPs* (as Bertoin Le Gall 2000).

Questions

How are organized the growth and the decay locally in the population? Do all families evolve at the same scale or some initial individuals (super-individuals) have progenies growing faster than all the others?

Definition (CSBP)

A positive Markov process $(X_t(x), t \geq 0)$ *with* $X_0(x) = x \geq 0$ *is a CSBP if for any* $y \in \mathbb{R}_+$

$$(X_t(x+y), t \geq 0) \stackrel{d}{=} (X_t(x), t \geq 0) + (\tilde{X}_t(y), t \geq 0)$$

where $(\tilde{X}_t(y), t \geq 0)$ *is an independent copy of* $(X_t(y), t \geq 0)$.

This ensures the existence of a map $t \mapsto v_t(\lambda)$ s.t.

$$\mathbb{E}[e^{-\lambda X_t(x)}] = \exp(-x v_t(\lambda)) \text{ and } v_{s+t}(\lambda) = v_s \circ v_t(\lambda),$$

Theorem (Jirina (58), Lamperti (67))

There exists Ψ *of the form*

$$\Psi(q) = \frac{\sigma^2}{2} q^2 + \gamma q + \int_0^{+\infty} \left(e^{-qx} - 1 + qx 1_{\{x \leq 1\}}\right) \pi(\mathrm{d}x)$$

such that $\frac{\mathrm{d}v_t(\lambda)}{\mathrm{d}t} = -\Psi(v_t(\lambda))$.

Asymptotic behaviors

Proposition (Grey 74)

- *Supercritical case:* $\Psi'(0) \in [-\infty, 0[$,
  - *The largest root of* Ψ, *called* ρ, *is in* $(0, +\infty]$
  - $X_t(x) \underset{t\to\infty}{\longrightarrow} 0$ *with probability* $e^{-x\rho}$
  - $X_t(x) \underset{t\to\infty}{\longrightarrow} +\infty$ *with probability* $1 - e^{-x\rho}$

  Non-explosion: $\mathbb{P}(\forall t, X_t(x) < \infty) = 1 \Longleftrightarrow \int_0 \frac{\mathrm{d}q}{|\Psi(q)|} = +\infty.$
- *Subcritical case:* $\Psi'(0) \geq 0$,
  - *The largest root of* Ψ *is* 0
  - $X_t(x) \underset{t\to\infty}{\longrightarrow} 0$ *a.s.*

  Persistence: $\mathbb{P}(\forall t; X_t(x) > 0) = 1 \Longleftrightarrow \int^{+\infty} \frac{\mathrm{d}q}{\Psi(q)} = +\infty$

Continuous population model

Definition (Flow of CSBPs: Bertoin Le Gall 2000, Dawson Li 2012, Duquesne Labbé 2014)

Let N_Ψ a measure on $\mathcal{D}(\mathbb{R}_+^\star, \bar{\mathbb{R}}_+)$ s.t. "$N_\Psi(\cdot) = \lim_{x\to 0} \frac{1}{x}\mathbb{P}_x^\Psi(\cdot)$."[ab] Consider $\mathcal{N} = \sum_{i\in I} \delta_{(x_i, X^i)}$ a PPP over $\mathbb{R}_+ \times \mathcal{D}$ with intensity $\mathrm{d}x \otimes N_\Psi(\mathrm{d}X)$. For all $x \geq 0$, let $X_0(x) = x$ and for all $t > 0$,

$$X_t(x) = \sum_{x_i \leq x} X_t^i$$

- *for all $t \geq 0$ $(X_t(x), x \geq 0)$ is a (càdlàg) subordinator with Laplace exponent $\lambda \mapsto v_t(\lambda)$*
- *for any $y \geq x$, $(X_t(y) - X_t(x), t \geq 0)$ is a CSBP(Ψ) started from $y - x$, independent of $(X_t(x), t \geq 0)$.*

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[a] in the infinite variation case.

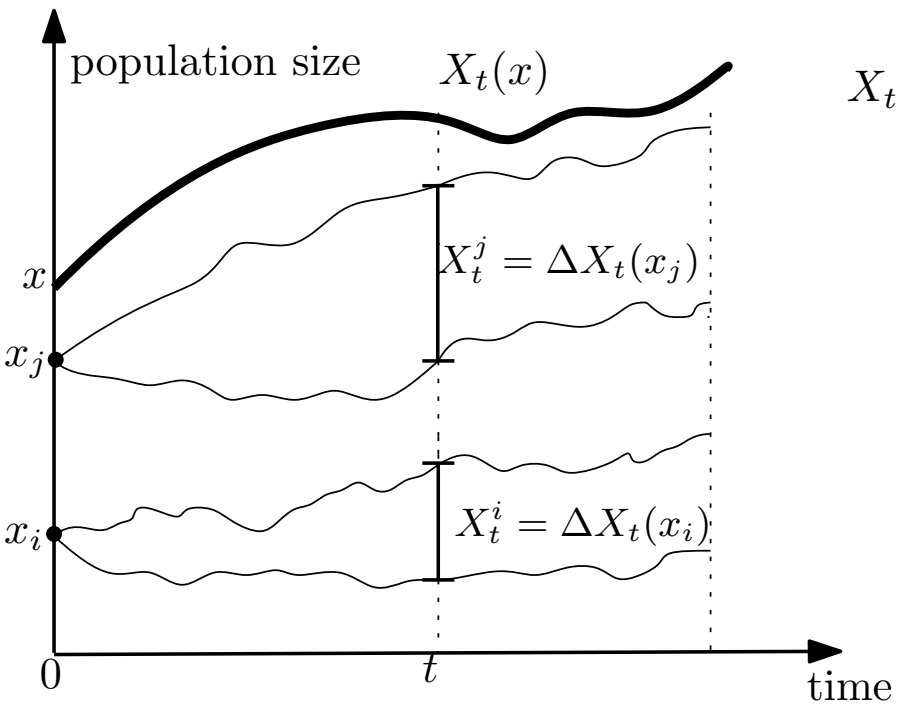
[b] In the finite variation case, there is an other Poisson representation

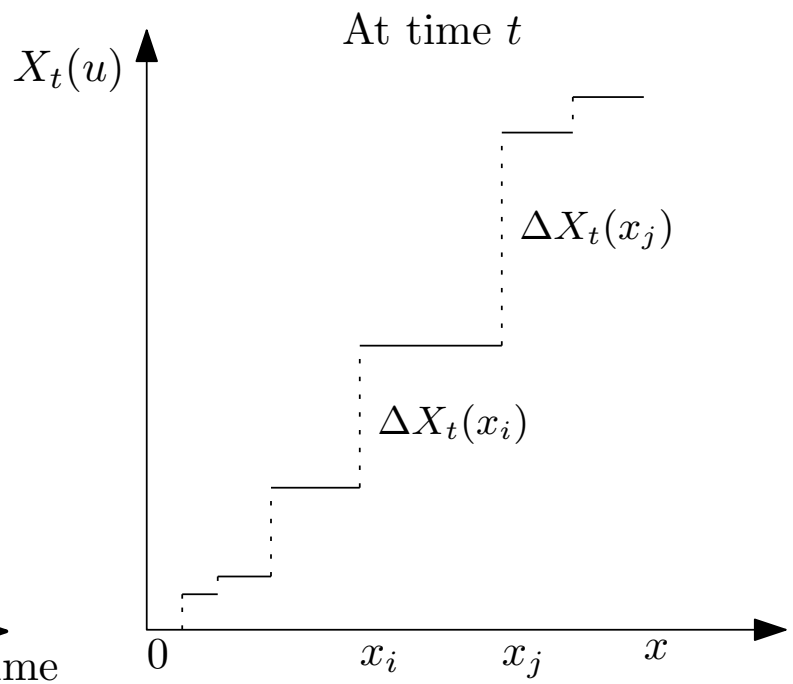
The flow $(X_t(x), t \geq 0, x \geq 0)$ provides a **continuous population**:

- The individual y is a descendant at time t of the individual x living at time 0 if

$$X_t(x-) < y < X_t(x)$$

- $\Delta X_t(x) = X_t(x) - X_t(x-)$ is the progeny of x at time t.





Super-individuals

In a *non-explosive* CSBP with **infinite mean** ($\Psi(u)/u \underset{u\to 0+}{\to} -\infty$) and in a *persistent* CSBP with **infinite variation** ($\Psi(u)/u \underset{u\to+\infty}{\to} +\infty$), some individuals have a progeny that overwhelms the total progeny of all individuals *below* them.

Definition

The individual x is a **super-individual** *if* $\lim_{t\to+\infty} \frac{\Delta X_t(x)}{X_t(x-)} = +\infty$ *a.s.*

Denote by $\mathcal{S}$ the set of super-individuals

$$\mathcal{S} := \left\{ x > 0;\ \lim_{t\to+\infty} \frac{\Delta X_t(x)}{X_t(x-)} = +\infty \right\}.$$

There is an order between super-individuals: if $x_1, x_2 \in \mathcal{S}$ and $x_1 \le x_2$, then $\frac{\Delta X_t(x_1)}{\Delta X_t(x_2)} \le \frac{X_t(x_2-)}{\Delta X_t(x_2)} \underset{t\to\infty}{\longrightarrow} 0$.

Supercritical CSBP with finite mean

Definition (Bertoin et al. 2008)

An individual x is **prolific** *if* $\Delta X_t(x) \underset{t\to\infty}{\longrightarrow} +\infty$ *a.s.*

$$\mathcal{P} := \{x > 0; \Delta X_t(x) \underset{t\to\infty}{\longrightarrow} +\infty\}$$

Proposition (Grey 74+Bertoin et al. 2008+ Duquesne Labbé 2014)

Assume $\Psi'(0+) \in (-\infty, 0)$. *Almost-surely, for all* $x > 0$,

$$v_{-t}(\lambda)X_t(x-) \underset{t\to\infty}{\longrightarrow} W^\lambda_{x-} \text{ and } v_{-t}(\lambda)X_t(x) \underset{t\to\infty}{\longrightarrow} W^\lambda_x$$

where $(W^\lambda_x, x \geq 0)$ *is a càdlàg subordinator*

- $\lambda \mapsto v_{-t}(\lambda)$, *the inverse of* $\lambda \mapsto v_t(\lambda)$
- $\mathcal{P} = \{x > 0; \Delta W^\lambda_x > 0\}$ *and* $\mathcal{S} \cap \mathcal{P}$ *is degenerate.*

Supercritical CSBP with infinite mean

Theorem (preliminary version, Grey 77, F. Ma 16)

Suppose $\Psi'(0+) = -\infty$ *and* $\int_0 \frac{\mathrm{d}u}{\Psi(u)} = -\infty$. *Fix* $\lambda_0 \in (0, \rho)$, *and define* $G(y) := \exp\left(-\int_y^{\lambda_0} \frac{\mathrm{d}u}{\Psi(u)}\right)$ *for* $y \in (0, \rho)$. *Then, for all* $x \geq 0$, *almost-surely*

$$e^{-t} G\left(\frac{1}{X_t(x)} \wedge \rho\right) \underset{t\to+\infty}{\longrightarrow} Z_x.$$

- G *is decreasing and slowly varying at* 0
- $\{Z_x = 0\} = \{X_t(x) \underset{t\to\infty}{\longrightarrow} 0\}$
- $\mathbb{P}(Z_x \leq z) = \exp(-xG^{-1}(z))$ *with* $G^{-1}(z) = v_{\log(1/z)}(\lambda_0)$.

Example (Neveu's mechanism)

$\Psi(u) = u \log u$ for which $\rho = 1$. Fix $\lambda_0 = \frac{1}{e}$, $G(z) = \log(1/z)$

Question

What is the nature of the process $(Z_x, x \geq 0)$*?*

Definition (extremal process="subordinator for the max operator")

A process $(Z_x, x \geq 0)$ *is an extremal-F process if*

$$\begin{cases} \mathbb{P}(Z_x \leq z) = F(z)^x \\ Z_{x+y} = Z_x \vee Z'_y \text{ a.s. where } Z'_y \perp\!\!\!\perp (Z_u)_{0 \leq u \leq x} \text{ and } Z'_y \stackrel{d}{=} Z_y \end{cases}$$

Lemma

$(Z_x, x \geq 0)$ *is an extremal-F process with* $F(z) = e^{-v_{\log(1/z)}(\lambda_0)}$.

Proof.

$X_t(x+y) = X'_t(y) + X_t(x)$ with $X'_t(y) = X_t(x+y) - X_t(x)$.
$\Longrightarrow e^{-t} G(\frac{1}{X_t(x+y)}) \geq e^{-t} G(\frac{1}{X_t(x)}) \vee e^{-t} G(\frac{1}{X'_t(y)})$ thus

$Z_{x+y} \geq Z_x \vee Z'_y$ a.s. but $Z_{x+y} \stackrel{d}{=} Z_x \vee Z'_y$ □

Fact ("Lévy-Itô decomposition" of extremal processes)

Consider a PPP with intensity $dx \otimes \mu$ over $\mathbb{R}_+ \times \mathbb{R}$. The process of its records is a càdlàg extremal-F process with $F(z) = e^{-\bar{\mu}(z)}$.

Theorem (F. Ma 2016, supercritical part)

There exists $\mathcal{M} := \sum_{i \in I} \delta_{(x_i, Z_i)}$ a PPP$(\mathrm{d}x \otimes \mu(\mathrm{d}z))$ with $\bar{\mu}(z) = v_{\log(1/z)}(\lambda_0)$ such that almost-surely, for all $x \geq 0$

$$e^{-t} G\left(\frac{1}{X_t(x)} \wedge \rho\right) \underset{t \to +\infty}{\longrightarrow} Z_x = \sup_{x_i \leq x} Z_i$$

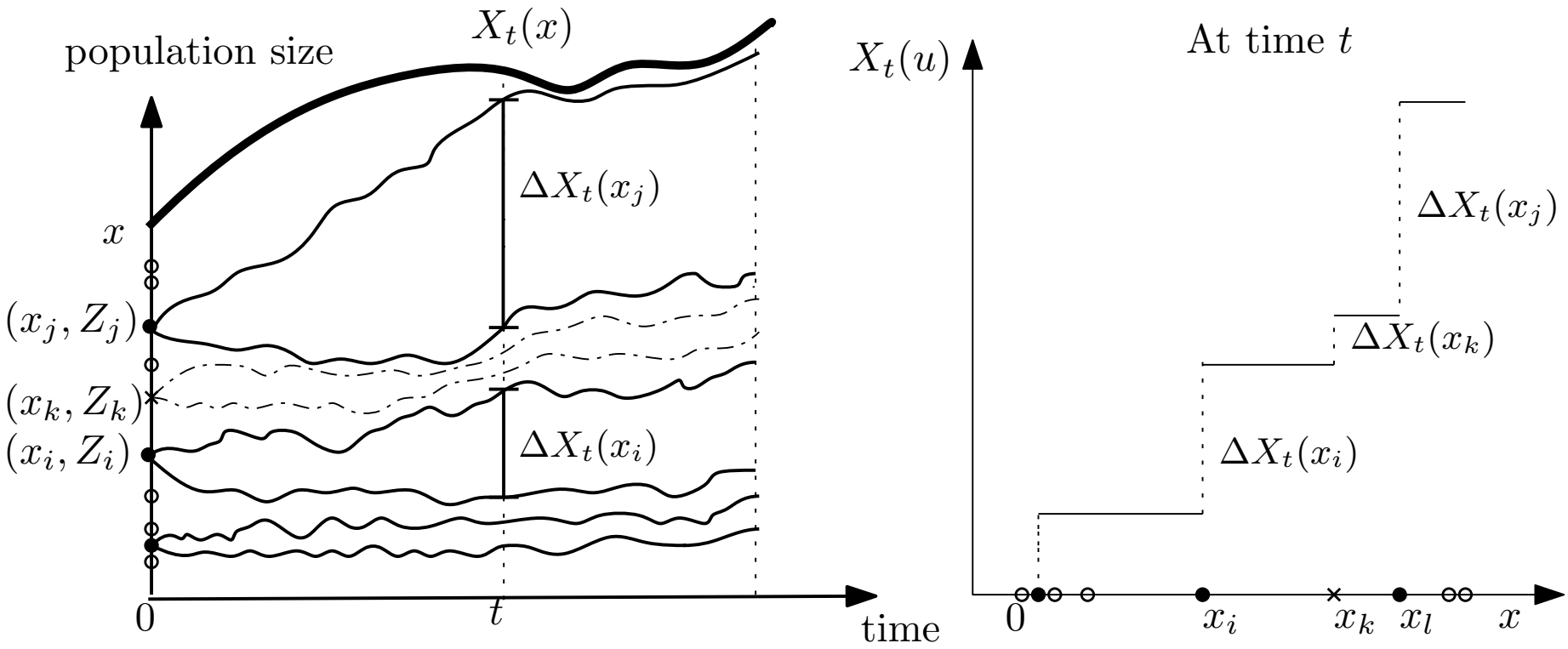
μ has total mass $\bar{\mu}(0) = \rho \in (0, +\infty]$ and has no atom.

Moreover,

Proposition

Almost-surely, if $Z_i > Z_j$ then $\Delta X_t(x_j)/\Delta X_t(x_i) \underset{t \to \infty}{\longrightarrow} 0$,
$\mathcal{P} = \{x_i; Z_i > 0, i \in I\}$ and $\mathcal{S} \cap \mathcal{P} = \{x > 0; \Delta Z_x > 0\}$ a.s.

Poisson representation



- $\circ$ denotes (x_l, Z_l) such that $Z_l = 0$ *(non prolific)*
- $\times$ denotes (x_i, Z_i) not a partial record (*prolific non superprolific*)
- $\bullet$ denotes (x_i, Z_i) partial record: (*superprolific*)

Subcritical case

Assume $\Psi'(0+) \geq 0$. Since for all $x \geq 0$, $X_t(x) \underset{t\to\infty}{\longrightarrow} 0$ a.s. there is no prolific individual in the population. Recall

$$\mathcal{S} := \left\{ x > 0; \lim_{t\to+\infty} \frac{\Delta X_t(x)}{X_t(x-)} = +\infty \right\}.$$

A super-individual is an individual whose *decay is much slower than the decay of all individuals below it.*

Subcritical CSBP with finite variation

Definition (variation)

For any Ψ,

$$\lim_{u\to+\infty} \frac{\Psi(u)}{u} =: \mathbf{d} = +\infty 1_{\{\sigma>0\}} + \gamma + \int_0^1 x\pi(\mathrm{d}x) \in \mathbb{R}\cup\{+\infty\}.$$

Proposition (Grey 74 + Duquesne Labbé 2014)

Assume $\mathbf{d}\in\mathbb{R}$. *For all* x, $(X_t(x), t\geq 0)$ *is persistent and almost-surely, for all* $x\geq 0$

$$v_{-t}(\lambda)X_t(x) \underset{t\to+\infty}{\longrightarrow} V_x^\lambda \text{ and } v_{-t}(\lambda)X_t(x-) \underset{t\to+\infty}{\longrightarrow} V_{x-}^\lambda.$$

where $(V_x^\lambda, x\geq 0)$ *is a càdlàg subordinator. Thus* $\mathcal{S}$ *is degenerate.*

Subcritical process with infinite variation

Theorem (F. Ma 2016, subcritical part)

Suppose $\mathbf{d} = +\infty$ *and* $\int^{+\infty} \frac{\mathrm{d}u}{\Psi(u)} = +\infty$. *Fix* $\lambda_0 \in (0, +\infty)$ *and define* $G(y) := \exp\left(-\int_{\lambda_0}^{y} \frac{\mathrm{d}u}{\Psi(u)}\right)$ *on* $(0, +\infty)$.

There exists $\mathcal{M} := \sum_{i \in I} \delta_{(x_i, Z_i)}$ *a PPP*$(\mathrm{d}x \otimes \mu(\mathrm{d}z))$ *with* $\bar{\mu}(z) = G^{-1}(z) = v_{\log(z)}(\lambda_0)$ *such that almost-surely*

$$e^t G\left(\frac{1}{X_t(x)}\right) \underset{t \to +\infty}{\longrightarrow} Z_x = \sup_{x_i \leq x} Z_i \text{ for all } x \geq 0.$$

$\mu(0, \infty) = \infty$ *and* μ *has no atom.*

Proposition

$$\mathcal{S} = \{x > 0; \Delta Z_x > 0\} \text{ a.s.}$$

Corollary (Supercritical case (Grey 79 for GW chains))

Consider two independent CSBPs $(X_t(x), t \geq 0)$, $(Y_t(y), t \geq 0)$ *non-explosive with infinite mean and same mechanism. Conditionally on* $\{X_t(x) \underset{t\to+\infty}{\longrightarrow} +\infty\} \cap \{Y_t(y) \underset{t\to+\infty}{\longrightarrow} +\infty\}$,

$$\frac{X_t(x)}{Y_t(y)} \underset{t\to+\infty}{\longrightarrow} \begin{cases} +\infty & \text{with probability } \frac{x}{x+y} \\ 0 & \text{with probability } \frac{y}{x+y}. \end{cases}$$

Corollary (Subcritical case)

Consider two independent subcritical persistent CSBPs $(X_t(x), t \geq 0)$, $(Y_t(y), t \geq 0)$ *with infinite variation and same mechanism*

$$\frac{X_t(x)}{Y_t(y)} \underset{t\to+\infty}{\longrightarrow} \begin{cases} +\infty & \text{with probability } \frac{x}{x+y} \\ 0 & \text{with probability } \frac{y}{x+y}. \end{cases}$$

Eve property

Duquesne and Labbé (2014) have considered the following question:

Question

*Does the population (encoded by a flow of CSBPs) concentrates on the progeny of a single individual? In other words, fix the initial size x, is there an individual $e \in [0, x]$ (**the Eve**), such that*

$$\frac{\Delta X_t(e)}{X_t(x)} \underset{t\to\infty}{\longrightarrow} 1 \text{ a.s.?}$$

Corollary (Duquesne and Labbé 2014)

In the case of infinite variation and infinite mean, the population has an Eve. In our framework, the Eve corresponds to the last super-individual in $[0, x]$.

Neveu case

Consider $(X_t(x), t \geq 0)$ a CSBP of Neveu. It is non-explosive with infinite mean and persistent with infinite variation. A well-known result, attributed to Neveu (1992) (shown in Fleischmann, Sturm (2004)), states that for any fixed x

$$e^{-t} \log X_t(x) \underset{t \to +\infty}{\longrightarrow} Z_x \text{ a.s.}$$

where Z_x has a Gumbel law over $\mathbb{R}$. By combining our results, we get:

Proposition

Almost-surely for all $x \geq 0$,

$$e^{-t} \log X_t(x) \underset{t \to +\infty}{\longrightarrow} Z_x$$

where $(Z_x, x \geq 0)$ is an extremal-Λ process with $\Lambda(z) = e^{-e^{-z}}$ for $z \in \mathbb{R}$.

Conclusion and references

- In the non-persistent and explosive cases, extremal processes still arise, but through the times of explosion and absorption.
- When the reproduction has infinite mean, the infinite divisibility of the flow $(X_t(x), t \geq 0, x \geq 0)$ becomes the *max*-infinite divisibility of the process $(Z_x, x \geq 0)$. Less clear in the subcritical setting...

Bertoin, Le Gall, *The Bolthausen-Sznitman coalescent and the genealogy of CSBPs*, PTRF (2000)

Duquesne, Labbé, *On the Eve property for CSBP*, Electron. J. Probab. (2014)

Neveu, *A CSBP in relation with the GREM model of spin glass theory*, Rapport interne (unprinted:-(Ecole Polytechnique, 1992.

- D. R. Grey, *Asymptotic behaviour of CSBPs*, JAP (1974),
- D. R. Grey, *Almost-sure convergence in Markov branching process with infinite mean*, J. Appl. Probability **14** (1977), 702–716.
- D. R. Grey,*On regular branching processes with infinite mean*, SPA (1978/79),

Thank you!!

Sketch of proof

Fix x.

1. For all $\lambda \in (0, \rho)$, $v_{-t}(\lambda) X_t(x) \underset{t\to\infty}{\longrightarrow} W_x^\lambda \in \{0, \infty\}$ a.s. and $\mathbb{P}(W_x^\lambda = \infty) = e^{-x\lambda}$.
2. If $\lambda' \geq \lambda$ then $W_x^{\lambda'} \geq W_x^\lambda$
3. $\Lambda_x := \inf\{\lambda \in (0, \rho) \cap \mathbb{Q}; W_x^\lambda = +\infty\}$ is a random variable!
4. Let $\lambda' \in (0, \rho)$. If $\lambda < \Lambda_x < \lambda'$, then for large t :

$$\begin{aligned} & v_{-t}(\lambda) \leq 1/X_t(x) \text{ and } v_{-t}(\lambda') \geq 1/X_t(x) \\ \Longrightarrow\ & G(v_{-t}(\lambda)) \geq G(1/X_t(x)) \geq G(v_{-t}(\lambda')) \\ \Longrightarrow\ & G(\lambda) \leq e^{-t} G(1/X_t(x)) \leq G(\lambda') \end{aligned}$$

5. If $\Lambda_x = \rho$ then $X_t(x) \underset{t\to\infty}{\longrightarrow} 0$.

This yields the a.s convergence: $e^{-t} G(1/X_t(x) \wedge \rho) \underset{t\to\infty}{\longrightarrow} Z_x$