# *Lévy forward price approach for multiple yield curves and low/negative interest rates*

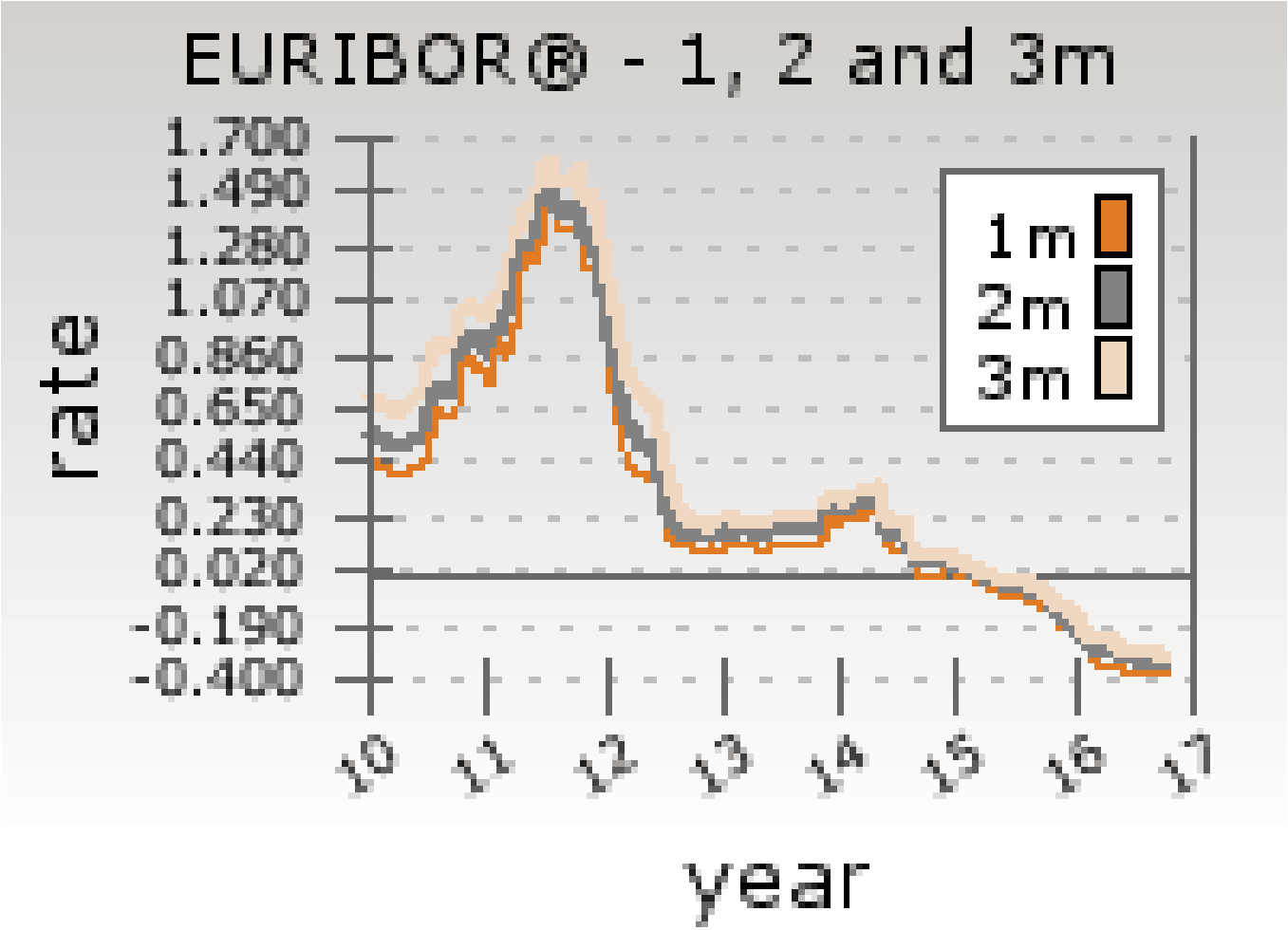
Zorana Grbac

LPMA, Université Paris Diderot

Joint work with E. Eberlein and Ch. Gerhart

Berlin-Paris Workshop on Stochastic Analysis with Applications in Biology and Finance
Berlin, 2-4 November 2016

# *Euribor rates*



# *The multiple-curve spreads*

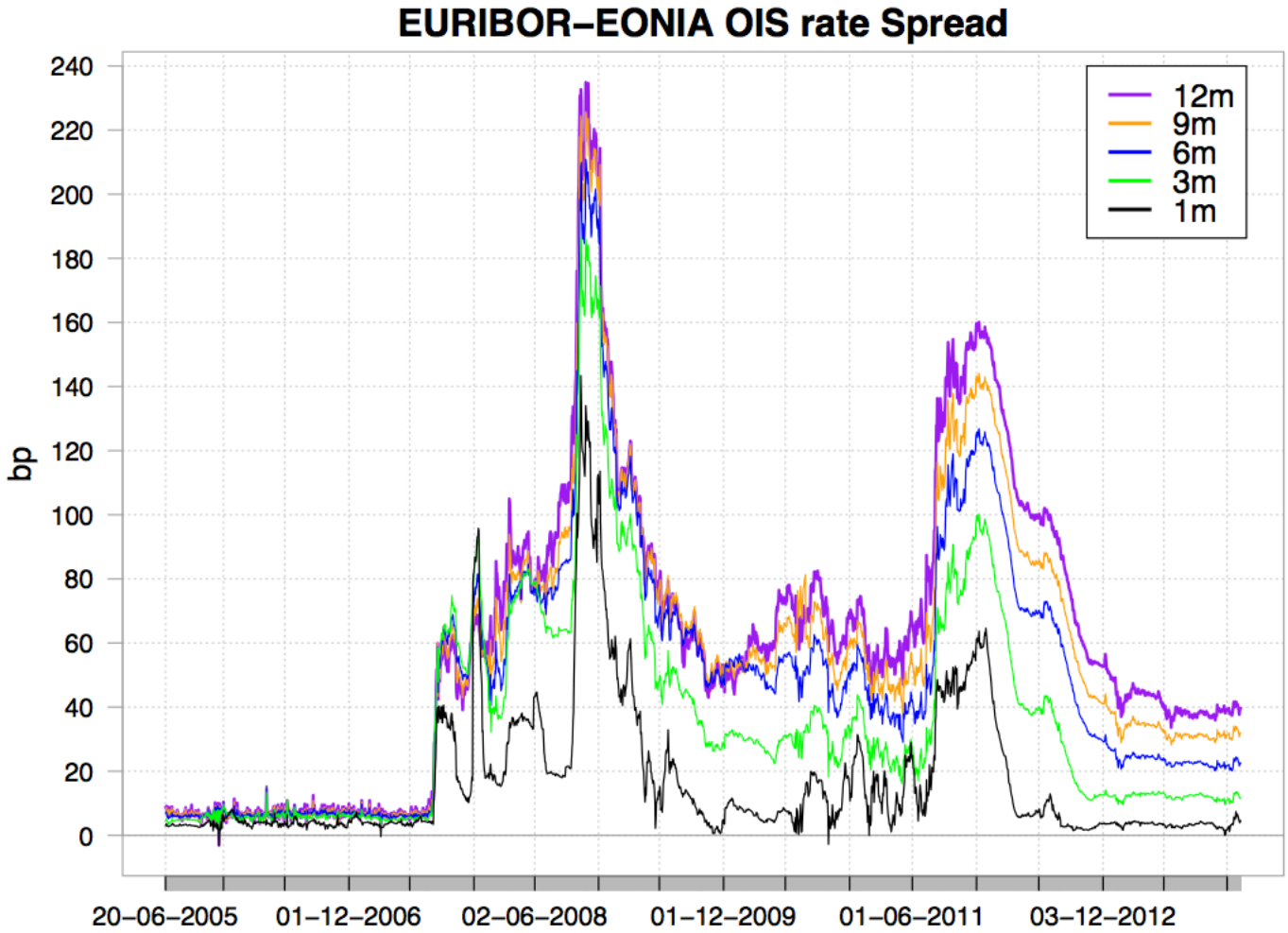


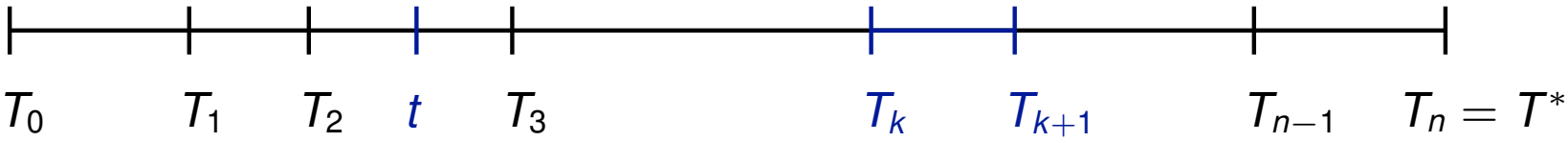

FIGURE 1. Evolution of the spreads between EURIBOR and EONIA OIS rates for different tenors.

# *Motivation*

- in the current market situation we observe negative Euribor rates for all tenors (even 12 months), which are very low in absolute value
- the spreads between the Euribor rates and the Eonia-based OIS rate still remain important and they continue to be positive, even though both rates have negative values
- moreover, the spreads are monotone with respect to tenor: the spreads corresponding to the Euribor rates with longer tenors are higher than those corresponding to the Euribor rates with shorter tenors
- ideally one would like a model capable of reproducing these features, which is analytically tractable and flexible enough to provide a good fit to market option data

# *Notation and introductory remarks - single curve case*

Discrete tenor structure: $0 = T_0 < T_1 < \ldots < T_n = T^*$, with $\delta = T_{k+1} - T_k$
(typically $\delta =$ 1month, 3 months,.., 12 months)




Zero coupon bonds: $B(\cdot, T_1), \ldots, B(\cdot, T_n)$

Forward Libor rate at time $t \leq T_k$ for the accrual period $[T_k, T_{k+1}]$

$$L(t, T_k) = \frac{1}{\delta}\left(\frac{B(t, T_k)}{B(t, T_{k+1})} - 1\right)$$

Forward price at time $t \leq T_k$

$$F(t, T_k) = \frac{B(t, T_k)}{B(t, T_{k+1})} = 1 + \delta L(t, T_k)$$

# *Introductory remarks and notation - single curve case*

Two approaches for modeling of the Libor rates:

- modeling directly the Libor rates (Libor market model (LMM) and various extensions):

$$L(t, T_k) = L(0, T_k) \exp X_t$$

- modeling the forward price process (Eberlein and Özkan (2005), Keller-Ressel et al (2013)):

$$F(t, T_k) = (1 + \delta L(0, T_k)) \exp X_t, \qquad L(t, T_k) = \frac{1}{\delta}(F(t, T_k) - 1)$$

where $X$ is some semimartingale.

- the forward price approach is particularly known for its analytical tractability due to the fact that the modeling quantity corresponds exactly to the density process in the forward measure changes (see for example Glau et al. (2016))
- however, in this approach the Libor rates can become negative, unless specific classes of processes are used $\Rightarrow$ in the current market situation this is an advantage!

# *Introductory remarks and notation - single curve case*

Moreover, if in a small time interval from $t_1$ to $t_2$ the driving process in the exponent changes its value by a small amount $\Delta$, then in the Libor model we have

$$L(t_2, T_k) = L(t_1, T_k) \exp \Delta \sim L(t_1, T_k) + \Delta L(t_1, T_k)$$

$$\Rightarrow L(t_2, T_k) - L(t_1, T_k) \sim L(t_1, T_k)\Delta$$

whereas in the forward price model

$$L(t_2, T_k) = \frac{1}{\delta}((1 + \delta L(t_1, T_k)) \exp \Delta - 1) \sim L(t_1, T_k) + \frac{\Delta}{\delta}$$

$$\Rightarrow L(t_2, T_k) - L(t_1, T_k) \sim \frac{\Delta}{\delta}$$

- In the Libor model, Libor rates change by an amount that is relative to their level $\Rightarrow$ when the rates are low, large movements of the driving process are needed for small changes in the rates $\Rightarrow$ problems in calibration (exploding volatilities in the LMM, big jumps in the Lévy model)
- On the contrary, the increment of the Libor rates in the forward price model does not depend on the actual level of the rate $\Rightarrow$ increased flexibility leading to excellent calibration

# *Advantages of the forward price approach - single curve case*

- the forward price model allows to accommodate negative interest rates with a lower bound given by

$$L(t, T_k) = \frac{1}{\delta}(F(t, T_k) - 1) \geq -\frac{1}{\delta}$$

- it possesses excellent calibration properties (evidenced in Eberlein and Kluge (2006, 2007) and Eberlein and Koval (2006)), even in the presence of very low interest rates

- it can be seen as a shifted Libor model (which is also scaled), which has been one of the financial industry answers to calibration problems in the situation of low and negative interest rates

# *Multiple curve modeling*

As we have seen from the graph, we need to model multiple interest rate curves:

- the reference rate is the OIS forward rate $L^{OIS}(t, T_k)$ bootstrapped from the OIS curve via OIS zero-coupon bonds $B^{OIS}(t, T_k)$

  We have

$$F^{OIS}(t, T_k) = \frac{B^{OIS}(t, T_k)}{B^{OIS}(t, T_{k+1})} = 1 + \delta L^{OIS}(t, T_k)$$

  where $F^{OIS}(t, T_k)$ is the OIS forward price process

- in addition, let $m \in \mathbb{N}$ be the number of Euribor rates. For every $i \in \{1, \dots, m\}$, we consider an equidistant discrete tenor structure $\mathscr{T}^i = \{T_0^i, \dots, T_{n_i}^i\}$ corresponding to curve $i$, where $n_i, n \in \mathbb{N}$, $\mathscr{T}^i \subset \mathscr{T} := \{T_0, \dots, T_n\}$ and $0 \leq T_0^i = T_0 < T_1^i < \cdots < T_{n_i}^i = T_n = T^*$. As before, the year fractions between the dates $T_{k-1}^i$ and $T_k^i$ are denoted by $\delta^i = \delta^i(T_{k-1}^i, T_k^i)$ for all $k \in \{1, \dots, n_i\}$. Furthermore, we postulate that $\mathscr{T}^m \subset \cdots \subset \mathscr{T}^1 \subset \mathscr{T}$.

- We have the forward Euribor rates $L^i(t, T_k^i)$, for various tenors $\delta^i$, defined by

$$L^i(t, T_k^i) = \mathbb{E}_{T_{k+1}^i}\left[L^i(T_k^i, T_k^i) | \mathscr{F}_t\right]$$

  using the spot Euribor rates $L^i(T_k^i, T_k^i)$

# *The multiple curve spreads*

The spreads between the relevant quantities of the multiple term structure and OIS curve can be considered in two ways as

1. additive (forward) spreads (plotted in the graph)

$$s^i(t, T_k^i) := L^i(t, T_k^i) - L^{OIS}(t, T_k^i)$$

   or

2. multiplicative (forward) spreads

$$S^i(t, T_k^i) := \frac{1 + \delta^i L^i(t, T_k^i)}{1 + \delta^i L^{OIS}(t, T_k^i)} = \frac{1 + \delta^i L^i(t, T_k^i)}{F^{OIS}(t, T_k^i)}.$$

In the forward price approach it is more convenient to model multiplicative (forward) spreads. Hence, we have

$$1 + \delta^i L^i(\cdot, T_k^i) = S^i(\cdot, T_k^i) F^{OIS}(\cdot, T_k^i). \tag{1}$$

The modeling choices for the spreads below are inspired by a credit risk framework with ratings from Eberlein and G. (2013)

# *Summary of the three possible modeling choices*

Assume that the OIS rates are modeled as for a given semimartingale

$$F^{OIS}(t, T_k) = F^{OIS}(0, T_k) \exp\left(\int_0^t \lambda^{OIS}(s, T_k) \mathrm{d}X_s + \mathit{drift}\right)$$

*(a)* the most tractable choice

$$S^i(t, T_k^i) = S^i(0, T_k^i) \exp\left(\int_0^t \gamma^i(s, T_k^i) \mathrm{d}X_s + \mathit{drift}\right)$$

*(b)* positive spreads

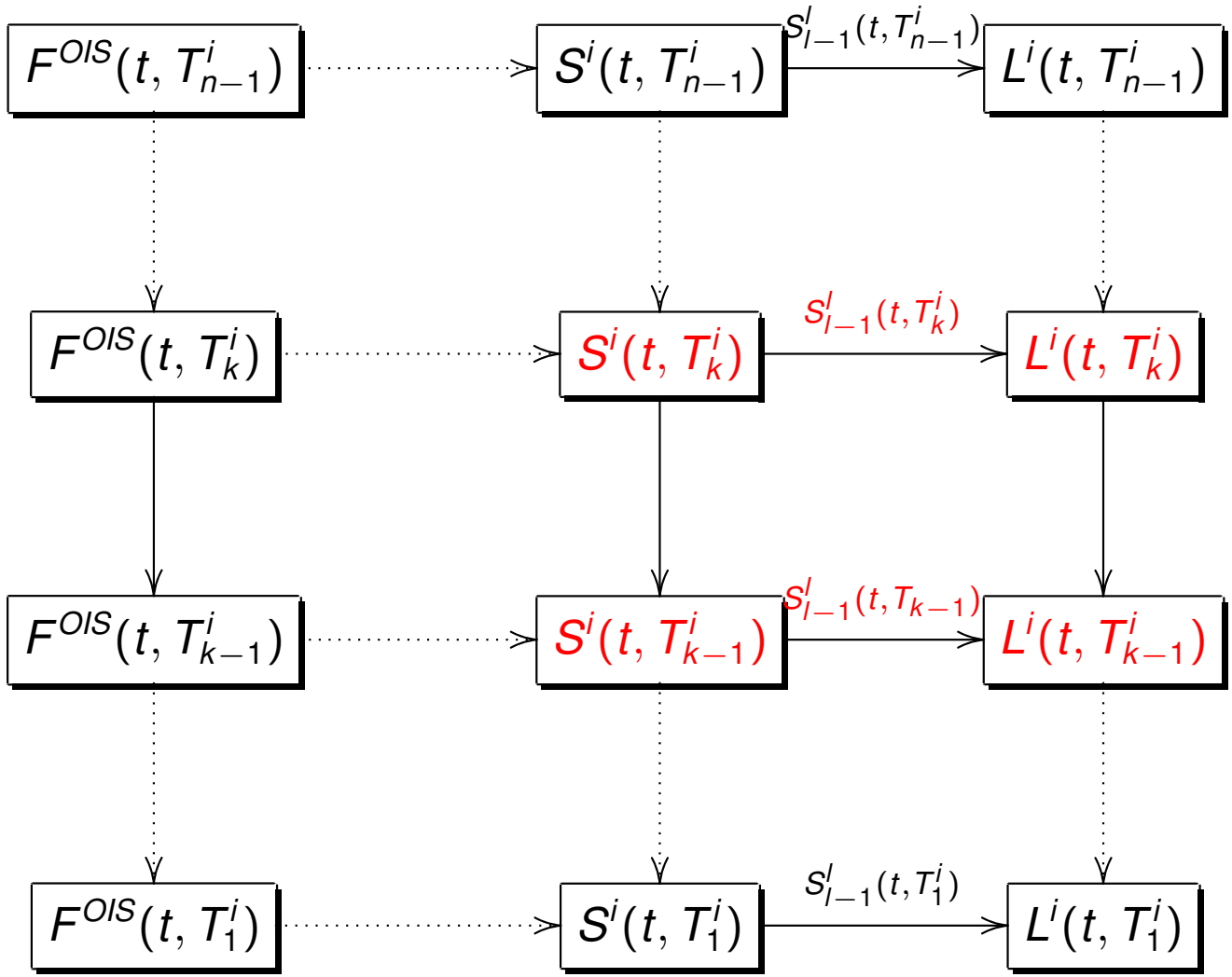
$$S^i(t, T_k^i) = S^i(0, T_k^i) \left(1 + \exp\left(\int_0^t \bar{\gamma}^i(s, T_k^i) \mathrm{d}X_s + \mathit{drift}\right)\right)$$

*(c)* positive and ordered spreads

$$S^i(t, T_k^i) = S^i(0, T_k^i) \prod_{l=1}^{j} S_{l-1}^l(t, T_k^i)$$

and

$$S_{l-1}^l(t, T_k^i) = S_{l-1}^l(0, T_k^i) \left(1 + \exp\left(\int_0^t \bar{\gamma}_{l-1}^l(s, T_k^i) \mathrm{d}X_s + \mathit{drift}\right)\right)$$




OIS forward price Spread $i$ Libor rate $i$

# *The Lévy driving process*

Let $(\Omega, \mathcal{F}_{T^*}, \mathbb{F} = (\mathcal{F}_t)_{0 \le t \le T^*}, \mathbb{P}_{T^*})$ be a complete stochastic basis.

- as driving process take a time-inhomogeneous Lévy process $X = (X^1, \ldots, X^d)$ whose Lévy measures satisfy an integrability condition:

$$\sup_{0 \le s \le T^*} \int_{|x|>1} \exp\langle u, x \rangle F_s^{T^*}(\mathrm{d}x) < \infty \quad \left( u \in [-(1+\varepsilon)M, (1+\varepsilon)M]^d \right).$$

- $X$ is a special semimartingale with canonical decomposition

$$X_t = \int_0^t b_s \mathrm{d}s + \int_0^t \sqrt{c_s} \mathrm{d}W_s^{T^*} + \int_0^t \int_{\mathbb{R}^d} x(\mu - \nu^{T^*})(\mathrm{d}s, \mathrm{d}x),$$

where $W^{T^*}$ denotes a $\mathbb{P}_{T^*}$-standard Brownian motion and $\mu$ is the random measure of jumps of $X$ with $\mathbb{P}_{T^*}$-compensator $\nu^{T^*}(\mathrm{d}s, \mathrm{d}x) = F_s^{T^*}(\mathrm{d}x)\mathrm{d}s$. We assume that $b = 0$.

# A. The OIS forward price approach (backward induction)

This approach is inspired by Eberlein and Özkan (2005) forward price approach in the single curve setup.

Fix an arbitrary tenor $i = 1, \ldots, m$: going backwards, we have for each $T_k^i$

*(i)* define the forward measure $\mathbb{P}_{T_{k+1}^i}$ via

$$\left.\frac{\mathrm{d}\mathbb{P}_{T_{k+1}^i}}{\mathrm{d}\mathbb{P}_{T^*}}\right|_{\mathcal{F}_t} = \frac{B^{OIS}(0,T^*)}{B^{OIS}(0,T_{k+1}^i)}\frac{B^{OIS}(t,T_{k+1}^i)}{B^{OIS}(t,T^*)} = \prod_{l=k+1}^{n-1}\frac{F^{OIS}(t,T_l^i)}{F^{OIS}(0,T_l^i)}.$$

*(ii)* the dynamics of the forward price $F^{OIS}(\cdot, T_k^i)$ under this measure

$$F^{OIS}(t,T_k^i) = F^{OIS}(0,T_k^i)\exp\left(\int_0^t b^{OIS}(s,T_k^i)\mathrm{d}s + \int_0^t \sigma^{OIS}(s,T_k^i)\mathrm{d}X_s^{T_{k+1}^i}\right), \quad (2)$$

where

$$X_t^{T_{k+1}^i} = \int_0^t \sqrt{c_s}\mathrm{d}W_s^{T_{k+1}^i} + \int_0^t\int_{\mathbb{R}^d} x(\mu - \nu^{T_{k+1}^i})(\mathrm{d}s,\mathrm{d}x)$$

with $\mathbb{P}_{T_{k+1}^i}$-Brownian motion $W^{T_{k+1}^i}$ and

$$\nu^{T_{k+1}^i}(\mathrm{d}s,\mathrm{d}x) = \exp\left(\sum_{l=k+1}^{n-1}\langle\lambda^{OIS}(s,T_l^i),x\rangle\right)\nu^{T^*}(\mathrm{d}s,\mathrm{d}x).$$

# *B. The multiplicative spreads*

In the specification (a):

we set for the dynamics of the spread $S^i(t, T_k^i)$

$$S^i(t, T_k^i) = S^i(0, T_k^i) \exp\left( \int_0^t b^i(s, T_k^i) \mathrm{d}s + \int_0^t \gamma^i(s, T_k^i) \mathrm{d}X_s^{T_{k+1}^i} \right),$$

The drift term $b^i(s, T_k^i)$ is chosen such that

$S^i(\cdot, T_k^i)$ is a $\mathbb{P}_{T_k^i}$-martingale

$\Leftrightarrow$

$L^i(\cdot, T_k^i)$ is a $\mathbb{P}_{T_{k+1}^i}$-martingale.

Then

$$\begin{aligned} 1 + \delta^i L^i(t, T_k^i) =& (1 + \delta^i L^i(0, T_k^i)) \exp\Big( \int_0^t (b^{OIS}(s, T_k^i) + b^i(s, T_k^i)) \mathrm{d}s \\ &+ \int_0^t (\lambda^{OIS}(s, T_k^i) + \gamma^i(s, T_k^i)) \mathrm{d}X_s^{T_{k+1}^i} \Big) \end{aligned}$$

# *B. The multiplicative spreads*

In the specification (c):

We set for the dynamics of the spread $S^l_{l-1}(t, T^i_k)$

$$S^l_{l-1}(t, T^i_k) = S^l_{l-1}(0, T^i_k) \frac{1 + \sum_{i=1}^{l} \exp\left( \int_0^t \bar{\gamma}^i_{i-1}(s, T^i_k) dX_s^{T^i_k} + \int_0^t \bar{b}^i_{i-1}(s, T^i_k) ds \right)}{1 + \sum_{i=1}^{l-1} \exp\left( \int_0^t \bar{\gamma}^i_{i-1}(s, T^i_k) dX_s^{T^i_k} + \int_0^t \bar{b}^i_{i-1}(s, T^i_k) ds \right)},$$

where we have

$$\begin{aligned} \bar{b}^i_{i-1}(t, T^i_k) = & -\frac{1}{2} \langle \bar{\gamma}^i_{i-1}(t, T^i_k), c_t \bar{\gamma}^i_{i-1}(t, T^i_k)^{\mathsf{T}} \rangle \\ & - \int_{\mathbb{R}^d} \left( e^{\langle \bar{\gamma}^i_{i-1}(t, T^i_k), x \rangle} - 1 - \langle \bar{\gamma}^i_{i-1}(t, T^i_k), x \rangle \right) F_t^{T^i_k}(dx) \end{aligned}$$

The drift is chosen in such a way that

$$S^i(t, T^i_k) = S^i(0, T^i_k) \prod_{l=1}^{j} S^l_{l-1}(t, T^i_k)$$

is a $\mathbb{P}_{T^i_k}$-martingale

# *Caplet pricing*

As an example we investigate the most advanced model ($c$). The time-$t$ price of a caplet, for $t \leq T$ and $T + \delta^l = T_k$ for some $k \in \{1, \ldots, n\}$ with tenor $\delta^l$, is given by

$$\begin{aligned}
\text{Cpl}(t, T, \delta^l, K) =& \delta^l B^{OIS}(t, T_k)\mathbb{E}_{T_k}\left[\left(L^l(T, T_k) - K\right)^+ \middle| \mathscr{F}_t\right] \\
=& B^{OIS}(t, T_k)\mathbb{E}_{T_k}\left[\delta^l\left(L^l(T, T_k) - K\right)^+ \middle| \mathscr{F}_t\right] \\
=& B^{OIS}(t, T_k)\mathbb{E}_{T_k}\left[\left(1 + \delta^l L^l(T, T_k) - (1 + \delta^l K)\right)^+ \middle| \mathscr{F}_t\right] \\
=& B^{OIS}(t, T_k)\mathbb{E}_{T_k}\left[\left(F^{OIS}(T, T_k)\prod_{i=1}^{l} S^i_{i-1}(T, T_k) - \tilde{K}^l\right)^+ \middle| \mathscr{F}_t\right] \\
=& B^{OIS}(t, T_k)(Z^k_t)^{-1}\mathbb{E}_{T^*}\left[Z^k_T\left(F^{OIS}(T, T_k)\prod_{i=1}^{l} S^i_{i-1}(T, T_k) - \tilde{K}^l\right)^+ \middle| \mathscr{F}_t\right]
\end{aligned} \tag{3}$$

where $\tilde{K}^l = 1 + \delta^l K$ and

$$Z^k_T = \begin{cases} \prod_{j \in \mathscr{J}^{T^*}_{T_k}} \frac{F^{OIS}(T, T_{j-1}, T_j)}{F^{OIS}(0, T_{j-1}, T_j)}, & T_k < T^* \\ 1, & T_k = T^*. \end{cases}$$

# *Caplet pricing*

We make the following standing assumption:

**Assumption**

$\mathbb{VOL}$ *For every* $l \in \{1, \ldots, m\}$ *and all* $T \in [0, T^*]$*, the volatility functions are decomposable in the following way:*

$$\lambda^{OIS}(t, T) = \lambda_1^{OIS}(T)\lambda(t)$$

*and*

$$\bar{\gamma}_{l-1}^{l}(t, T) = \bar{\gamma}_{1,l-1}^{l}(T)\lambda(t)$$

*where* $\lambda_1^{OIS} : [0, T^*] \to \mathbb{R}_+$*,* $\bar{\gamma}_{1,l-1}^{l} : [0, T^*] \to \mathbb{R}_+$ *and* $\lambda : [0, T^*] \to \mathbb{R}^d$ *are deterministic, continuous and bounded in terms of*

$$|\lambda^k(t)| \leq M^{'}$$

*for every* $t \in [0, T^*]$ *and a constant* $M^{'} < M$ *and every* $k \in \{1, \ldots, d\}$*.*

# *Caplet pricing*

Let us define $Y_T = \int_0^T \lambda(s) dX_s^{T^*}$ and consider its extended characteristic function $\varphi_{Y_T}$ under $P_{T^*}$ which can be expressed as

$$\varphi_{Y_T}(z) = \exp\left(\int_0^T \theta_s\left(iz\lambda(s)\right) ds\right).$$

Recall that the extended cumulant of $X^{T^*}$ with respect to $P_{T^*}$ is given by

$$\theta_t(z) = \frac{1}{2}\langle z, c_t z\rangle + \int_{\mathbb{R}^d}\left(e^{\langle z,x\rangle} - 1 - \langle z, x\rangle\right) F_t^{T^*}(dx)$$

for any $z \in \mathbb{C}$ where it is defined.

# *Caplet pricing*

- we wapply the Fourier based valuation method from Eberlein et al. (2010) to determine the pricing formula for the present price of a caplet.
- write the needed expectation as an expectation of a payoff function $f_K^{k,l}$ applied to $Y_T$. Since $Z_0^k \equiv 1$ we get the formula for $t = 0$ in the form

$$\mathrm{Cpl}(0, T, \delta^l, K) = B^{OIS}(0, T_k)\mathbb{E}_{T^*}\left[f_K^{k,l}(Y_T)\right].$$

# *Caplet pricing*

For any $k \in \{1, \dots, n-1\}$, we have

$$f_K^{k,l}(x) := \Bigg( \tilde{D}^{OIS}(T,T,T_k) \exp\Big( \sum_{j \in \mathscr{J}_T^{T_n}} \lambda_1^{OIS}(T_{j-1}) x \Big) + \sum_{i=1}^{l} \tilde{D}^{i}(T,T,T_k) \exp\Big( [ \sum_{j \in \mathscr{J}_T^{T_n}} \lambda_1^{OIS}(T_{j-1}) + \bar{\gamma}_{1,i-1}^{i}(T) ] x \Big) - \tilde{K}^{l} A^{OIS}(T,T_k) \exp\Big( \sum_{j \in \mathscr{J}_{T_k}^{T_n}} \lambda_1^{OIS}(T_{j-1}) x \Big) \Bigg)^{+}$$

and

$$f_K^{n,l}(x) := \Bigg( D^{OIS}(T,T,T_n) \exp\Big( \sum_{j \in \mathscr{J}_T^{T_n}} \lambda_1^{OIS}(T_{j-1}) x \Big) + \sum_{i=1}^{l} D^{i}(T,T,T_n) \exp\Big( \Big[ \sum_{j \in \mathscr{J}_T^{T_n}} \lambda_1^{OIS}(T_{j-1}) + \bar{\gamma}_{1,i-1}^{i}(T) \Big] x \Big) - \tilde{K}^{l} \Bigg)^{+},$$

where we set

# *Caplet pricing*

$$\tilde{D}^{OIS}(T,T,T_k) = D^{OIS}(T,T,T_k)A^{OIS}(T,T_k)C^{OIS}(T,T_k)$$
$$\tilde{D}^{i}(T,T,T_k) = D^{i}(T,T,T_k)A^{OIS}(T,T_k)C^{i}(T,T_k)$$

with

$$A^{OIS}(T,T_k) = \exp\left(\int_0^T \Big[\sum_{j\in\mathscr{J}_{T_k}^{T^*}} b^{OIS}(s,T_{j-1}) + \mathbf{1}_{\{k\leq n-2\}} \sum_{j\in\mathscr{J}_{T_k}^{T_{n-1}}} \langle\lambda^{OIS}(s,T_{j-1}), w(s,T_j,T^*)\rangle\Big] ds\right),$$

$$C^{OIS}(T,T_k) = \exp\left(\int_0^T \sum_{j\in\mathscr{J}_{T}^{T_k}} \langle\lambda^{OIS}(s,T_{j-1}), w(s,T_k,T^*)\rangle ds\right)$$

and

$$C^{i}(T,T_k) = \exp\left(\int_0^T \sum_{j\in\mathscr{J}_{T}^{T_k}} \langle\lambda^{OIS}(s,T_{j-1}) + \bar{\gamma}^{i}_{i-1}(s,T), w(s,T_k,T^*)\rangle ds\right).$$

# *Caplet pricing*

**Proposition**

*The present price of the caplet can be calculated by*

$$\mathrm{Cpl}(0,T,\delta^l,K) = \frac{B^{OIS}(0,T_k)}{\pi}\int_0^\infty \mathrm{Re}\left(\varphi_{X_T}(u-iR)\hat{f}_K^{k,l}(iR-u)\right)du \tag{4}$$

*where* $R = 1 + \frac{M-M'}{M'}$, $\hat{f}_K^{k,l}$ *denotes the (extended) Fourier transform of* $f_K^{k,l}$ *that admits the representation*

$$\begin{aligned}\hat{f}_K^{k,l}(z) = e^{izx_k}\Bigg[ &- \tilde{D}^{OIS}(T,T,T_k)\frac{e^{\sum_{j\in\mathscr{J}_T^{T_n}}\lambda_1^{OIS}(T_{j-1})x_k}}{\sum_{j\in\mathscr{J}_T^{T_n}}\lambda_1^{OIS}(T_{j-1})+iz}\\ &- \sum_{j=1}^{l}\tilde{D}^{j}(T,T,T_k)\frac{e^{[\sum_{j\in\mathscr{J}_T^{T_n}}\lambda_1^{OIS}(T_{j-1})+\bar{\gamma}^j_{1,j-1}(T)]x_k}}{\sum_{j\in\mathscr{J}_T^{T_n}}\lambda_1^{OIS}(T_{j-1})+\bar{\gamma}^j_{1,j-1}(T)+iz}\\ &+ \tilde{K}^l A^{OIS}(T,T_k)\frac{e^{\sum_{j\in\mathscr{J}_{T_k}^{T_n}}\lambda_1^{OIS}(T_{j-1})x_k}}{\sum_{j\in\mathscr{J}_{T_k}^{T_n}}\lambda_1^{OIS}(T_{j-1})+iz}\Bigg]\end{aligned}$$

*for any* $k \in \{1,\ldots,n-1\}$ *and*

# *Caplet pricing*

*Proposition*

$$\hat{f}_K^{n,l}(z) = e^{izx_n}\Big[ - D^{OIS}(T,T,T_k)\frac{e^{\sum_{j\in\mathscr{J}_T^{T_n}}\lambda_1^{OIS}(T_{j-1})x_n}}{\sum_{j\in\mathscr{J}_T^{T_n}}\lambda_1^{OIS}(T_{j-1})+iz}$$

$$- \sum_{j=1}^{l} D^j(T,T,T_k)\frac{e^{[\sum_{j\in\mathscr{J}_T^{T_n}}\lambda_1^{OIS}(T_{j-1})+\bar{\gamma}^j_{1,j-1}(T)]x_n}}{\sum_{j\in\mathscr{J}_T^{T_n}}\lambda_1^{OIS}(T_{j-1})+\bar{\gamma}^j_{1,j-1}(T)+iz} + \frac{\tilde{K}^l}{iz}\Big],$$

*where* $z \in \mathbb{C}$ *with*

$$\sum_{j\in\mathscr{J}_T^{T_n}}\lambda_1^{OIS}(T_{j-1}) + \max_{i=1,\dots,l}\bar{\gamma}^i_{1,i-1}(T) < \mathrm{Im}(z)$$

*for all* $T \in \mathscr{T}$, $x_k$ *is the unique root of the function*

$$h^{k,l}(x) = \tilde{D}^{OIS}(T,T,T_k)\exp\Big(\sum_{j\in\mathscr{J}_T^{T_n}}\lambda_1^{OIS}(T_{j-1})x\Big)$$

$$+ \sum^{l}\tilde{D}^i(T,T,T_k)\exp\Big([\sum\lambda_1^{OIS}(T_{j-1})+\bar{\gamma}^i_{1,i-1}(T)]x\Big)$$

# *Concluding remarks*

- the forward measure changes are related to the OIS curve and the tractable OIS forward price model $\Rightarrow$ the driving process remains a time-inhomogeneous Lévy process under all forward measures
- specification (a) is the most analytically tractable one: the pricing formulas are of the same simplicity as in the single curve case; however it does not provide positive multi-curve spreads
- specification (b) ensures the positivity of the multi-curve spreads at a cost of slightly more involved pricing formulas
- specification (c) ensures both the positivity and the monotonicity of the spreads with respect to tenor
- Even in the most complicated specification (c), the pricing of caplets does not require the usual freezing of the drift

# *Concluding remarks*

- an alternative to using Lévy processes is to consider a local volatility model for the OIS forward price and the spreads, such as

$$dF^{OIS}(t, T_k) = F^{OIS}(t, T_k) \left( b^{OIS}(t, T_k)\mathrm{d}t + \sigma(t, F^{OIS}(t, T_k))\mathrm{d}W_t^{T_{k+1}} \right)$$

  (work in progress with D. Krief and P. Tankov)

- pricing formulas using asymptotic expansions

# *References*

This work is based on the following papers:

E. Eberlein, Ch. Gerhart and Z. G. (2016). Multiple-curve Lévy forward price model. Preprint, University of Freiburg.

E. Eberlein and Z. G. (2013). The rating-based Lévy Libor model. *Mathematical Finance 23(4)*, 591-626.

E. Eberlein and F. Özkan (2005). The Lévy Libor model. *Finance and Stochastics 9(3)*, 327-348.

Thank you for your attention