Reduction of complexity in an heterogeneous population with two timescales

Berlin-Paris Workshop

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Aggregated mortality data

Mortality data allows to forecast aggregate indicators (annual death probability, life expectancy..) and is used to:

- Decision making for state pension reforms, insurance and regulatory reserving policies.
- Pricing of financial products indexed on national mortality indexes.

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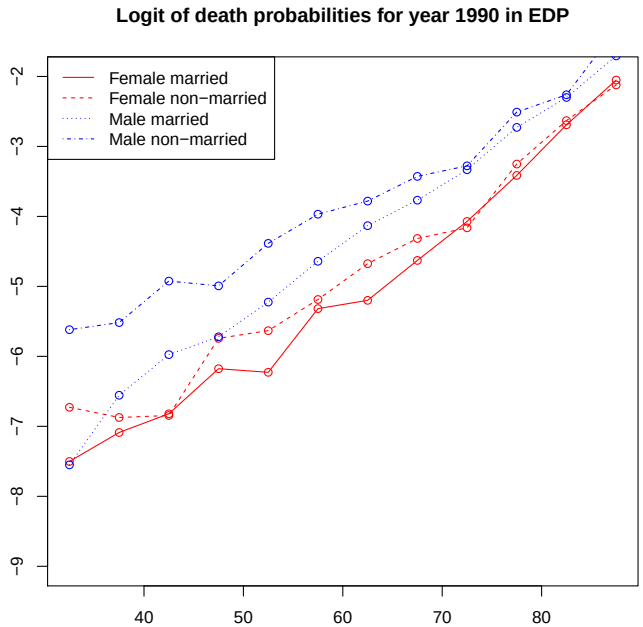
Mortality data is often provided on a national basis:

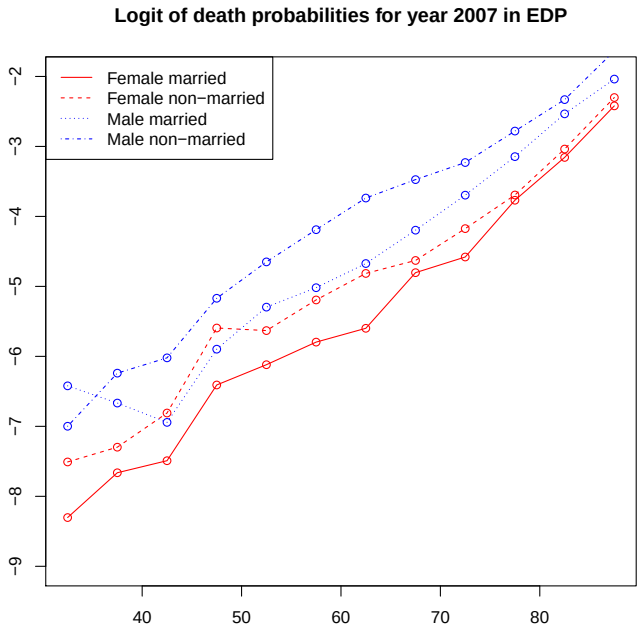
- High amount of data at the national level $\rightarrow$ lower variance of estimations.
- **But** bigger populations implies higher heterogeneity. Trade-off:
  - Behaviors deviate further from average $\rightarrow$ increase of variance.

Heterogeneity

Longevity is known to depend on individuals characteristics:

- gender, socioeconomic category, diploma level, place of living...





"Historic mortality improvement rates have varied by socio-economic classes and these variations can be as high as the variation by gender"

Longevity basis risk report (2014).

Dynamic Heterogeneity

Additional complexity: Heterogeneity of the population changes over time, temporal instabilities.

- Fertility trends change also with individuals characteristics.
- Individuals change of characteristics.

$\Rightarrow$ **Observed mortality result of complex demographic mechanisms**

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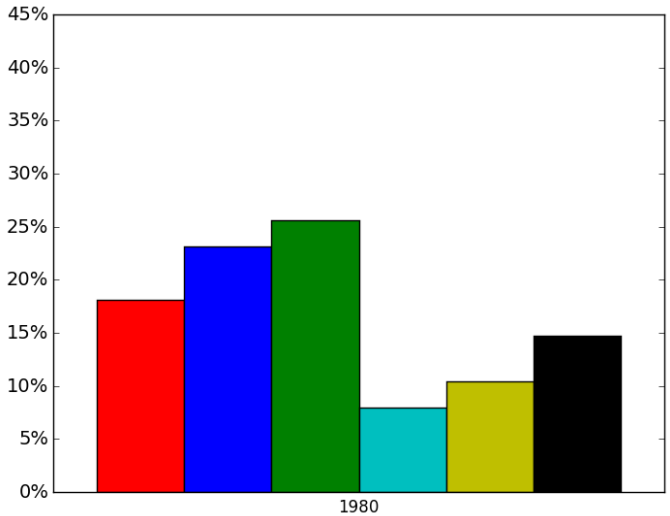
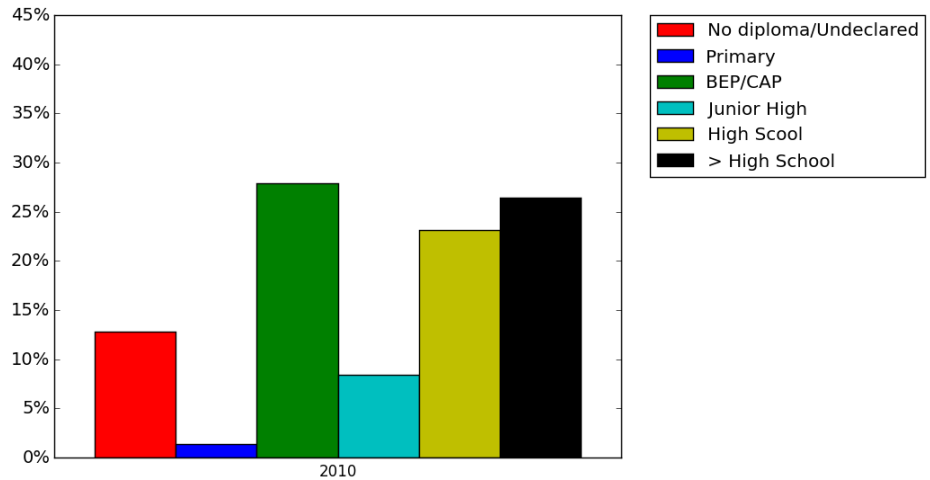
⇒ **Observed mortality result of complex demographic mechanisms**

Furthermore, these changes can happen at a fast pace.

Evolution of French diploma repartition for 30-45 years old (EDP,INSEE).





Dynamics with several timescales

More "micro" description of the population helps to understand aggregated variables.

- Goal of aggregation: model complex link between the micro and macro aggregated variables.
- Separation of timescales between different types of phenomenons: permits to make some averaging approximations and thus to better understand the aggregation process.

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Some examples of multiscale dynamics with averaging:

- Derivatives in finance : Fouque, Papanicolaou and Sircar (2000)
- Operation research: Yin and Zhang (2004)
- Chemical reactions: Ball, Kurtz et al.(2006), Faggionato (2008)
- Ecology and evolution: Taylor and Véber (2009), Champagnat and Lambert (2007), Méléard and Tran(2012), Auger (2012)

Our framework

Study of a stochastic population model when "composition changes" occur at a fast pace in comparison to the demographic time scale of a life time.

- Pathwise representation of populations as solution of Stochastic differential system (thinning).
- New thinning representation more adapted to separation of timescales.
- Use this new representation to reduce the problem to a sequence of simple slow-fast systems.

1 Birth Death Swap systems

2 New construction of BDS systems

3 Averaging result in the presence of two timescales

Plan

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Population process

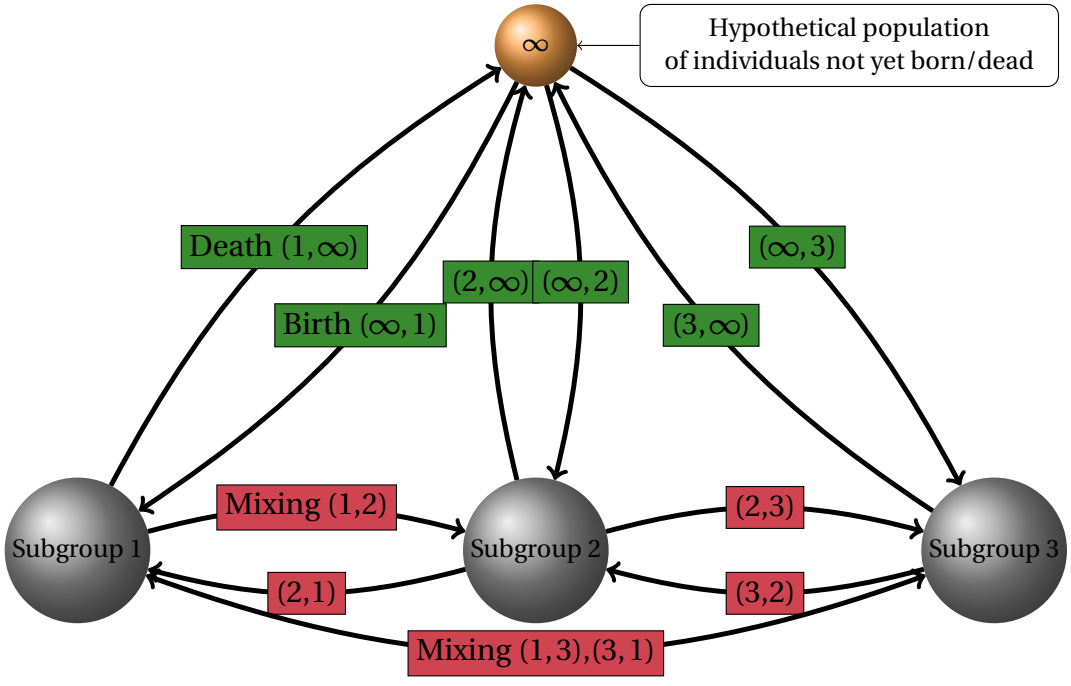
- We consider a population divided in p homogeneous subgroups: individuals in the same subgroup share the same set of characteristics and have the same "demographic" behavior.
- Population process: describes the evolution of the number of individuals in each subgroup. *càdlàg jump* process $Z = (Z^1, .., Z^p) \in \mathbb{N}^p$, adapted to a given filtration $(\mathcal{G}_t)_{t\geq 0}$ (not necessarily canonical) where Z_t^i is the number of individual in subgroup i at time t.
- Canonical basis $(\mathbf{e}_i)_{\{1\leq i\leq p\}}$.

Set of all possible events

Two kinds of events occur:

- Swap events $(i,j) \in \mathcal{I}^s$: an individual changes of characteristics and moves from a subgroup i to j.
  - New composition of the population: $\epsilon^{ij}(z) = z + (\mathbf{e}_j - \mathbf{e}_i)$.
- Demographic events (∞, i),$(i, \infty) \in \mathcal{I}^{dem}$: birth or death in one of the subgroups. A "hypothetical" infinite population $\{\infty\}$ of individuals not born yet and of individuals who died is defined, with $\mathbf{e}_\infty = \mathbf{0}$.
  - New composition of the population : $\epsilon^{\infty i}(z) = z + (\mathbf{e}_i - \mathbf{e}_\infty)$ (birth) or $\epsilon^{i\infty}(z) = z + (\mathbf{e}_\infty - \mathbf{e}_i)$.
- **Impossible events:** No swap or death events can happen an empty population: $\epsilon^{i\beta}(z) = z + (\mathbf{e}_\beta - \mathbf{e}_i)\mathbb{1}_{\{z_i \neq 0\}}$.

Events in the population (II)



- To each event $\gamma = (\alpha, \beta) \in \mathcal{I} = \mathcal{I}^s \cup \mathcal{I}^{dem}$ is associated a jump $\phi(\gamma)$ in the population: $\phi(\gamma) = \phi(\alpha, \beta) = \mathbf{e}_\beta - \mathbf{e}_\alpha$.
- Notation: let $\mathbf{n} = (n^\gamma)_{\gamma \in \mathcal{I}}$. $\phi \odot \mathbf{n}$ denotes the product in the matrix sense of ϕ and $\mathbf{n}$: $\phi \odot \mathbf{n} = \sum_{\gamma \in \mathcal{I}} \phi(\gamma) n_\gamma \in \mathbb{Z}^p$.

Generating counting processes

- Z can be written as the sum of its jumps:

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- Population process can be expressed as a linear combination of a counting processes:

$$N_t^{\alpha\beta} = \sum_{0 < s \leq t} \mathbb{1}_{\{\Delta Z_s = \phi(\alpha,\beta)\}} \mathbb{1}_{\{Z_{s^-}^{\alpha} \neq 0\}}, \qquad Z_t = Z_0 + \phi \odot \mathbf{N}_t \tag{1}$$

- $\mathbf{N} = (N^{\gamma})_{\gamma \in \mathcal{I}}$ is a a multivariate counting process (no components jump together) called the *generating counting processes (GCPs).*
- Decomposition used for continuous time Markov chain applied to the study of chemical reactions (Kurtz and Anderson (2015)).

Multivariate intensity of GCPs

Advantage: Multivariate counting processes easily characterized by their intensity of their components.

- $(\Omega, \mathcal{G}, \mathbb{P}, \{\mathcal{G}_t\})$, with $\mathcal{P}(\mathcal{G}_t)$ σ-algebra of predictable processes.
- N have the $\{\mathcal{G}_t\}$-intensity $\mu \in \mathcal{P}(\mathcal{G}_t)$ if $N_t - \int_0^t \mu_s ds$ is a $\{\mathcal{G}_t\}$-martingale ($\mathbb{E}[N_{t+dt} - N_t | \mathcal{G}_t] \simeq \mu_t dt$).

Goal: Build $\mathbf{N} = (N^\gamma)$ with the right multivariate intensity $\boldsymbol{\mu} = (\mu^\gamma)$ so that Z is a well-defined populations process:

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- $\boldsymbol{\mu}$ depend on $\mathbf{N}$ only through the population process Z.
- $\boldsymbol{\mu}$ should depends of additional randomness.
- restriction of the support of $\boldsymbol{\mu}$: no impossible events, $Z \in \mathbb{N}^p$.

Birth Death Swap system (BDSs)

Definition (BDS system)

1. **Multivariate intensity:** Let $\boldsymbol{\mu}(t,\omega,z) = (\mu^{\gamma}(t,\omega,z))_{\gamma\in\mathcal{I}}$ be a multivariate non-negative predictable functional such that:
   a) $\mu^{i\beta}(t,\omega,z)\mathbb{1}_{\{z_i=0\}} = 0, \quad \forall\, i \in [\![1,p]\!], \beta \in [\![1,p]\!] \bigcup \{\infty\}.$
   b) Absorbing state population: $\mu^{\gamma}(t,\omega,\mathbf{0}) = 0, \quad \forall\, \gamma \in \mathcal{I}.$

   Such $\boldsymbol{\mu}(t,z)$ is called a Birth Death Swap functional.
2. **BDS system**: Let Z_0 be a random variable in $\mathbb{N}^p$ and $\mathbf{N}$ be a multivariate process indexed by $\mathcal{I}$. To $\mathbf{N}$ and initial population Z_0 is associated the process $Z = Z_0 + \phi \odot \mathbf{N}$.

   The couple $(\mathbf{N}, Z)$ is called a Birth Death Swap system (BDSs) of intensity functional $\boldsymbol{\mu}(t,z)$ i.f.f:

   $\mathbf{N}$ has $\{\mathcal{G}_t\}$-multivariate intensity $\boldsymbol{\mu}(t, Z_{t^-}) = \boldsymbol{\mu}(t, Z_0 + \phi \odot \mathbf{N}_{t^-})$.

The thinning procedure

Thinning permits to represent pure jump processes as strong solutions of stochastic differential systems driven by Poisson measures.

Extensive literature on thinning:

- Population Dynamics:
  - Birth-Death processes (homogeneous populations): Garcia (1995)...
  - Heterogeneous (spatial) populations: Fournier and Méléard (2004), Garcia and Kurtz (2006), Mélard and Tran (2009), Bezborodov(2015),El Karoui and Boumezoued (2016)...
- Spatial counting processes :
  - Brémaud et Massoulié (1996), Massoulié (1998), Delattre, Fournier and Hoffman (2016)...

The thinning procedure

Idea:realize counting processes with complex intensities as projections of extended Poisson random measures.

- Space-time Poisson process: random measure Q on $\mathbb{R}^+ \times \mathbb{R}^+$ of $\mathcal{G}_t$-intensity $dt \otimes d\theta$ such that for disjoints sets $A_1, .., A_n$, $|A_i| = \int_{A_i} d\theta < \infty$, the family of projected measures:

$$Q_t(A_i) = \int_{[0,t]\times A_i} Q(ds, d\theta) = Q([0,t] \times A_i) \quad i = 1..n.$$

are independent Poisson processes of intensity $|A_i|$.

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- **Martingale property**: $f(t,\theta)$ predictable process such that $\int_{[0,t]\times E} f(s,\theta) ds d\theta < \infty$:

$Q_t(f) = \int_0^t \int_E f(s,\theta) Q(ds, d\theta) - \int_0^t \int_E f(s,\theta) ds d\theta$ is a $\{\mathcal{G}_t\}$-martingale.

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- **Projection on predictable random sets:** if $f(t,\theta) = \mathbb{1}_{[0,\lambda_t]}(\theta)$,

$N_t = Q_t(f) = Q_t([0,\lambda_t])$ is a *counting process* with $\mathcal{G}_t$-intensity λ_t

Pathwise representation of BDS system

- **Driving multivariate Poisson measures** family of (p+1)p independent Poisson measures $\mathbf{Q}(ds, d\theta) = (Q^{\gamma}(ds, d\theta))_{\gamma \in \mathcal{I}}$.

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- **Birth Death Swap system associated to $(\mu, Z_0, \mathbf{Q})$:** For each event γ, the thinning procedure is applied to Q^γ in order to obtain a counting process N^γ of intensity $\lambda_t^\gamma = \mu^\gamma(t, Z_0 + \sum_\gamma \phi(\gamma) N_{t-}^\gamma)$:

$$dN_t^\gamma = Q^\gamma(ds, [0, \mu^\gamma(t, Z_{t-})]), \quad Z_t = Z_0 + \phi \odot \mathbf{N}_t$$

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$$dN_t^\gamma = Q^\gamma(ds, [0, \mu^\gamma(t, Z_{t-})]), \quad Z_t = Z_0 + \phi \odot \mathbf{N}_t$$

- The BDS system is solution of a multivariate differential equation driven by $\mathbf{Q}$:

$$Z_t = Z_0 + \phi \odot \mathbf{N}_t, \quad d\mathbf{N}_t = \mathbf{Q}(dt, [0, \boldsymbol{\mu}(t, Z_{t-})]). \tag{2}$$

Existence of solutions of stochastic differential system (SDS)?

Existence of solutions the SDS

(2) Do the generating counting processes stay finite in finite time?

Assumptions needed on intensities of generating counting processes:

Assumption

(i) *Sublinear growth of aggregate birth intensity:*

$$\forall z \in \mathbb{N}^p, \quad \mu^b(t,z) = \sum_{i=1}^{p} \mu^{\infty i}(t,z) \leq \bar{\lambda}_t < z, \mathbf{1} > + \bar{f}_t.$$

(ii) $\forall (\gamma, z) \in \mathcal{I} \times \mathbb{N}^p$, $t \geq 0$, $\mathbb{E}[\int_0^t \mu^\gamma(s,z)ds] < \infty$. *Furthermore* $\mathbb{E}[\int_0^t \bar{\lambda}_s ds]$, $\mathbb{E}[\int_0^t \bar{f}_s ds] < \infty$.

- Hypothesis weaker than in Massoulié (1998).
- Sublinear growth: pathwise comparison with a birth process which don't explode: Bezborodov (2015) (Markov), K., El Karoui (work in progress).

Pure Swap systems

BDS systems can be degenerated into Swap systems (individuals only change of characteristics):

- Realized as solution of stochastic differential system driven by $\mathbf{Q}^s$:

$$X_t = \zeta_D \mathbb{1}_{\{D \leq t\}} + \phi \odot \mathbf{N}^s_t, \quad d\mathbf{N}^s_t = \mathbf{Q}^s(dt, [0, \boldsymbol{\mu}^s(t, X_t^-)]) \qquad (3)$$

- Swap process is generated by a vector $\mathbf{N}^s$ indexed by swap events in $\mathcal{I}^{swap}$ which counts the number of swap processes.
- Initial condition have been replaced by $\zeta_D \mathbb{1}_{\{D \leq t\}}$: the swap system start at random time D in state ζ_D.
- $(X, \mathbf{N}^s)$ is said to be solution of $\mathcal{S}^{\boldsymbol{\mu}^s}(\mathbb{1}_{\{D \leq t\}}, \zeta_D, \mathbf{Q}^s)$.
- After time D, the swap system has finite state space of population of size $< \zeta_D, \mathbf{1} >$.

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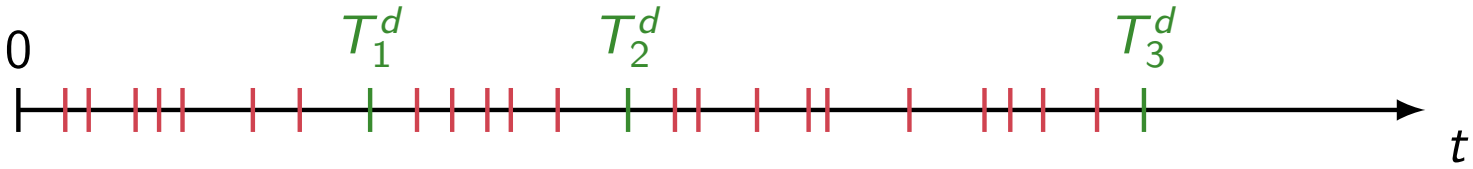
Separation of timescales

Separation of timescale assumption: swap event intensities >> demographic intensities.



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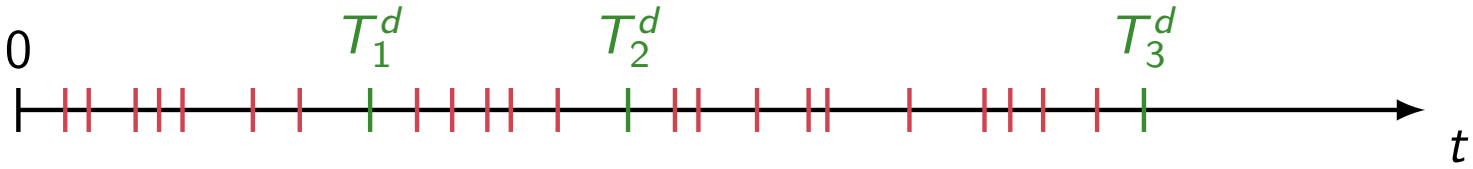

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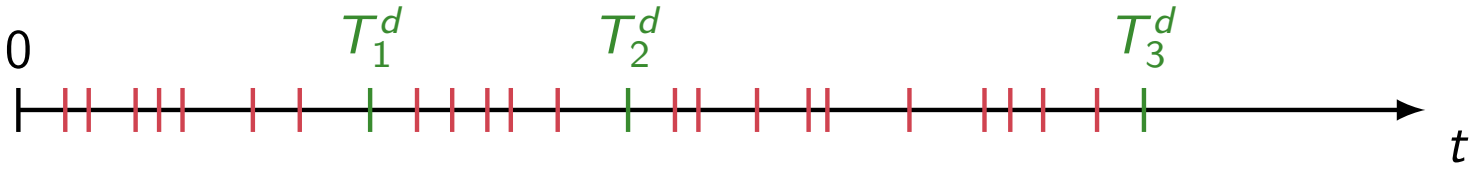

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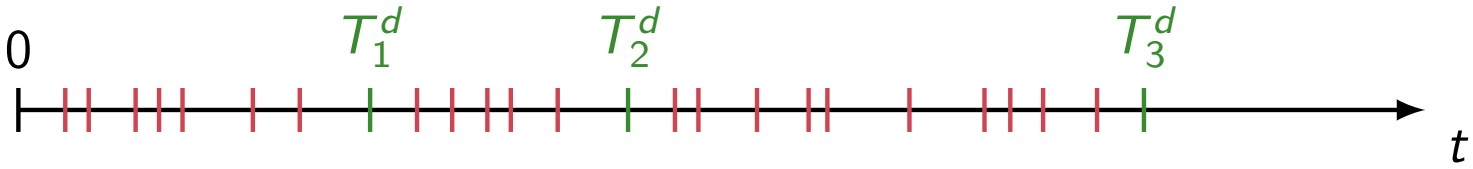

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- Recursive construction by pasting of "swap excursions" at demographic event times.

Switch process

- Pasting of two Markov processes (Meyer (1975)): killing and conditional rebirth of processes (distributional point of view).
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A **Switch process** defines both a pasting date and an event from a swap process ξ and a demographic intensity functional $\boldsymbol{\mu}^{\mathrm{dem}} = (\boldsymbol{\mu}^b, \boldsymbol{\mu}^d)$:

- The switch process is a multivariate counting process, $\mathbf{K}(\xi) = (\mathbf{K}^b(\xi), \mathbf{K}^d(\xi))$, defined by thinning of a multivariate Poisson measure $\mathbf{Q}^{\mathrm{dem}}$:

$$d\mathbf{K}_t(\xi) = \mathbf{Q}^{\mathrm{dem}}(dt, [0, \boldsymbol{\mu}^{\mathrm{dem}}(t, \xi_{t-})]). \tag{4}$$

- Total number of switch event: $K_t^{\natural}(\xi) =< \mathbf{K}_t, \mathbf{1} >$.

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$$d\xi_t^e = \mathbb{1}_{\{K_{t-}^\natural(\xi)=0\}} \left(d\xi_t + d\mathbf{K}_t^b(\xi) - d\mathbf{K}_t^d(\xi) \right)$$

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- Switch event and date serve as initial state for the next excursion.

BDS construction algorithm

- Algorithm given a BDS intensity functional $\boldsymbol{\mu}$, an initial population Z_0 and a sequence of i.i.d Poisson measures $(\mathbf{Q}^n) = (\mathbf{Q}^{s,n}, \mathbf{Q}^{\mathrm{dem,n}})$.

BDS system construction algorithm

1. **First step:**
   - Define the swap system $(X^0, \mathbf{N}^{s,0})$ solution of $\mathcal{S}^{\mu^s}(0, Z_0, \mathbf{Q}^{s,0})$, $\mathbf{K}(X^0)$ the switch process obtained by thinning of $\mathbf{Q}^{\mathrm{dem},0}$.
   - The population process is identified to the swap excursion $X^{0,e}$ on the first demographic interval:

$$\{J_{t-}(Z) = 0\} = \{K^{\natural}_{t-}(X^0) = 0\}, \quad \mathbf{1}_{\{J_{t-}(Z)=0\}} Z_t = X^{0,e}_t$$

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2. **Second step:**
   - $(X^{(1)}, \mathbf{N}^{s,1})$ solution of $\mathcal{S}^{\mu^s}(\mathbf{1}_{\{J_t, Z_t(Z)>0\}}, \mathbf{Q}^{s,1})$, and $\mathbf{K}(X^1)$, $X^{1,e}$.
   - $\{K^{\natural}_{t-}(X^0) = 0\} \subset \{K^{\natural}_{t-}(X^1) = 0\}$.
   - On the second demographic interval:

     $$\{J_{t-}(Z) \leq 1\} = \{K^{\natural}_{t-}(X^1) = 0\}, \quad \mathbf{1}_{\{J_{t-}(Z)=1\}} Z_t = X^{1,e}_t$$

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     $\ldots$

BDS system equation

- The population process is obtained by pasting of all swap excursions:

$$J_t(Z) = \sum_{k\geq 0} \mathbb{1}_{\{K_t^\natural(X^k)>0\}}, \quad Z_t = \sum_{k\geq 0} \mathbb{1}_{\{J_{t-}(Z)=k\}} X_t^{k,e}$$

- Switch processes are defined by a thinning equation and all swap processes verify stochastic differential systems:

$$X_t^k = \zeta_t^k + \phi^s \odot \mathbf{N}_t^{s,k}, \quad d\mathbf{N}_t^{s,k} = \mathbf{Q}^{s,k}(dt, [0, \boldsymbol{\mu}^s(t, X_{t-}^k)]).$$

- As a consequence, the built BDS system also verifies a SDS:

$$Z_t = Z_0 + \phi \odot \mathbf{N}_t, \quad d\mathbf{N}_t = \sum_{k\geq 0} \mathbb{1}_{\{J_{t-}(Z)=k\}} \mathbf{Q}^k(dt, \boldsymbol{\mu}(t, Z_{t-})]).$$

Plan

1 Birth Death Swap systems

2 New construction of BDS systems

3 Averaging result in the presence of two timescales

Aggregated demographic processes

- Usual observed processes: counting processes N^b and N^d which count the number of births and deaths in the global population:

$$N^b = \sum_{i=1}^{p} N^{\infty i} \qquad N^d = \sum_{i=1}^{p} N^{i\infty}.$$

- Counting processes with intensity functional:

$$\mu^b(t,z) = \sum_{i=1}^{p} \mu^{(\infty,i)}(t,z), \quad \mu^d(t,z) = \sum_{i=1}^{p} \mu^{(i,\infty)}(t,z)$$

- Aggregated process $\bar{Z} = \sum Z^i = Z_0 + N^b - N^d$ is not a usual "autonomous" Birth and Death process:
  - Birth and death intensities depend on the whole structure of the population.
  - Due to swap events, they are are not constant between two events.

What can we say when swap events do not happen at the same timescale than demographic events ?

Slow fast system

Hypothesis: swap intensities >> demographic intensities.

- **Assumptions:** The swap intensities are deterministic functions $\frac{1}{\epsilon}\mu^s(z)$. The population with two timescales Z^ϵ now depends on a small parameter ϵ.

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Slow-fast dynamic of pure swap process and its switch process:

$$\begin{cases} X_t^\epsilon = X_{\frac{t}{\epsilon}}, \\ \mathbf{K}_t^\epsilon(X^\epsilon) = \int_0^t \mathbf{Q}^{\mathrm{dem}}(ds, [0, \mu^{\mathrm{dem}}(s, \omega, X_{\frac{s}{\epsilon}}^-)]). \end{cases} \tag{5}$$

Convergence of the Switch process

Assumption

$\forall n \in \mathbb{N}$, the pure swap process restricted to $\mathcal{U}_n$ X is ergodic and admits a unique stationary distribution ν_n. For all bounded functions f:

$$\frac{1}{t}\int_0^t f(X_s)ds \underset{t\to+\infty}{\longrightarrow} \nu_n(f) = \int_{\mathcal{U}_n} f(x)\nu_n(dx) \quad a.s,$$

$$\mathbb{E}[f(X_t)] \underset{t\to+\infty}{\longrightarrow} \nu_n(f).$$

Proposition: *Let $X_0^\epsilon \in \mathcal{U}_n$. Under these assumptions, the switch process $\mathbf{K}^\epsilon(X^\epsilon)$ of the slow-fast system (5) converge in distribution towards a multivariate counting process $\mathcal{K}$ such that:*

$\mathcal{K}$ is a multivariate counting processes of intensity $\nu_n(\boldsymbol{\mu}^{\mathrm{dem}}(t,\cdot))$

First step of the BDS construction algorithm

- At the limit,the first demographic event D_0 (first jump of $K^{\natural}(X^0)$) happens with intensity:

$$\lambda_t^{dem}(\bar{Z}_0) = \nu_{\bar{Z}_0}(\mu^b(t,\cdot) + \mu^d(t,\cdot)),$$

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- Averaging effect:
  - The intensity first demographic event do not depend of the first swap system dynamic anymore.
  - Intensity depends on the initial state of the population only through its size.

Convergence of the aggregated process

- Result can be generalized to aggregated process $\bar{Z}^\epsilon =< Z^\epsilon, \mathbf{1} >$.
- When timescale of Swap system becomes instantaneous: aggregated process converges to an averaged "autonomous" process.

Theorem (Convergence of the aggregated process)

Let $\bar{Z}^{\bar{z}_0,\epsilon}$ be the aggregated process starting with an initial size of $\bar{z}_0$. $\bar{Z}^{\bar{z}_0,\epsilon}$ converge in distribution towards a Birth and Death process $\bar{\mathcal{Z}}^{\bar{z}_0}$ with birth and death intensities defined respectively by:

$$\lambda^b(t,\bar{z}) = \nu_{\bar{z}}(\mu^b(t,\cdot)) \quad \lambda^d(t,\bar{z}) = \nu_{\bar{z}}(\mu^d(t,\cdot)) \quad \forall \bar{z} \in \mathbb{N}.$$

Conclusion and perspectives

- Dynamic of the aggregated process can be approximated by a simpler autonomous process.
- The aggregated birth and death intensities are "averaged" with respect to the stationary distribution of the swap process.
- Perspectives:
  - Understand the link between this limit and an "intermediate" limit given by quasi stationary distributions.
  - Add an age structure and additional dependence on the past to the BDS system.

Thank You For Your Attention