A Multiscale-Analysis of Travelling Waves in Stochastic Reaction-Diffusion Equations

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Stochastic Bistable Reaction-Diffusion Equations

$$dv(t,x) = \nu \Delta v(t,x)\, dt + bf(v(t,x))\, dt + \varepsilon\, dW(t,x), \quad t > 0, \quad x \in \mathbb{R} \quad (1)$$

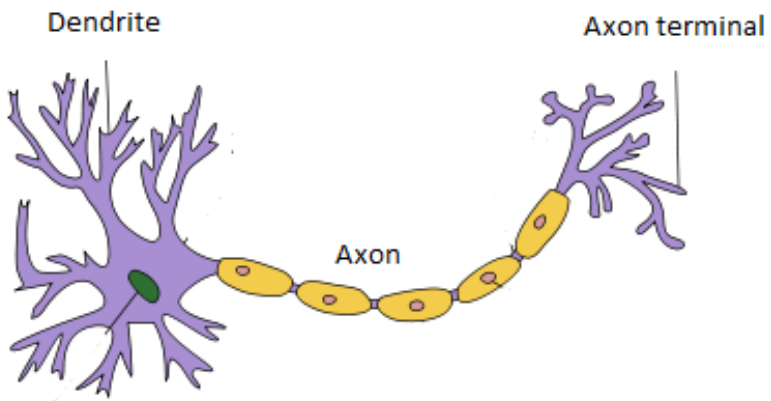
- $\nu, b, \varepsilon > 0$

$f \in C^1(\mathbb{R})$ is a bistable reaction function, i.e.

$$\begin{aligned} & f(0) = f(a) = f(1) = 0 \quad \text{for some } a \in (0,1) \\ & f(x) < 0 \quad \text{for } x \in (0,a)\,, f(x) > 0 \text{ for } x \in (a,1) \qquad \textbf{(A1)} \\ & f'(0) < 0, f'(a) > 0, f'(1) < 0\,. \end{aligned}$$

Stochastic Bistable Reaction-Diffusion Equations

- Nagumo equation with $f(v) = v(1 - v)(v - a)$
- simplification of the FitzHugh-Nagumo system, which is the prototype of an excitable system
- model for the propagation of an action potential through the axon of a single neuron



[US National Cancer Institute. http://training.seer.cancer.gov/module]

Stochastic Bistable Reaction-Diffusion Equations

$\varepsilon = 0$

- existence of a travelling wave solution

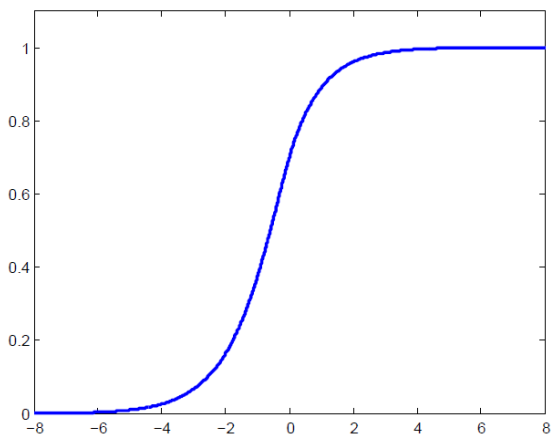
$$v(t,x) = \hat{v}(x + ct)$$

with a non-decreasing C^2-function $\hat{v}$ satisfying

$$c\,\hat{v}_x = \nu\,\hat{v}_{xx} + b\,f(\hat{v}) \qquad \text{(TW)}$$

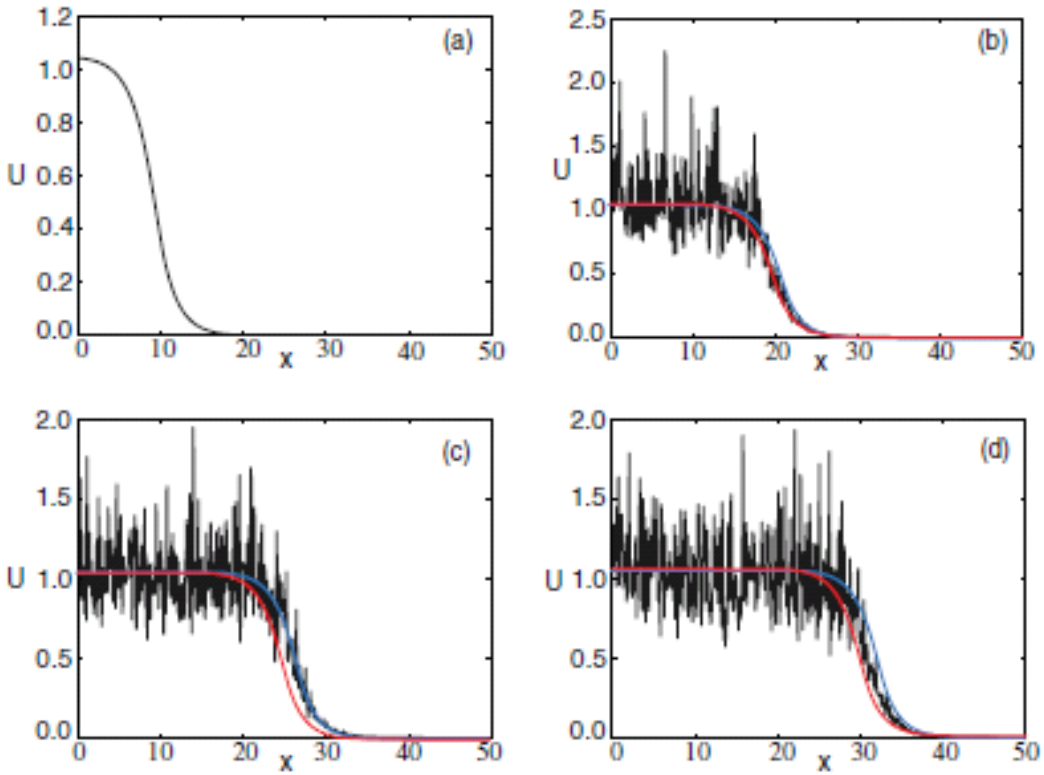
for some wavespeed $c \in \mathbb{R}$ and boundary conditions
$\hat{v}(-\infty) = 0$, $\hat{v}(+\infty) = 1$

$\hat{v}$



Phenomenology for Neural Fields

$$dv(x,t) = \Big[-v(x,t) + \int_{-\infty}^{\infty} w(x-y)\, F(v(y,t))\, dy \Big]\, dt + \varepsilon dW^{Q}(t,x)$$



$$v(t,x) = \hat{v}(x + ct + C(t)) + \varepsilon\, u(t,x)$$

P.C. Bressloff and M.A. Webber, *Front Propagation in Stochastic Neural Fields*, SIAM J. Appl. Dyn. Syst., 11, No. 2, 2012.

Decomposition of the Solution

decompose $v = \hat{v} + u$ where u satisfies

$$\begin{aligned} du(t,x) = \nu \Delta u(t,x)\, dt + b\big(f(u(t,x) + \hat{v}(x+ct)) - f(\hat{v}(x+ct))\big)\, dt \\ + \varepsilon dW(t,x) \end{aligned} \tag{2}$$

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- (2) can now be solved on $L^2(\mathbb{R})$
- however, for stability analysis below we need an additional control wrt. to the measure

$$\rho(x) = Ze^{-\frac{c}{\nu}x}, \quad Z > 0$$

Decomposition of the Solution

The density ρ is naturally associated to the equation in two aspects

(i) allows us to separate the two spatial scales: dynamics of phase and fluctuations decouple in $L^2(\rho)$

(ii) spectral properties of the system can be proven in $L^2(\rho)$, which allows us to derive stability properties

Realisation on $L^2(1+\rho)$

$$H^1(1+\rho) \hookrightarrow L^2(1+\rho) \hookrightarrow H^{1,*}(1+\rho)$$

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$$_{H^{1,*}}\langle \Delta v, u \rangle_{H^1} = -\int_{\mathbb{R}} v_x \left(u\,(1+\rho)\right)_x dx$$

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- let $G : \mathbb{R} \times H^1(1+\rho) \to H^{1,*}(1+\rho)$ be given by

$$_{H^1}\langle v, G(t,u)\rangle_{H^{1,*}} = \int_{\mathbb{R}} v(x)\, b\big(f(u(x)+\hat{v}(x+ct)) - f(\hat{v}(x+ct))\big)(1+\rho(x))\, dx.$$

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- $W(t)$, $t \in [0,T]$, Q-Wiener process on $H^1(1+\rho)$

Realisation as Stochastic Evolution Equation

Theorem

Under some technical assumptions on f, for each $T > 0$, $\varepsilon > 0$ and each $\eta \in L^2(1+\rho)$ there exists a unique variational solution $u \in L^2(\Omega; C([0,T]; L^2(1+\rho))) \cap L^2(\Omega \times [0,T]; H^1(1+\rho))$, almost surely satisfying

$$u(t) = u^0 + \int_0^t \nu \Delta u(s)\, ds + \int_0^t G(s, u(s))\, ds + \varepsilon W(t), \quad t \in [0,T] \tag{3}$$

$$u^0 = \varepsilon \eta$$

Here the Bochner integrals are defined in $H^{1,*}(1+\rho)$.

$v := u + \hat{v}$ is a solution of (1)

Random Phase-Adaptation of Gradient Type

Aim: Minimise

$$C \mapsto \|v(\cdot, t) - \hat{v}(\cdot + ct + C)\|$$

over all possible phases C.

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over all possible phases C. Let $m > 0$. Consider

$$\begin{aligned} \dot{C}^m(t) &= m\, B(t, C^m(t)), \quad t \in [0, T] \\ C(0) &= 0 \end{aligned} \tag{4}$$

where

$$B(t, C) = \langle v(t, \cdot) - \hat{v}(\cdot + ct + C), \hat{v}_x(\cdot + ct + C) \rangle_{\rho(\cdot + ct + C)}$$

Theorem (Well-posedness)

P-almost surely there exists a unique adapted solution $C^m \in C^1([0, T])$ of the (pathwise) ODE (4).

Random Phase-Adaptation of Gradient Type

Replace

$$v(t) = \hat{v}(\cdot + ct) + u(t)$$

by

$$v(t) = \hat{v}(\cdot + ct + C^m(t)) + u^m(t),$$

where u^m is given by

$$u^m(t) = u(t) + \hat{v}(\cdot + ct) - \hat{v}(\cdot + ct + C^m(t)) = v(t) - \hat{v}(\cdot + ct + C^m(t))$$

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- the system (C^m, u^m) is coupled
- of which orders wrt. ε are C^m and u^m?
- can we characterise their behaviour?
- stability of this decomposition?

The Frozen-Wave Setting

Let $H = L^2(\mathbb{R})$ and $V = H^1(\mathbb{R})$

$$V \hookrightarrow H = H^* \hookrightarrow V^*$$

The Frozen-Wave Setting

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Linearise equation around travelling wave:

$$du(t) = [\mathscr{L}_t u(t) + R(t,u)]\ dt + \varepsilon\, dW(t)$$

with

$$\mathscr{L}_t : V \to V^*, \quad \mathscr{L}_t u = \nu \Delta u + b f'(\hat{v}(\cdot + ct))\, u, \quad t \in [0,T]$$

and

$$R : [0,T] \times V \to V^*,$$

$$R(t,u) = b\left(f(u + \hat{v}(\cdot + ct)) - f(\hat{v}(\cdot + ct)) - f'(\hat{v}(\cdot + ct))u\right)$$

$\to$ study stability in moving frame picture, where wave solution is frozen by moving instead the coordinates

The Frozen-Wave Setting

For $h : [0, T] \to V$ let $h^{\#}(x, t) := h(x - ct, t)$. Then for $g \in V$, by Itô's lemma

$$\langle u^{\#}(t), g\rangle = \langle u^{\#}(0), g\rangle + \int_0^t {}_{V^*}\langle \mathscr{L}^{\#} u^{\#}(s), g\rangle_V \, ds + \int_0^t {}_{V^*}\langle R^{\#}(u^{\#}(s)), g\rangle_V \, ds + \varepsilon \int_0^t \langle g, dW^{\#}(s)\rangle$$

- frozen-wave operator $\mathscr{L}^{\#} : V \to V^*$

$$\mathscr{L}^{\#} u = \nu \Delta u + b f'(\hat{v}) u - c \partial_x u$$

- frozen remainder $R^{\#} : V \to V^*$

$$R^{\#}(u) = b\,(f(u + \hat{v}) - f(\hat{v}) - f'(\hat{v}) u)$$

The Frozen-Wave Setting

adjoint operator

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The Frozen-Wave Setting

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- $\mathscr{L}^{\#} \hat{v}_x = 0$ by (TW)
- $\mathscr{L}^{\#,*} \Psi = 0$ for $\Psi(x) = e^{-\frac{c}{\nu} x} \, \hat{v}_x(x)$

Spectral Gap

additional assumption:

$\exists v_* \in (a, 1)$ such that f is strictly convex on $[0, v_*)$ and strictly concave on $(v_*, 1]$. **(A2)**

Proposition

Assume (**A1**) - (**A2**). *There exist* $\kappa > 0$, $C_* > 0$ *such that for* $u \in V$ *with* $u\rho \in V$

$$v^* \langle \mathscr{L}^{\#} u, u\rho \rangle_V \leq -\kappa \|u\|_\rho^2 + C_* \langle \hat{v}_x, u \rangle_\rho^2$$

i.e. the flow generated by $\mathscr{L}^{\#}$ is contracting on the orthogonal complement of $\hat{v}_x$

Spectral Gap

Corollary

$\mathscr{L}^{\#}$ generates a contraction semigroup $(P_t^{\#})_{t\geq 0}$ on $\hat{v}_x^{\perp} \subset L^2(\rho)$ satisfying

$$\|P_t^{\#} u\|_{\rho} \leq e^{-\kappa t} \|u\|_{\rho}$$

Lemma

$(\mathscr{L}_t)$ generates an evolution family $(P_{t,s})_{0\leq s\leq t\leq T}$ on $H^1(1+\rho)$ with

$$\|P_{t,s} h\|_{H^1(1+\rho)} \leq e^{L_*(t-s)} \|h\|_{H^1(1+\rho)}$$

for a constant $L_* > 0$.

Multiscale Decomposition

- want to identify highest order terms in C^m and u^m

Lemma

The dynamically adapted wave speed
$\dot{C}^m(t) = c^m(t) = m\,\langle u^m(t), \Psi(\cdot + ct + C^m(t))\rangle$ *solves the SDE*

$$
\begin{aligned}
c^m(t) = c^m(0) &+ m \int_0^t - c^m(s) + c^m(s)\langle u^m(s), \Psi_x(\cdot + cs + C^m(s)\rangle \, ds \\
&+ m \int_0^t v^* \langle R^m(s, u^m(s)), \Psi(\cdot + cs + C^m(s))\rangle_V \, ds \\
&+ \varepsilon m \int_0^t \langle \Psi(\cdot + cs + C^m(s)), dW(s)\rangle \\
c^m(0) &= \varepsilon m \langle \eta, \Psi\rangle
\end{aligned}
$$

Multiscale Decomposition

First-order speed: c_0^m

$$\begin{aligned} dc_0^m(t) &= -m\, c_0^m(t)\, dt + m\langle \Psi(\cdot + ct), dW(t)\rangle, \quad t \in [0, T] \\ c_0^m(0) &= m\langle \eta, \Psi\rangle \end{aligned}$$

First-order fluctuations: u_0^m

$$\begin{aligned} du_0^m(t) &= [\mathscr{L}_t u_0^m(t) - c_0^m(t)\hat{v}_x(\cdot + ct)]\; dt + dW(t), \quad t \in [0, T] \\ u_0^m(0) &= \eta \end{aligned}$$

Multiscale Decomposition

First-order speed: c_0^m

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First-order fluctuations: u_0^m

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$$u_0^m(0) = \eta$$

- first order phase adaptation $C_0^m(t) = \int_0^t c_0^m(s)\, ds$
- to first order dynamics of C^m and u^m decouple!

Multiscale Decomposition

Set

$$\tau_{q,\varepsilon} = \inf\{t \in [0,T] : \|u(t)\|_{H^1(1+\rho_t)} \geq \varepsilon^{1-q}\}$$
$$\tau^m_{q,\varepsilon} = \inf\{t \in [0,T] : |C^m_0(t)| \geq \varepsilon^{-q}\}$$

Theorem

Let $q < \frac{1}{2}$. On $\{\tau_{q,\varepsilon} \wedge \tau^m_{q,\varepsilon} = T\}$ the stochastic travelling wave v can be decomposed into

$$v(t) = \hat{v}(\cdot + ct + \varepsilon C^m_0(t)) + \varepsilon u^m_0(t) + \varepsilon r^m(t)$$

with

$$\|r^m\|_T \leq \alpha(T)\varepsilon^{1-2q}$$

where the constant $\alpha(T)$ is independent of m and ε. Moreover,

$$\lim_{\varepsilon \to 0} P[\tau_{q,\varepsilon} \wedge \tau^m_{q,\varepsilon} = T] = 1$$

Immediate Relaxation

Aim: Minimise

$$C \mapsto \|v(\cdot, t) - \hat{v}(\cdot + ct + C)\|$$

over all possible phases C.

$$\dot{C}^m(t) = m\, B(t, C^m(t)), \quad t \in [0, T]$$
$$C(0) = 0$$

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- identify $\lim_{m\to\infty} C_0^m$ and $\lim_{m\to\infty} u_0^m$ corresponding to immediate relaxation to the right phase

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Π_t: projection onto orthogonal complement of $\hat{v}_x(\cdot + ct)$ in $L^2(\rho(\cdot + ct))$

$$\Pi_t h = h - \langle h, \hat{v}_x(\cdot + ct)\rangle_{\rho(\cdot+ct)}\, \hat{v}_x(\cdot + ct)$$

Immediate Relaxation

Lemma

For any $\delta > 0$ *almost surely*

$$\sup_{\delta \le t \le T} |C_0^m(t) - C_0(t)| \underset{m\to\infty}{\longrightarrow} 0$$

as well as

$$\sup_{\delta \le t \le T} \|u_0^m(t) - u_0(t)\|_{H^1(1+\rho_t)} \underset{m\to\infty}{\longrightarrow} 0$$

where

$$C_0(t) = \langle \eta, \Psi \rangle + \int_0^t \langle \Psi(\cdot + cs)\,,\, dW(s) \rangle \quad \text{for } t > 0$$

$$C_0(0) = 0$$

and

$$u_0(t) = P_{t,0}\Pi_0\eta + \int_0^t P_{t,s}\Pi_s dW(s) \quad \text{for } t > 0$$

$$u_0(0) = \eta$$

Multiscale Decomposition

$$\tau_{q,\varepsilon} = \inf\{t \in [0,T] : \|u(t)\|_{H^1(1+\rho_t)} \geq \varepsilon^{1-q}\}$$
$$\tau^{\infty}_{q,\varepsilon} = \inf\{t \in [0,T] : |C_0(t)| \geq \varepsilon^{-q}\} \wedge T$$

Theorem (Stability)

On $\{\tau_{q,\varepsilon} \wedge \tau^{\infty}_{q,\varepsilon} = T\}$ the following multiscale decomposition of the stochastic travelling wave v holds:

$$v(t) = \hat{v}(\cdot + ct + \varepsilon C_0(t)) + \varepsilon u_0(t) + \varepsilon r(t)$$

with

$$\|r\|_T \leq \alpha(T)\, \varepsilon^{1-2q}$$

where $\alpha(T)$ is a positive constant. Moreover, in the small-noise limit the above representation holds for almost every path $\omega \in \Omega$, i.e.

$$P[\tau_{q,\varepsilon} \wedge \tau^{\infty}_{q,\varepsilon} = T] \underset{\varepsilon \to 0}{\longrightarrow} 1.$$

Properties of First-Order Approximations

Diffusiveness

$$\begin{aligned}\mathrm{Var}(C_0(t)) &= \mathrm{Var}\left(\int_0^t \langle \Psi(\cdot + cs), dW(s)\rangle\right) \\ &= \int_0^t \langle \Psi(\cdot + cs), Q\Psi(\cdot + cs)\rangle \, ds \approx \langle \Psi, Q\Psi\rangle \, t\end{aligned}$$

Orthogonality

$$\langle u_0(t), \hat{v}_x(\cdot + ct)\rangle_{\rho_t} = \langle \Pi_0 \eta, \Psi\rangle + \int_0^t \langle \Psi(\cdot + cs), \Pi_s dW(s)\rangle = 0$$

Proposition (Minimisation)

For $t < \tau_{q,\varepsilon} \wedge \tau^{\infty}_{q,\varepsilon}$ the function $a \mapsto \|v - \hat{v}(\cdot + ct + \varepsilon a)\|_{\rho_t}$ is locally minimal to order ε at $a = C_0(t)$.

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