A regularity structure for rough volatility

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joint work with C. Bayer, P. Friz, P. Gassiat, B. Stemper

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$$\frac{\mathrm{d}S_t}{S_t} = \sigma \mathrm{d}B_t$$

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stochastic volatility

Idea: Make σ random: $\sigma > 0 \to \sigma_t(\omega) > 0$.

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S. Heston:

$$\frac{\mathrm{d}S_t}{S_t} = \sigma_t \,\mathrm{d}B_t = \sigma_t \,\mathrm{d}(\rho W_t + \overline{\rho}\overline{W}_t)$$

with

- independent Wiener processes $W, \overline{W}$, $\rho^2 + \overline{\rho}^2 = 1$.
- σ_t satisfies an SDE driven by W.

rough volatility

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$$\log \sigma_t \sim \hat{W}_t$$

$\hat{W}_t$ fractional BM with $H < \frac{1}{2}$ ($H \sim 0.1$).

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We will from now on pursue this approach!

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is well-defined as an Itô-integral, but critical.

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Thus: If W^ε is an approximation for W and $\widehat{W}^\varepsilon = \int_0^t K_H(t,s)\mathrm{d}W^\varepsilon$ we cannot expect

$$\int_0^T f\left(\widehat{W}^\varepsilon\right) \mathrm{d}W^\varepsilon$$

to converge!

Aim of this talk:
Approximation theory for

$$\int_0^T f(\hat{W})\mathrm{d}W$$

for $H \in (0, 1/2)$.

Idea: renormalization

Find (divergent) $(\ldots)_\varepsilon$ s.t.

$$\int_0^T f(\widehat{W}^\varepsilon)\mathrm{d}W^\varepsilon - (\ldots)_\varepsilon \xrightarrow{\varepsilon\to 0} \int_0^T f(\widehat{W})\mathrm{d}W_t$$

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$(\dots)_\varepsilon \sim$ (infinite) Itô-Stratonovich correction.

How to define W^ε, $\hat{W}^\varepsilon$

Definition

Let $\delta^\varepsilon : \mathbb{R}^2 \to \mathbb{R}$ be such that

- $\delta^\varepsilon(x,y) = \delta^\varepsilon(y,x)$
- $|\delta^\varepsilon| \lesssim \varepsilon^{-1}$, $\operatorname{supp} \delta^\varepsilon(x,\cdot) \subseteq B(x, c\cdot\varepsilon)$, $c > 0$
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$$\hat{W}^\varepsilon(t) = \int_0^t K_H(t,r)\mathrm{d}W^\varepsilon(t) = \int_0^t K_H(t,r)\dot{W}^\varepsilon(t)\mathrm{d}t$$

Examples for δ^ε

- **Mollifier**: Take compactly supported, symmetric ρ such that $\int \rho(x)\mathrm{d}x = 1$ and set

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- **Wavelets**: truncated expansion in compactly supported (father) wavelets

$$\delta^\varepsilon(x,y) = \sum_{k,l\in\mathbb{Z}} \phi_{k,N}(x)\phi_{l,N}(y)$$

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special case: Haar wavelets $\phi = \mathbf{1}_{[0,1)}$.

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Recall: We want

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How to derive $(\ldots)_\varepsilon$?

Take partition $[0, T) = \bigcup_{l=1}^{N} [s_l, s_{l+1})$

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Given $|s_{l+1} - s_l| = \mathcal{O}(\varepsilon)$ we approximate f by Taylor series.

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For $|s_{l+1} - s_l| = \mathcal{O}(\varepsilon)$ expand around s_l

$$\int_{s_l}^{s_{l+1}} f(\hat{W}^\varepsilon(t))\,\mathrm{d}W^\varepsilon(t)$$

$$= \sum_{k=0}^{K} \frac{1}{k!} f^{(k)}(\hat{W}^\varepsilon(s_l)) \int_{s_l}^{s_{l+1}} (\hat{W}^\varepsilon(t) - \hat{W}^\varepsilon(s_l))^k\,\mathrm{d}W^\varepsilon(t) + \text{error term}$$

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The core of the convergence problem is

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We fix this by using Wiener chaos decomposition

Wiener chaos decomposition

We set

$$P^n = \overline{\{p(W(t_1), \ldots, W(t_m)) \,|\, p \text{ polynomial on } \mathbb{R}^m \text{ of degree } \leq n\}}^{L^2(\mathbb{P})}$$

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Wiener chaos decomposition:

$$L^2\left(\mathbb{P}, \sigma(W(t),\ t \geq 0)\right) = \bigoplus_{n=0}^{\infty} \mathcal{H}^n$$

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$$X = \sum_m X_m,\ X_m \in \mathcal{H}^m$$

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We will use

$$\boxed{U^k V = U^k \diamond V + k\mathbb{E}[UV]V^{k-1}}$$

for $U, V \in \mathcal{H}^1$.

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But

$$\int_{s_l}^{s_{l+1}} (\hat{W}^\varepsilon(t) - \hat{W}^\varepsilon(s_l))^k \diamond \dot{W}^\varepsilon(t)\mathrm{d}t \to \int_{s_l}^{s_{l+1}} (\hat{W}(t) - \hat{W}(s_l))^k \mathrm{d}W(t)$$

Wick renormalization

$$\int_{s_k}^{s_{k+1}} (\hat{W}^\varepsilon(t) - \hat{W}^\varepsilon(s_k))^k \, \mathrm{d}W^\varepsilon(t)$$
$$= \int_{s_k}^{s_{k+1}} (\hat{W}^\varepsilon(t) - \hat{W}^\varepsilon(s_k))^k \cdot \dot{W}^\varepsilon(t) \mathrm{d}t$$

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$$\boxed{U^k V = U^k \diamond V + k\mathbb{E}[UV]V^{k-1}}$$

$$= \int_{s_k}^{s_{k+1}} (\hat{W}^\varepsilon(t) - \hat{W}^\varepsilon(s_k))^k \diamond \dot{W}^\varepsilon(t) \mathrm{d}t$$

$$+k \int_{s_k}^{s_{k+1}} \underbrace{\mathbb{E}[(\hat{W}^\varepsilon(t) - \hat{W}^\varepsilon(s_k))\dot{W}^\varepsilon(t)]}_{R^\varepsilon(t)+o_\varepsilon(1)} \cdot (\hat{W}^\varepsilon(t) - \hat{W}^\varepsilon(s_k))^{k-1} \mathrm{d}t$$

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with $R^\varepsilon(t) = \mathbb{E}[\hat{W}^\varepsilon(t)\dot{W}^\varepsilon(t)]$.

computation of R^ε

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$$\text{" } \longrightarrow \text{ " } K_H(t, t) = |t - t|^{H-1/2} = \infty$$

Wick renormalization

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$$\longrightarrow \int_{s_k}^{s_{k+1}} (\hat{W}(t) - \hat{W}(s_k)) \, \mathrm{d}W(t)$$

Putting things together

Theorem

Let $K = \max\{j \cdot H - 1/2 \leq 0\}$, $f \in C^{K+1}(\mathbb{R})$ *with* $\mathbb{E}[\sup_{t\in[0,T]} |f^{(k)}(\hat{W}(t))|^p] < \infty$, $p \in [1,\infty)$, $k \leq K+1$. *Then we have*

$$\sum_{l=0}^{\lfloor T2^N \rfloor} \sum_{k=0}^{K} f(\hat{W}^\varepsilon(s_k)) \int_{s_k}^{s_{k+1}} (\hat{W}^\varepsilon(t) - \hat{W}^\varepsilon(s_k))^k \, \mathrm{d}W^\varepsilon(t) - (\ldots)_\varepsilon$$

$$\xrightarrow{\varepsilon\to 0} \int_0^T f(\hat{W}(t)) \mathrm{d}W(t)$$

in $L^p(\mathbb{P})$ *for* $p \in [1,\infty)$.

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$$\sum_{l=0}^{\lfloor T2^N \rfloor} \sum_{k=0}^{K} f(\hat{W}^\varepsilon(s_k)) \Big(\int_{s_k}^{s_{k+1}} (\hat{W}^\varepsilon(t) - \hat{W}^\varepsilon(s_k))^k \, \mathrm{d}W^\varepsilon(t) - k \int_{s_k}^{s_{k+1}} R^\varepsilon(t) (\hat{W}^\varepsilon(t) - \hat{W}^\varepsilon(s_k))^{k-1} \mathrm{d}t \Big)$$
$$\xrightarrow{\varepsilon \to 0} \int_0^T f(\hat{W}(t)) \mathrm{d}W(t)$$

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The case of Haar wavelets

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with

$$\hat{\phi}_{k,N}(t) = \frac{\mathbf{1}_{t>k\mathbf{2}^{-N}\mathbf{2}^{N/2}}}{\mathbf{1}+H-\mathbf{1/2}}\left(|t-k\cdot 2^{-N}|^{1+H-1/2} - |t-\min((k+1)2^{-N},t)|^{1+H-1/2}\right)$$

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Further

$$R^{\varepsilon}(t) = \mathbb{E}[\hat{W}^{\varepsilon}(t)\dot{W}^{\varepsilon}(t)] = \frac{\mathbf{2}^{N}}{\mathbf{1}+H-\mathbf{1/2}}|t-\lfloor t2^{N}\rfloor 2^{-N}|^{1+H-1/2}$$

The case of Haar wavelets

$$\int_0^t f(\hat{W}(r))\mathrm{d}W(r) \approx$$

$$\sum_{l=0}^{\lfloor t2^N \rfloor} \sum_{k=0}^{K} \frac{1}{k!} f^{(k)}(\hat{W}^\varepsilon(t_l)) 2^{N/2} Z_l \int_{t_l}^{t_{l+1}} \mathrm{d}r \left(\hat{W}^\varepsilon(r) - \hat{W}^\varepsilon(t_l)\right)^k -$$

$$- \sum_{k=1}^{K} \frac{1}{(k-1)!} f^{(k)}(\hat{W}^\varepsilon(t_l)) \int_{t_l}^{t_{l+1}} \mathrm{d}r\, R^\varepsilon(r) \left(\hat{W}^\varepsilon(r) - \hat{W}^\varepsilon(t_l)\right)^{k-1}$$

Thank you for your attention!