The generalized opportunity process for utility maximization and applications

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The utility maximization problem

The opportunity process for power utility

The generalized opportunity process and main results

Stability of utility maximization

Financial Market

- $T > 0$ finite time horizon
- $\left(\Omega, \mathcal{G}, \mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P}\right)$ filtered probability space.
- n stocks with discounted semimartingale price process $S = (S^1, \cdots, S^n)$.

$$\mathcal{M}^e(S) := \{\mathbb{Q} \sim \mathbb{P} : S \text{ is a } \mathbb{Q} - \text{local martingale}\}.$$

Assumption: $\mathcal{M}^e(S) \neq \emptyset$.

Utility maximization problem

- $x > 0$: initial capital of an agent in the market.
- $U : (0, +\infty) \to \mathbb{R}$ a strictly increasing, concave and C^1-function with $AE[U] = \limsup_{z \to +\infty} \frac{zU'(z)}{U(z)} < 1$ and

$$U'(0) = +\infty, \quad U'(+\infty) = 0.$$

- $\mathcal{X}(x)$ set of admissible wealth processes with initial capital x

$$\mathcal{X}(x) \quad := \quad \left\{ X \geq 0 : X = x + \int_0^{\cdot} \pi dS \right\}.$$

Problem: Find $\widehat{X} \in \mathcal{X}(x)$ such that

$$u(x) = \max_{X \in \mathcal{X}(x)} \mathbb{E}\left[U(X_T)\right] = \mathbb{E}\left[U(\widehat{X}_T)\right]$$

Duality approach

- Convex conjugate to U: $V(y) = \sup_{z>0}(U(z) - zy), y > 0$.
- $\mathcal{Y}$: set of supermartingale deflators for S i.e.

$$\mathcal{Y} := \{Y : Y \geq 0 \text{ and } YX \text{ supermartingale } \forall X \in \mathcal{X}(1)\}.$$

Dual problem: find $\widehat{Y} \in \mathcal{Y}$ such that

$$v(y) := \inf_{Y \in y\mathcal{Y}} \mathbb{E}[V(yY_T)] = \mathbb{E}\left[V(y\widehat{Y}_T)\right].$$

Kramkov and Schachermayer'99: $\widehat{X}$ and $\widehat{Y}$ exist. Moreover,

$\widehat{Y}_T = U'(\widehat{X}_T)$ and $\widehat{X}\widehat{Y}$ a uniformly integrable (U.I.) martingale.

Muckhenhoupt's condition and reverse Hölder's inequality

$\mathcal{T}(\mathbb{F})$: set of $\mathbb{F}$-stopping times valued in $[0, T]$.

- **Def**: Let $Y > 0$ be a càdlàg semimartingale, $r > 1$ and $q \in (0, 1)$. Y satisfies:
  - Muckhenhoupt's condition (A_r) if there exists $K_1 > 0$ s.t. for every $\sigma \in \mathcal{T}(\mathbb{F})$:

$$\mathbb{E}\left[\left(\frac{Y_\sigma}{Y_T}\right)^{\frac{1}{r-1}}\middle|\mathcal{F}_\sigma\right] \leq K_1.$$

  - reverse Hölder's inequality (b_q^-) if there exists $K_2 > 0$ s.t. for every $\sigma \in \mathcal{T}(\mathbb{F})$:

$$\mathbb{E}\left[\left(\frac{Y_T}{Y_\sigma}\right)^{q}\middle|\mathcal{F}_\sigma\right] \geq K_2.$$

- **Remark**: (A_r) is stronger than (b_q^-). If Y satisfies (b_q^-) and $\mathbb{E}[Y_T] < +\infty$, then Y is of class (D), i.e. $\{Y_\sigma,\ \sigma \in \mathcal{T}(\mathbb{F})\}$ is U.I.

Objective and motivation

Objective: give necessary and/or sufficient conditions for $\widehat{Y}$ to satisfy (A_r) or (b_q^-) for some $r > 1$ or $q \in (0,1)$.

Motivation:

- Assume that $\widehat{Y}$ is a local martingale satisfying (A_r) for some $r > 1$. In this case, $\widehat{Y}_T = \widehat{Y}_0 \frac{d\widehat{\mathbb{Q}}}{d\mathbb{P}}$ for some $\widehat{\mathbb{Q}} \in \mathcal{M}^e(S)$. $\widehat{\mathbb{Q}}$ is a pricing measure which reflects the agent risk preference and its initial position.
- The integrability properties of $\widehat{Y}$ provide a basis to investigate its continuity properties w.r.t. risk preference and initial capital in appropriate topologies:
  1. continuity of the terminal values $\widehat{Y}_T$ in the topology of L^1-convergence.
  2. continuity of the process $\widehat{Y}$ in the topology of uniform convergence.

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Opportunity process for power utility

Let $p \in (-\infty, 0) \cup (0, 1)$ and U be given by

$$U(z) = \frac{z^p}{p}, \ z > 0.$$

For $X \in \mathcal{X}(x)$, let $\mathcal{X}_t(X) := \{\widetilde{X} \in \mathcal{X}(x) : \widetilde{X}_s = X_s, s \in [0, t]\}$.

Proposition (Nutz'10)

- *There exists a unique càdlàg semimartingale $L(p)$ such that*

$$L_t(p)\frac{1}{p}X_t^p = \operatorname*{ess\,sup}_{\widetilde{X} \in \mathcal{X}_t(X)} \mathbb{E}\left[U(\widetilde{X}_T)\middle|\mathcal{F}_t\right] = u(t, X_t), \ \forall X \in \mathcal{X}(x).$$

- $\frac{1}{p}L_t(p)x^p = u(t, x)$ *and* $L_T(p) = 1$.
- $L(p)$ *is a supermartingale for* $p \in (0, 1)$ *and a submartingale for* $p \in (-\infty, 0)$.

Opportunity process for power utility (continued)

- $\widehat{Y} = \widehat{X}^{p-1} L(p) = U'(\widehat{X}) L(p)$.
- For $p \in (0,1)$ and $\sigma \in \mathcal{T}(\mathbb{F})$,

$$(L_\sigma(p))^{\frac{1}{1-p}} = \mathbb{E}\left[\left(\frac{\widehat{Y}_\sigma}{\widehat{Y}_T}\right)^{\frac{1}{\frac{1}{p}-1}} \middle| \mathcal{F}_\sigma\right] \leq \mathbb{E}\left[\left(\frac{Y_\sigma}{Y_T}\right)^{\frac{1}{\frac{1}{p}-1}} \middle| \mathcal{F}_\sigma\right] \quad \forall Y \in \mathcal{Y},$$

$L(p)$ is bounded away from 0 and ∞ if and only if $(A_{\frac{1}{p}})$ holds for $\widehat{Y}$ or some $Y \in \mathcal{Y}$.

- For $p \in (-\infty, 0)$, $q = \frac{p}{p-1} \in (0,1)$ and $\sigma \in \mathcal{T}(\mathbb{F})$

$$(L_\sigma(p))^{\frac{1}{1-p}} = \mathbb{E}\left[\left(\frac{\widehat{Y}_T}{\widehat{Y}_\sigma}\right)^{q} \middle| \mathcal{F}_\sigma\right] \geq \mathbb{E}\left[\left(\frac{Y_T}{Y_\sigma}\right)^{q} \middle| \mathcal{F}_\sigma\right], \forall\, Y \in \mathcal{Y}.$$

$L(p)$ is bounded away from 0 and ∞ if and only if $\widehat{Y}$ satisfies (b_q^-) or (b_q^-) holds for some $Y \in \mathcal{Y}$.

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Extension to general utility functions

Def: Let $a, b, C > 0$ with $a \leq b$. U satisfies condition $(G_{a,b,C})$ if

$$\frac{1}{C}\left(\frac{y}{x}\right)^a \leq \frac{U'(x)}{U'(y)} \leq C\left(\frac{y}{x}\right)^b, y \geq x.$$

$$a \leq -\frac{zU''(z)}{U'(z)} \leq b \text{ for all } z > 0 \Longrightarrow (G_{a,b,1}) \text{ is satisfied.}$$

Theorem (Kramkov and Weston'16)

Assume that $(G_{a,b,C})$ holds for U with $a \in (0,1)$ and there exists $Z \in \mathcal{Y}$ satisfying $(A_{\frac{1}{1-a}})$. Then $\widehat{Y}$ satisfies (A_r) with $r = 1 + \frac{b}{1-a}$.

Issues:

- The necessity for the condition (A_r) is not addressed.
- The condition (A_r) becomes too restrictive for $a > 1$ and if one is only interested in the class (D) property of $\widehat{Y}$.

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Generalized opportunity process

Observation: $\widehat{Y}$ admits a multiplicative decomposition $\widehat{X}\widehat{Y}$ martingale and $\widehat{Y}_T = U'(\widehat{X}_T) \Longrightarrow$

$$\widehat{Y}_t = \frac{\widehat{Y}_t \widehat{X}_t}{\widehat{X}_t} = U'(\widehat{X}_t) \frac{\mathbb{E}\left[U'(\widehat{X}_T)\widehat{X}_T \middle| \mathcal{F}_t\right]}{\widehat{X}_t U'(\widehat{X}_t)} = U'(\widehat{X}_t) L_t.$$

- $L := \frac{\widehat{Y}}{U'(\widehat{X})}$ is refereed to as the generalized opportunity process.
- L is càdlàg, strictly positive and $L_T = 1$.
- For $U = \log$, $L = 1$ since $\widehat{X}U'(\widehat{X}) = 1$.

$\longrightarrow$ Our analysis on the conditions (A_r) and (b_q^-) for $\widehat{Y}$ is based on L.

Muckhenhoupt's (A_r)-condition

Theorem

Suppose that there exist $a, b, C > 0$ with $a \leq b$ s.t. U satisfies $(G_{a,b,C})$.

1. *If there exists $Y \in \mathcal{Y}$ satisfying (A_k) for some $k > 1$ and L is bounded away from 0 and ∞. Then $\widehat{Y}$ satisfies (A_r) with*

$$r = 1 + bk.$$

2. *Assume that $\widehat{Y}$ satisfies (A_r) for some $r > 1$, then L is bounded away from 0.*
3. *Assume that $a = b \in (0, 1)$. The following are equivalent:*
   - 3.1 *L is bounded away from 0 and ∞.*
   - 3.2 *$\widehat{Y}$ satisfies $(A_{\frac{1}{1-a}})$.*

Comments: If $a \in (0, 1)$ and $k = \frac{1}{1-a}$, then $r = 1 + \frac{b}{1-a}$ and we recover the result of Kramkov and Weston'16.

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Boundedness of L

Proposition

Assume that U satisfies $(G_{a,b,C})$. Consider the following assertions:

B1. *$a \in (0,1)$ and there exists $Z \in \mathcal{Y}$ satisfying $(A_{\frac{1}{1-a}})$.*

B2. *$a \geq 1$ and there exists $Z \in \mathcal{Y}$ satisfying (A_k) for some $k > 1$.*

If either B1. or B2. hold, then there exist $K_1 > 0, K_2 > 0$ depending only on a, b, C and k such that

$$K_1 \leq L \leq K_2.$$

Idea of the proof:

1. For $a \in (0,1)$, the growth of L is controlled by $L(1-a)$.
2. For $a \geq 1$, L is bounded from above.

Boundedness of L and reverse Hölder's inequality for $\widehat{Y}$

Theorem

Suppose that U satisfies $(G_{a,b,C})$ with $a > 1$. Let $q \in (0,1)$, the following assertions are equivalent:

i) *L is bounded away from 0 and ∞.*

ii) *(b_q^-) holds for $\widehat{Y}$.*

iii) *(b_q^-) holds for some $Y \in \mathcal{Y}$.*

iv) *$L(p)$ is bounded away from 0 and ∞ for all $p < 0$.*

Idea of the proof:

- i) implies ii) relies on an estimate of the form $\mathbb{E}\left[\left(\frac{\widehat{Y}_T}{\widehat{Y}_\sigma}\right)^q \middle| \mathcal{F}_\sigma\right] \geq K L_\sigma^{\frac{q}{a-1}}$ where $K > 0$.
- iv) implies i) is based on inequalities between L and $L(p)$.

Continuous BMO-martingales

All local martingales are assumed to be continuous.

Definition: A local martingale M with $M_0 = 0$, is a BMO martingale if

$$||M||_{BMO} := \sup_{\sigma \in \mathcal{T}(\mathbb{F})} ||\mathbb{E}\left[\langle M\rangle_T - \langle M\rangle_\sigma \middle| \mathcal{F}_\sigma\right]||_\infty < +\infty.$$

Kazamaki'94: Let M be an $\mathbb{F}$-local martingale with $M_0 = 0$ and

$$\mathcal{E}(M) := \exp\left(M - \frac{1}{2}\langle M\rangle\right).$$

The following assertions are equivalent:

i) M is a BMO martingale.

ii) $\Gamma = \mathcal{E}(M)$ satisfies (A_r) for some $r > 1$.

iii) $\Gamma = \mathcal{E}(M)$ satisfies (b_q^-) for all $q \in (0, 1)$.

Example: continuous stock price process

- Let W be a $\mathbb{R}^n$-valued Brownian motion. We assume that

$$dS_t = S_t(dW_t + \mu_t dt), \text{ where } \mu \in L^2_{\text{loc}}.$$

  - $Z^\mu = \mathcal{E}(-\mu \cdot W) \in \mathcal{Y}$.
  - $\widehat{Y}$ is a local martingale and $\widehat{Y} = \mathcal{E}\big(-\mu \cdot W + \widehat{N}\big)$ with $\widehat{N}$ a local martingale orthogonal to W, i.e. $\big[W, \widehat{N}\big] = 0$.

Proposition: Suppose that U satisfies $(G_{a,b,C})$ with $a > 1$. The following are equivalent:

1. $\mu \cdot W \in BMO$.
2. $\widehat{Y}$ satisfies (A_r) for some $r > 1$.
3. $\widehat{Y}$ satisfies (b_q^-) for all $q \in (0, 1)$.
4. L is bounded away from 0 and ∞.

Stability of utility maximization

Theorem

Let U and S be as in the example. Let $(U_m)_{m\in\mathbb{N}}$ and $(x_m)_{m\in\mathbb{N}} \subseteq (0,+\infty)$ be a sequence of utility functions and initial capitals. For each $m \in \mathbb{N}$, let $\widehat{Y}^m$ and L^m be the dual minimizer and gen. opportunity process associated to U_m and x_m.
Assume that:

1. $\lim_{m\to+\infty} U_m = U$ *pointwise and* $\lim_{m\to+\infty} x_m = x$.
2. *for each* $m \in \mathbb{N}$, U_m *satisfies* $(G_{a,b,C})$.
3. $a > 1$ *and* $\mu \cdot W \in BMO$.

Then, there exists $r' > 1$ *such that*

$$\lim_{m\to+\infty} \mathbb{E}\left[\sup_{t\in[0,T]} |\widehat{Y}^m_t - \widehat{Y}_t|^{r'}\right] = 0.$$

Sketch of the proof

1. $\widehat{Y}_T^m \to \widehat{Y}_T$ in probability and $\widehat{Y}_0^m \to \widehat{Y}_0$ by Kardaras and Žitković'11.
2. Show that there exists $r' > 1$ s.t. $\left(|\widehat{Y}_T^m - \widehat{Y}_T|^{r'}\right)_{m\in\mathbb{N}}$ is U.I.
   - As $a > 1$ and $Z^\mu = \mathcal{E}(\mu \cdot W)$ satisfies (A_k). There exists $K > 1$ s.t. with $r = 1 + bk$

$$\mathbb{E}\left[\left(\frac{\widehat{Y}_\sigma}{\widehat{Y}_T}\right)^{\frac{1}{r-1}}\middle|\mathcal{F}_\sigma\right] + \mathbb{E}\left[\left(\frac{Y_\sigma^m}{\widehat{Y}_T^m}\right)^{\frac{1}{r-1}}\middle|\mathcal{F}_\sigma\right] \leq K.$$

   - By Gehring's lemma, there exist $r' > 1$ and K_2, K_3 such that

$$\mathbb{E}\left[|\widehat{Y}_T^m - \widehat{Y}_T|^{r'}\right] \leq K_2(|Y_0^m + \widehat{Y}_0|^{r'}) \leq K_3.$$

3. Apply Doob's maximal inequality to the submartingale $|\widehat{Y}^m - \widehat{Y}|$ and use L^1-convergence of $|\widehat{Y}_T^m - \widehat{Y}_T|^{r'}$.

Thank you for your attention!