On the robustness of martingale representations and of BSDEs

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joint work with
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Let $(\Omega, \mathcal{G}, \mathbb{F}, \mathbb{P})$ be a stochastic basis, T be a fixed, finite time horizon and

- X be a square-integrable Lévy martingale
- $\xi \in \mathbb{L}^2(\Omega, \mathcal{F}_T, \mathbb{P})$ and
- f be globally Lipschitz with $\mathbb{E}\big[\int_0^T f^2(\cdot, 0, 0, \mathbf{0})\mathrm{d}t\big] < \infty$.

Then the BSDEJ

$$Y_t = \xi + \int_{(t,T]} f(s, Y_s, Z_s, U_s)\, \mathrm{d}s - \int_t^T Z_s\, \mathrm{d}X_s^c - \int_t^T \int_{\mathbb{R}} U_s(x)\, \widetilde{\mu}^X(\mathrm{d}s, \mathrm{d}x) \quad (1)$$

has a unique solution (Y, Z, U), see [TL] and [BBP97].

Remark

The solution is given by a fixed point argument. In particular

$$Y_t = \mathbb{E}\Big[\xi + \int_{(t,T]} f(s, \widetilde{Y}_s, \widetilde{Z}_s, \widetilde{U}_s)\, \mathrm{d}s \,\Big|\, \mathcal{F}_t\Big]$$

Weak approximation for (1): see [BE08] and [MPS14].

Assumption

C is a predictable, càdlàg and increasing process s.t.

$$\langle X^c \rangle_t^{ij} = \int_{(0,t]} \frac{\mathrm{d}\langle X^c \rangle_s^{ij}}{\mathrm{d}C_s} \mathrm{d}C_s \quad \textit{and} \quad \nu(\omega; \mathrm{d}t, \mathrm{d}x) = K(\omega, t; dx)\ \mathrm{d}C_t$$

where $\frac{\mathrm{d}\langle X^c \rangle}{\mathrm{d}C}$ and K possess suitable properties; see [JS13, Prop.II.2.9].

Given the data $(\mathbb{F}, T, X, \xi, f, C)$, we are interested in BSDEJs of the form

$$Y_t = \xi + \int_{(t,T]} f(s, Y_s, Z_s, U_s)\ \mathrm{d}C_s - \int_t^T Z_s\ \mathrm{d}X_s^c - \int_t^T \int_{\mathbb{R}^\ell} U_s(x)\ \widetilde{\mu}^X(\mathrm{d}s, \mathrm{d}x) - \int_t^T \mathrm{d}N_s \quad \text{(BSDEJ)}$$

i.e. we seek a quadruple (Y, Z, U, N) that satisfies the equation (BSDEJ) $\mathbb{P}$-almost surely on $[\![0, T]\!]$.

We generalise the work of [EKH97].

Existence and uniqueness of solution for $\mathcal{D}-$BSDEJ

Lemma (*A priori* estimates, [PPS16b])

Let the semimartingale y *with decomposition*

$$y_t = \xi + \int_{(t,T]} f_s \, \mathrm{d}C_s - \int_t^T \mathrm{d}\eta_s,$$

$A = \int_0^\cdot \alpha_s^2 \, \mathrm{d}C_s$ *with* $\Delta A \leq \Phi$ *and* $\hat{\beta} \in \mathbb{R}_+$ *such that*

$$\|\xi\|^2_{\mathbb{L}^2_{\hat{\beta}}} = \mathbb{E}\left[\mathrm{e}^{\hat{\beta} A_T} |\xi|^2\right] < \infty \quad \textit{and} \quad \left\| \frac{f}{\alpha} \right\|^2_{\mathbb{H}^2_{\hat{\beta}}} = \mathbb{E}\left[\int_{(0,T]} \mathrm{e}^{\hat{\beta} A_t} \frac{|f_t|^2}{\alpha_t^2} \mathrm{d}C_t \right] < \infty.$$

Then, for $\gamma, \delta \in (0, \hat{\beta}], \gamma \neq \delta$,

$$\|\alpha y\|^2_{\mathbb{H}^2_\delta} + \|\eta\|^2_{\mathcal{H}^2_\delta} \leq \widetilde{\Pi}^\Phi(\delta) \, \|\xi\|^2_{\mathbb{L}^2_\delta} + \Pi^\Phi(\gamma, \delta) \left\| \frac{f}{\alpha} \right\|^2_{\mathbb{H}^2_{\gamma \vee \delta}}.$$

Theorem ([PPS16b])

Let $\mathcal{D} := (\mathbb{F}, T, X, \xi, f, C)$ *be standard data under* $\hat{\beta}$. *If* $M(\hat{\beta}) < 1/3$, *then there exists a unique solution* $\mathcal{S} := (Y, Z, U, N)$ *of the equation* (BSDEJ).

[PPS16b] Existence and uniqueness results for BSDE with jumps: the whole nine yards $\rightsquigarrow$ arxiv/1607.04214.

Meaning of robustness

We use the term robustness to describe a continuity property. More specifically, given a sequence of data $\left(\mathcal{D}^k\right)_{k\in\mathbb{N}\cup\{\infty\}}$ with associated "output" the sequence $\left(\mathcal{S}^k\right)_{k\in\mathbb{N}\cup\{\infty\}}$, the robustness property consists of the following:

$$\text{if } \mathcal{D}^k \longrightarrow \mathcal{D}^\infty, \text{ then } \mathcal{S}^k \longrightarrow \mathcal{S}^\infty.$$

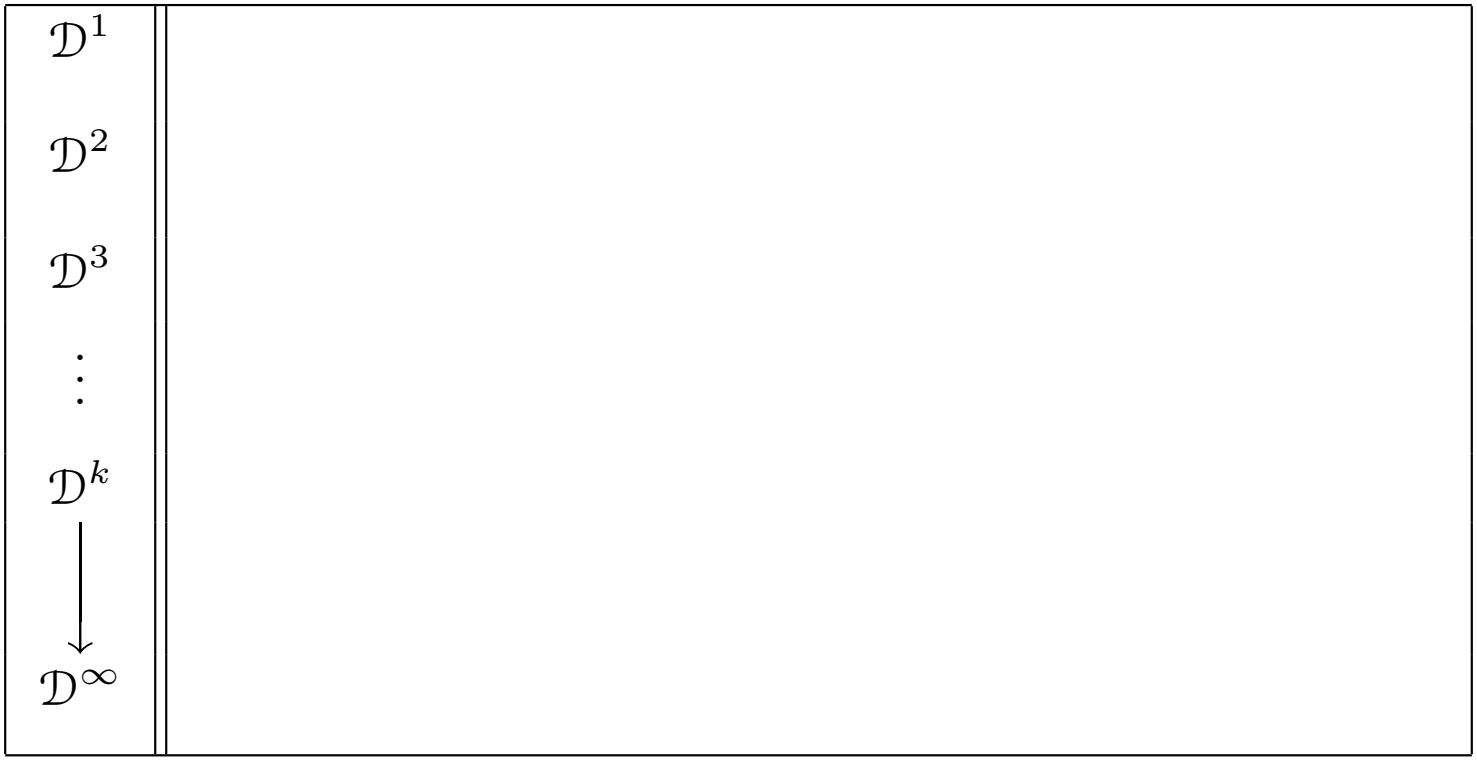
Remark

For this property the term stability has also been used in the literature.

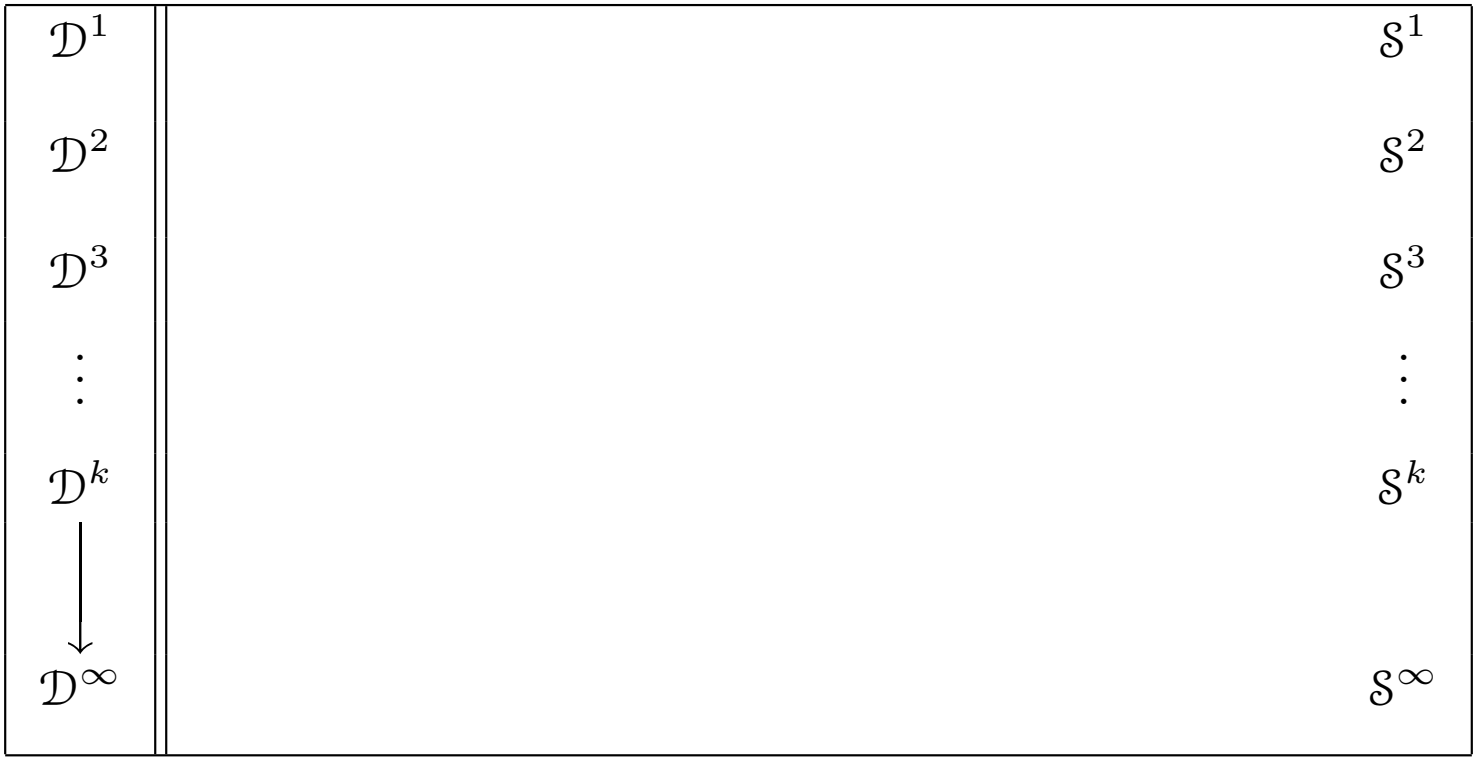
We used the term robustness as our work generalises

- [JMP00] Explicit form and *robustness* of martingale representations and
- [BDM02] On the *robustness* of BSDEs.

Apply Moore–Osgood Theorem



Apply Moore–Osgood Theorem



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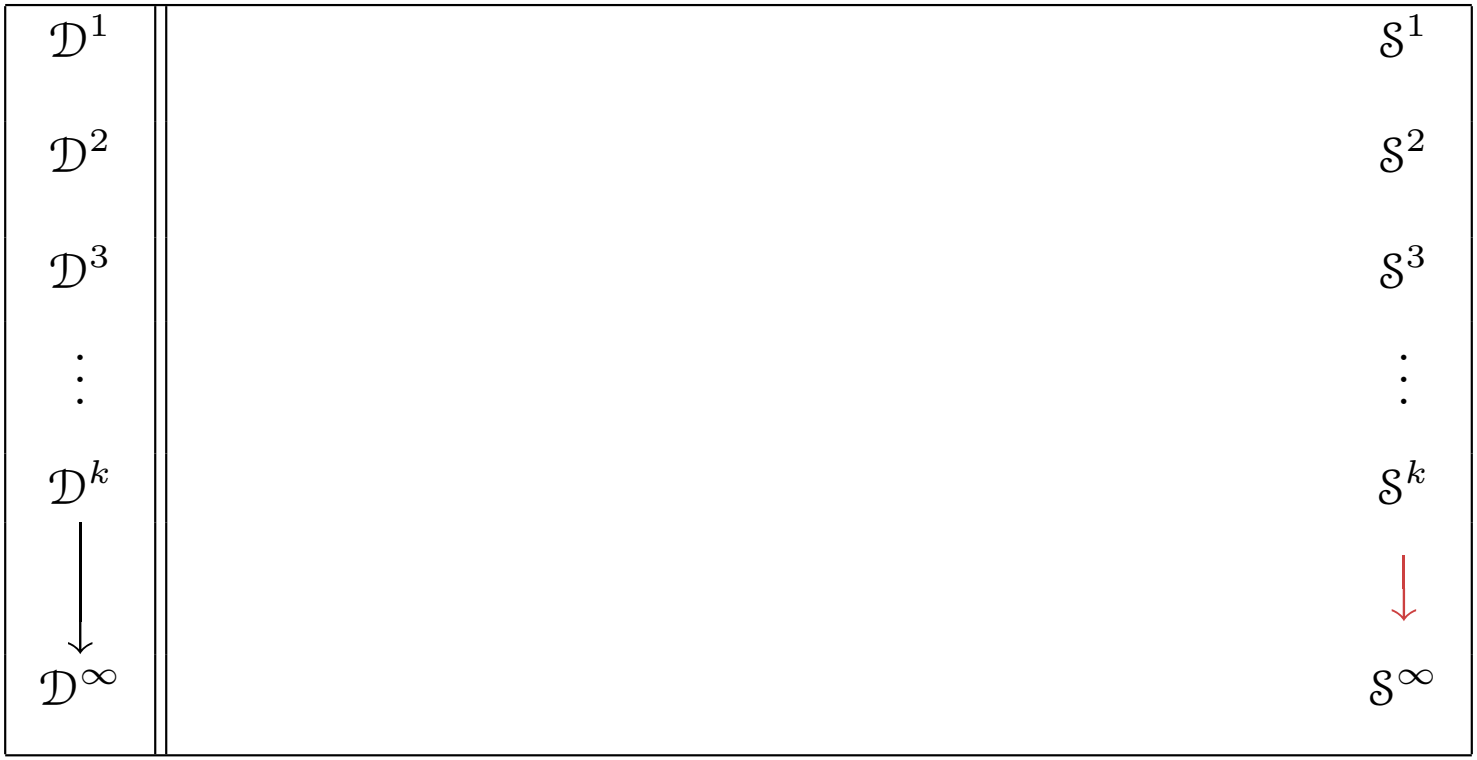

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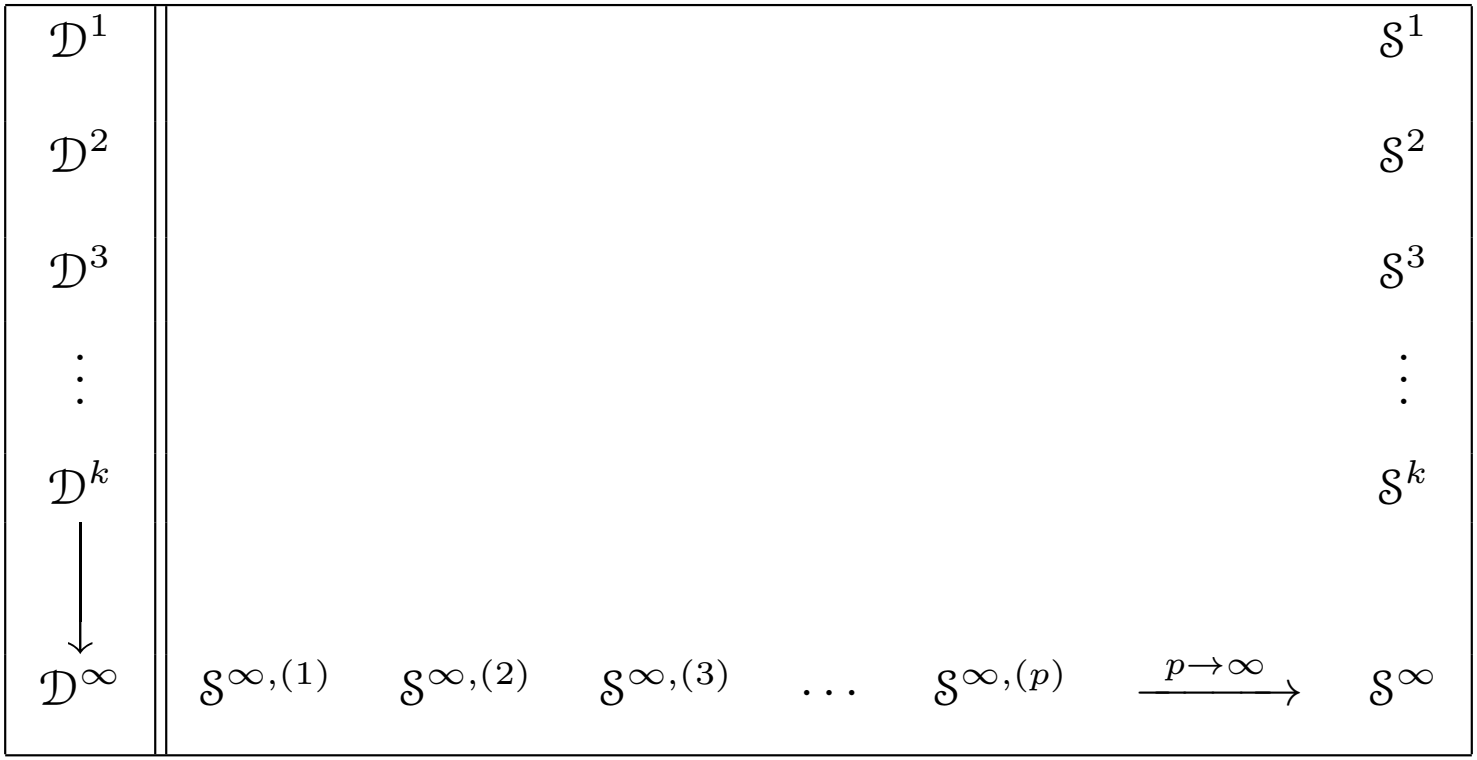

Apply Moore–Osgood Theorem

$\mathcal{D}^1$	$\mathcal{S}^{1,(1)}$	$\mathcal{S}^{1,(2)}$	$\mathcal{S}^{1,(3)}$	$\dots$	$\mathcal{S}^{1,(p)}$	$\xrightarrow{p\to\infty}$	$\mathcal{S}^1$
$\mathcal{D}^2$	$\mathcal{S}^{2,(1)}$	$\mathcal{S}^{2,(2)}$	$\mathcal{S}^{2,(3)}$	$\dots$	$\mathcal{S}^{2,(p)}$	$\xrightarrow{p\to\infty}$	$\mathcal{S}^2$
$\mathcal{D}^3$	$\mathcal{S}^{3,(1)}$	$\mathcal{S}^{3,(2)}$	$\mathcal{S}^{3,(3)}$	$\dots$	$\mathcal{S}^{3,(p)}$	$\xrightarrow{p\to\infty}$	$\mathcal{S}^3$
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$
$\mathcal{D}^k$	$\mathcal{S}^{k,(1)}$	$\mathcal{S}^{k,(2)}$	$\mathcal{S}^{k,(3)}$	$\dots$	$\mathcal{S}^{k,(p)}$	$\xrightarrow{p\to\infty}$	$\mathcal{S}^k$
$\downarrow$							
$\mathcal{D}^\infty$	$\mathcal{S}^{\infty,(1)}$	$\mathcal{S}^{\infty,(2)}$	$\mathcal{S}^{\infty,(3)}$	$\dots$	$\mathcal{S}^{\infty,(p)}$	$\xrightarrow{p\to\infty}$	$\mathcal{S}^\infty$

Theorem (Moore–Osgood)

Let (Γ, d) *a metric space and* $(\gamma_{k,p})_{k,p\in\mathbb{N}}$ *a sequence such that* $\lim\limits_{p\to\infty}\sup\limits_{k\in\mathbb{N}}\gamma_{k,p} = 0$ *and* $\lim\limits_{p\to\infty}\lim\limits_{k\to\infty}\gamma_{k,p} = 0$, *then* $\lim\limits_{k,p\to\infty}\gamma_{k,p} = 0$.

Apply Moore–Osgood Theorem

$\mathcal{D}^1$	$\mathcal{S}^{1,(1)}$	$\mathcal{S}^{1,(2)}$	$\mathcal{S}^{1,(3)}$	$\dots$	$\mathcal{S}^{1,(p)}$	$\xrightarrow{p\to\infty}$	$\mathcal{S}^1$
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$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$
$\mathcal{D}^k$	$\mathcal{S}^{k,(1)}$	$\mathcal{S}^{k,(2)}$	$\mathcal{S}^{k,(3)}$	$\dots$	$\mathcal{S}^{k,(p)}$	$\xrightarrow{p\to\infty}$	$\mathcal{S}^k$
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Apply Moore–Osgood Theorem

$\mathcal{D}^1$	$\mathcal{S}^{1,(1)}$	$\mathcal{S}^{1,(2)}$	$\mathcal{S}^{1,(3)}$	$\dots$	$\mathcal{S}^{1,(p)}$	$\xrightarrow{p\to\infty}$	$\mathcal{S}^1$
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$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$
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$Y_t + \text{LS-integral}_t = \text{UI-Martingale}_t$

$$Y_t^{k,(1)} = \xi^k + \int_{(t,T^k]} f^k(s,0,0,\mathbf{0})\,\mathrm{d}C_s^k - \int_t^{T^k} Z_s^{k,(1)}\,\mathrm{d}X_s^{k,c} - \int_t^{T^k}\int_{\mathbb{R}} U_s^{k,(1)}(x)\widetilde{\mu}^{X^k}(\mathrm{d}s,\mathrm{d}x) - \int_t^{T^k} \mathrm{d}N_s^{k,(1)}$$

$$Y_t^{k,(1)} + \int_{(0,t]} f^k(s,0,0,\mathbf{0})\,\mathrm{d}C_s^k$$

$Y_t + \text{LS-integral}_t = \text{UI-Martingale}_t$

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$$= \xi^k + \int_{(0,T^k]} f^k(s,0,0,\mathbf{0})\,\mathrm{d}C_s^k - \int_t^{T^k} Z_s^{k,(1)}\,\mathrm{d}X_s^{k,c} + \int_t^{T^k}\int_{\mathbb{R}} U_s^{k,(1)}(x)\widetilde{\mu}^{X^k} + \int_0^{T^k} \mathrm{d}N_s^{k,(1)}$$

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$$Y_t^{k,(1)} + \int_{(0,t]} f^k(s,0,0,\mathbf{0})\,\mathrm{d}C_s^k$$

$$= Y_0^{k,(1)} + \int_0^{T^k} Z_s^{k,(1)}\,\mathrm{d}X_s^{k,c} + \int_0^{T^k}\int_{\mathbb{R}} U_s^{k,(1)}(x)\widetilde{\mu}^{X^k} + \int_0^{T^k} \mathrm{d}N_s^{k,(1)}$$

$$- \int_t^{T^k} Z_s^{k,(1)}\,\mathrm{d}X_s^{k,c} + \int_t^{T^k}\int_{\mathbb{R}} U_s^{k,(1)}(x)\widetilde{\mu}^{X^k} + \int_0^{T^k} \mathrm{d}N_s^{k,(1)}$$

$Y_t + \text{LS-integral}_t = \text{UI-Martingale}_t$

$$\mathbf{Y}_t^{k,(1)} = \xi^k + \int_{(t,T^k]} f^k(s,0,0,\mathbf{0})\,\mathrm{d}C_s^k - \int_t^{T^k} Z_s^{k,(1)}\,\mathrm{d}X_s^{k,c} - \int_t^{T^k}\int_{\mathbb{R}} U_s^{k,(1)}(x)\widetilde{\mu}^{X^k}(\mathrm{d}s,\mathrm{d}x) - \int_t^{T^k} \mathrm{d}N_s^{k,(1)}$$

$$\mathbf{Y}_t^{k,(1)} + \int_{(0,t]} f^k(s,0,0,\mathbf{0})\,\mathrm{d}C_s^k = Y_0^{k,(1)} + \int_0^t Z_s^{k,(1)}\,\mathrm{d}X_s^{k,c} + \int_0^t\int_{\mathbb{R}} U_s^{k,(1)}(x)\widetilde{\mu}^{X^k} + \int_0^t \mathrm{d}N_s^{k,(1)}$$

$Y_t + \text{LS-integral}_t = \text{UI-Martingale}_t$

$$Y_t^{k,(1)} = \xi^k + \int_{(t,T^k]} f^k(s,0,0,\mathbf{0})\,\mathrm{d}C_s^k - \int_t^{T^k} Z_s^{k,(1)}\,\mathrm{d}X_s^{k,c} - \int_t^{T^k}\int_{\mathbb{R}} U_s^{k,(1)}(x)\widetilde{\mu}^{X^k}(\mathrm{d}s,\mathrm{d}x) - \int_t^{T^k} \mathrm{d}N_s^{k,(1)}$$

$$\begin{aligned} Y_t^{k,(1)} + \int_{(0,t]} f^k(s,0,0,\mathbf{0})\,\mathrm{d}C_s^k &= Y_0^{k,(1)} + \int_0^t Z_s^{k,(1)}\,\mathrm{d}X_s^{k,c} + \int_0^t\int_{\mathbb{R}} U_s^{k,(1)}(x)\widetilde{\mu}^{X^k} + \int_0^t \mathrm{d}N_s^{k,(1)} \\ &= \mathbb{E}\left[Y_{T^k}^{k,(1)} + \int_{(0,T^k]} f^k(s,0,0,\mathbf{0})\,\mathrm{d}C_s^k \,\middle|\, \mathcal{F}_t^k\right] \end{aligned}$$

$Y_t + \text{LS-integral}_t = \text{UI-Martingale}_t$

$$Y_t^{k,(1)} = \xi^k + \int_{(t,T^k]} f^k(s,0,0,\mathbf{0})\,\mathrm{d}C_s^k - \int_t^{T^k} Z_s^{k,(1)}\,\mathrm{d}X_s^{k,c} - \int_t^{T^k}\int_{\mathbb{R}} U_s^{k,(1)}(x)\widetilde{\mu}^{X^k}(\mathrm{d}s,\mathrm{d}x) - \int_t^{T^k} \mathrm{d}N_s^{k,(1)}$$

$$\begin{aligned} Y_t^{k,(1)} + \int_{(0,t]} f^k(s,0,0,\mathbf{0})\,\mathrm{d}C_s^k &= Y_0^{k,(1)} + \int_0^t Z_s^{k,(1)}\,\mathrm{d}X_s^{k,c} + \int_0^t\int_{\mathbb{R}} U_s^{k,(1)}(x)\widetilde{\mu}^{X^k} + \int_0^t \mathrm{d}N_s^{k,(1)} \\ &= \mathbb{E}\Big[\ \xi^k\ + \int_{(0,T^k]} f^k(s,0,0,\mathbf{0})\,\mathrm{d}C_s^k \Big| \mathcal{F}_t^k\Big] \end{aligned}$$

$Y_t + \text{LS-integral}_t = \text{UI-Martingale}_t$

$$\int_0^t Z_s^{k,(1)} \, \mathrm{d}X_s^{k,c} + \int_0^t \int_{\mathbb{R}} U_s^{k,(1)}(x) \widetilde{\mu}^{X^k} + \int_0^t \mathrm{d}N_s^{k,(1)}$$

$$= \mathbb{E}\Big[\quad \xi^k \quad + \int_{(0,T^k]} f^k(s,0,0,\mathbf{0}) \, \mathrm{d}C_s^k \Big| \mathcal{F}_t^k \Big]$$

The 1^{st} step of the induction

$$\mathbb{E}\left[\xi^k + \int_{(0,\infty)} f^k(s,0,0,\mathbf{0})\,\mathrm{d}C^k_s \middle| \mathcal{F}^k_\cdot\right] = Z^{k,(1)} \cdot X^{c,k} + U^{k,(1)} \star \widetilde{\mu}^{X^k} + N^{k,(1)}$$

The 1^{st} step of the induction

$$\begin{aligned}
\mathbb{E}\Big[\xi^1 + \int_{(0,\infty)} f^1(s,0,0,\mathbf{0})\,\mathrm{d}C^1_s \Big| \mathcal{F}^1_\cdot\Big] &= Z^{1,(1)} \cdot X^{c,1} + U^{1,(1)} \star \widetilde{\mu}^{X^1} + N^{1,(1)} \\
\mathbb{E}\Big[\xi^2 + \int_{(0,\infty)} f^2(s,0,0,\mathbf{0})\,\mathrm{d}C^2_s \Big| \mathcal{F}^2_\cdot\Big] &= Z^{2,(1)} \cdot X^{c,2} + U^{2,(1)} \star \widetilde{\mu}^{X^2} + N^{2,(1)} \\
\mathbb{E}\Big[\xi^3 + \int_{(0,\infty)} f^3(s,0,0,\mathbf{0})\,\mathrm{d}C^3_s \Big| \mathcal{F}^3_\cdot\Big] &= Z^{3,(1)} \cdot X^{c,3} + U^{3,(1)} \star \widetilde{\mu}^{X^3} + N^{3,(1)} \\
&\vdots \qquad\qquad \vdots \qquad\qquad \vdots \\
\mathbb{E}\Big[\xi^k + \int_{(0,\infty)} f^k(s,0,0,\mathbf{0})\,\mathrm{d}C^k_s \Big| \mathcal{F}^k_\cdot\Big] &= Z^{k,(1)} \cdot X^{c,k} + U^{k,(1)} \star \widetilde{\mu}^{X^k} + N^{k,(1)}
\end{aligned}$$

$$\mathbb{E}\Big[\xi^\infty + \int_{(0,\infty)} f^\infty(s,0,0,\mathbf{0})\,\mathrm{d}C^\infty_s \Big| \mathcal{F}^\infty_\cdot\Big] = Z^{\infty,(1)} \cdot X^{\infty,c} + U^{\infty,(1)} \star \widetilde{\mu}^{X^\infty} + N^{\infty,(1)}$$

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$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$\mathbb{E}\left[\xi^k + \int_{(0,\infty)} f^k(s,0,0,\mathbf{0})\,\mathrm{d}C^k_s \middle| \mathcal{F}^k_\cdot\right] = Z^{k,(1)} \cdot X^{c,k} + U^{k,(1)} \star \widetilde{\mu}^{X^k} + N^{k,(1)}$$

$$\downarrow$$

$$\mathbb{E}\left[\xi^\infty + \int_{(0,\infty)} f^\infty(s,0,0,\mathbf{0})\,\mathrm{d}C^\infty_s \middle| \mathcal{F}^\infty_\cdot\right] = Z^{\infty,(1)} \cdot X^{\infty,c} + U^{\infty,(1)} \star \widetilde{\mu}^{X^\infty} + N^{\infty,(1)}$$

The 1^{st} step of the induction

$$
\begin{aligned}
\mathbb{E}\Big[\xi^1 + \int_{(0,\infty)} f^1(s,0,0,\mathbf{0})\,\mathrm{d}C^1_s \Big| \mathcal{F}^1_\cdot\Big] &= Z^{1,(1)} \cdot X^{c,1} + U^{1,(1)} \star \widetilde{\mu}^{X^1} + N^{1,(1)} \\
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&\ \ \vdots \qquad\qquad\qquad\quad \vdots \qquad\qquad\quad \vdots \\
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&\ \ \downarrow \qquad\qquad\qquad\quad \downarrow \qquad\qquad\quad \downarrow \\
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\end{aligned}
$$

The 1^{st} step of the induction

$$
\begin{aligned}
\mathbb{E}\Big[\ \psi^1\ \Big|\mathcal{F}_\cdot^1\Big] &= Z^{1,(1)}\cdot X^{c,1} + U^{1,(1)}\star\widetilde{\mu}^{X^1} + N^{1,(1)}\\
\mathbb{E}\Big[\ \psi^2\ \Big|\mathcal{F}_\cdot^2\Big] &= Z^{2,(1)}\cdot X^{c,2} + U^{2,(1)}\star\widetilde{\mu}^{X^2} + N^{2,(1)}\\
\mathbb{E}\Big[\ \psi^3\ \Big|\mathcal{F}_\cdot^3\Big] &= Z^{3,(1)}\cdot X^{c,3} + U^{3,(1)}\star\widetilde{\mu}^{X^3} + N^{3,(1)}\\
&\ \ \vdots \qquad\qquad\qquad\qquad\qquad\qquad \vdots \qquad\qquad \vdots\\
\mathbb{E}\Big[\ \psi^k\ \Big|\mathcal{F}_\cdot^k\Big] &= Z^{k,(1)}\cdot X^{c,k} + U^{k,(1)}\star\widetilde{\mu}^{X^k} + N^{k,(1)}\\
&\ \ \downarrow \qquad\qquad\qquad\qquad\qquad\qquad \downarrow \qquad\qquad \downarrow\\
\mathbb{E}\Big[\ \psi^k\ \Big|\mathcal{F}_\cdot^\infty\Big] &= Z^{\infty,(1)}\cdot X^{\infty,c} + U^{\infty,(1)}\star\widetilde{\mu}^{X^\infty} + N^{\infty,(1)}
\end{aligned}
$$

Framework for robustness of martingale representations

Definition

We say that $\left(X^k, \mathbb{F}^k\right) \xrightarrow{\text{ext}} \left(X^\infty, \mathbb{F}^\infty\right)$ if for every $Z \in \mathbb{L}^1\left(\Omega, \mathcal{F}^\infty_\infty, \mathbb{P}\right)$ holds

$$\left(X^k, \mathbb{E}\left[Z\middle|\mathcal{F}^k_\cdot\right]\right) \xrightarrow{(\mathrm{J}_1(\mathbb{R}^2),\mathbb{P})} \left(X^\infty, \mathbb{E}\left[Z\middle|\mathcal{F}^\infty_\cdot\right]\right).$$

Assumptions

Let $X^k \in \mathcal{H}^2(\mathbb{F}^k)$, for $k \in \mathbb{N} \cup \{\infty\}$. The following hold

(M1) $(X^k, \mathbb{F}^k) \xrightarrow{\text{ext}} (X^\infty, \mathbb{F}^\infty)$,

(M2) *the filtration $\mathbb{F}^\infty$ is quasi-left-continuous,*

(M3) *X^∞ possesses the $\mathbb{F}^\infty-$Predictable Representation Property,*

(M4) $\psi^k \xrightarrow{\mathbb{L}^2(\mathcal{G})} \psi^\infty$.

Theorem ([PPS16a])

Under the described framework, let the orthogonal decompositions of $\mathbb{E}[\psi^k|\mathcal{F}^k]$ *w.r.t.* $(X^k, \mathbb{F}^k)$, *i.e. for* $k \in \mathbb{N}$

$$Y^k := \mathbb{E}[\psi^k|\mathcal{F}^k_\cdot] = \mathbb{E}[\psi^k|\mathcal{F}^k_0] + Z^k \cdot X^{k,c} + U^k \star \widetilde{\mu}^{X^k} + N^k$$

$$\text{and } Y^\infty := \mathbb{E}[\psi^\infty|\mathcal{F}^\infty_\cdot] = \mathbb{E}[\psi^\infty|\mathcal{F}^\infty_0] + Z^\infty \cdot X^{\infty,c} + U^\infty \star \widetilde{\mu}^{X^\infty}.$$

Then holds

$$\left(Y^k, Z^k \cdot X^{k,c} + U^k \star \widetilde{\mu}^{X^k}, N^k\right) \xrightarrow[k\to\infty]{(\mathrm{J}_1(\mathbb{R}^3),\mathbb{L}^2)} \left(Y^\infty, Z^\infty \cdot X^{\infty,c} + U^\infty(x) \star \widetilde{\mu}^{X^\infty}, 0\right)$$

and $$\left(\langle Y^k, X^k\rangle, \langle Y^k\rangle, \langle N^k\rangle\right) \xrightarrow[k\to\infty]{\mathbb{S}^1(\mathbb{R}^3)} \left(\langle Y^\infty, X^\infty\rangle, \langle Y^\infty\rangle, 0\right).$$

In forthcoming [PPS16a] BSDEs with jumps are robust, we generalise

- [JMP00] Explicit form and robustness of martingale representations
- [BDM02] On the robustness of BSDEs.

Theorem ([Mém03, Theorem 11])

Let $(S^k)_{k\in\mathbb{N}\cup\{\infty\}}$ be a sequence of $\mathbb{F}^k-$special semimartingales with $S^k = M^k + A^k$. Assume, moreover,

(S1) $(S^k, \mathbb{F}^k) \xrightarrow{\text{ext}} (S^\infty, \mathbb{F}^\infty)$,

(S2) S^∞ *is* $\mathbb{F}^\infty$*-quasi-left-continuous,*

(S3) *uniform integrability of* $\left([S^k]_\infty\right)_{k\in\mathbb{N}}$ *and tightness of* $\left(\mathrm{Var}(A^k)_\infty\right)_{k\in\mathbb{N}}$.

Then

$$(S^k, M^k, A^k) \xrightarrow{(\mathrm{J}_1(\mathbb{R}^2),\mathbb{P})} (S^\infty, M^\infty, A^\infty).$$

Remark

Recall that, according to our pevious discussion,

$$Y_t^{k,(1)} + \int_{(0,t]} f^k(s,0,0,\mathbf{0})\mathrm{d}C_s^k = M^{k,(1)}$$

Characterization of weak convergence: the atomless case

Theorem ([PPS16a])

Let $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$, $(C^k)_{k\in\mathbb{N}}$ be sequence of càdlàg, increasing and bounded functions and C^∞ be a continuous, increasing and bounded function such that

$$C^k \xrightarrow{\mathrm{J}_1} C^\infty.$$

Let, moreover, $\mathcal{A}$ be a relatively compact and uniformly bounded subset of $\mathbb{D}(\mathbb{R}_+;\mathbb{R})$. Then it holds

$$\lim_{k\to\infty} \sup_{\alpha\in\mathcal{A}} \left| \int_{[0,\infty)} \alpha_s \, \mathrm{d}C^k_s - \int_{[0,\infty)} \alpha_s \, \mathrm{d}C^\infty_s \right| = 0.$$

Assumptions

Let $(\mathcal{D}^k)_{k\in\mathbb{N}\cup\{\infty\}}$ *be a sequence of standard data. Furthermore assume*

(B1) $\left(X^{c,k}, X^{d,k}, \mathbb{F}^k\right) \xrightarrow{\text{ext}} \left(X^{\infty,c}, X^{\infty,d}, \mathbb{F}^\infty\right)$, *for* $X^{c,k} \perp\!\!\!\perp X^{d,k}$,

(B2) *the filtration* $\mathbb{F}^\infty$ *is quasi-left-continuous,*

(B3) X^∞ *possesses the* $\mathbb{F}^\infty-$*Predictable Representation Property,*

(B4) $\xi^k \xrightarrow[k\to\infty]{\mathbb{L}^2(\mathcal{G})} \xi^\infty$,

(B5) *We impose conditions such that we can obtain*

$$f^k\left(\cdot, W^k_\cdot, \overline{Z}_\cdot, \overline{U}_\cdot\right) \xrightarrow[k\to\infty]{(\mathrm{J}_1,\mathbb{P})} f^\infty\left(\cdot, W^\infty_\cdot, \overline{Z}_\cdot, \overline{U}_\cdot\right),$$

whenever $W^k \xrightarrow[k\to\infty]{(\mathrm{J}_1,\mathbb{P})} W^\infty$ *and* $\overline{Z}, \overline{U}$ *are suitable and fixed.*

(B6) $C^k \xrightarrow[k\to\infty]{(\mathrm{J}_1,\mathbb{P})} C^\infty$.

Theorem ([PPS16a])

Under the previous framework, let for $k \in \mathbb{N} \cup \{\infty\}$,

$$(Y^k, Z^k, U^k, N^k) \quad \textit{be the solution of the} \quad \mathcal{D}^k - (\text{BSDEJ}).$$

Then it holds

$$\left(Y^k, Z^k \cdot X^{c,k} + U^k \star \widetilde{\mu}^{X^k}, N^k\right) \xrightarrow{(\mathrm{J}_1(\mathbb{R}^3), \mathbb{L}^2)} \left(Y^\infty, Z^\infty \cdot X^{\infty,c} + U^\infty \star \widetilde{\mu}^{X^\infty}, 0\right)$$

and

$$\left(\langle Y^k, X^{c,k}\rangle, \langle Y^k, X^{d,k}\rangle, \langle Y^k\rangle, \langle N^k\rangle\right) \xrightarrow{\mathbb{S}^1(\mathbb{R}^4)} \left(\langle Y^\infty, X^{\infty,c}\rangle, \langle Y^\infty, X^{\infty,d}\rangle, \langle Y^\infty\rangle, 0\right).$$

In forthcoming [PPS16a] BSDEs with jumps are robust, we generalise

- [BDM02] On the robustness of BSDEs,
- [MPS14] Convergence of BSΔEs driven by random walks to BSDEs: the case of (in)finite activity jumps with general driver

The End!

Conlusion:

- a unified framework for existence and uniqueness of solution for BSΔEs as well as for BSDEs,
- suitable framework for the robustness property of martingale representations as well as of BSDE$-$semimartingales.

Outlook:

- weak approximation schemes for 2BSDEJ.

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