Alpha-CIR Model in Sovereign Interest Rate Modelling

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Motivation

- Current sovereign bond markets in the Euro zone:
  - persistency of low interest rates
  - significant fluctuations at local extent.

**Figure**: Long term interest rates of Euro area countries.
Modelling approaches

- Large fluctuations in financial data motivate the introduction of jumps in the interest rate dynamics: Eberlein & Raible (1999), Filipović, Tappe & Teichmann (2010)...


- Difficulty: jump presence v.s. trend of low interest rate
Plan of our work

- Objective: a new model of interest rate ($\alpha$-CIR model) for these seemingly puzzling phenomena in a unified and parsimonious framework.
- Jump diffusion model as natural extension of the CIR model, using the $\alpha$-stable branching processes
  - CIR model is the particular case with continuous path
- Integral representation to highlight the branching property:
  - limit of Hawkes processes: clustering and self-exciting properties;
  - link with CBI processes: exponential affine structure for bond price, Duffie, Filipović & Schachermayer (2001)
- The bond price decreases with the parameter $\alpha$, which allows to respond to the low interest rate behavior.
  - surprising result:
    - when $\alpha$ decreases, the tails are heavier. The “risks” become larger but the bond price increases....
  - comparaison with CIR model with “Poisson” $\alpha$-stable jumps.
  - same effect if jumps are in branching mechanism or immigration rate, but very low interest rates in $\alpha$-CIR model.
The $\alpha$-CIR model setup: Integral representation (Dawson-Li)

Integral form by using the random fields

$$
\begin{align*}
    r_t &= r_0 + \int_0^t a(b - r_s) \, ds + \sigma \int_0^t \int_0^{r_s} W(ds, du) \\
        &\quad + \sigma Z \int_0^t \int_0^{r_s} \int_{\mathbb{R}^+} \tilde{N}(ds, du, d\zeta),
\end{align*}
$$

- $W(ds, du)$: white noise on $\mathbb{R}_+^2$ with intensity $dsdu$,
- $\tilde{N}(ds, du, d\zeta)$: compensated Poisson random measure on $\mathbb{R}_+^3$ with intensity $dsdu\mu(d\zeta)$,
- $\mu(d\zeta)$ is a Lévy measure satisfying $\int_0^{\infty} (\zeta \wedge \zeta^2) \mu(d\zeta) < \infty$.

The $\alpha$-CIR model setup

We consider $\alpha$-CIR($a, b, \sigma, \sigma_Z, \alpha$) model for the short interest rate

$$r_t = r_0 + \int_0^t a (b - r_s) ds + \sigma \int_0^t \sqrt{r_s} dB_s + \sigma_Z \int_0^t r_s^{1/\alpha} dZ_s$$

(2)

- $B = (B_t, t \geq 0)$ a Brownian motion
- $Z = (Z_t, t \geq 0)$ a spectrally positive $\alpha$-stable compensate Lévy process with parameter $\alpha \in (1, 2]$ with
  $$\mathbb{E} \left[ e^{-qZ_t} \right] = \exp \left\{ - \frac{tq^\alpha}{\cos(\pi \alpha/2)} \right\}, \quad q \geq 0.$$
- $B$ and $Z$ are independent
- $Z_t$ follows the $\alpha$-stable distribution $S_\alpha(t^{1/\alpha}, 1, 0)$ with scale parameter $t^{1/\alpha}$, skewness parameter 1 and zero drift.
A natural extension of the CIR model: aggregate

- When $\sigma_Z = 0$, we recover the CIR model.
- When $\alpha = 2$, it also reduces to a CIR model but with volatility parameter $(\sigma^2 + 2\sigma_Z^2)^{1/2}$.
- The difference of $Z$ from a Brownian motion is controlled by the tail index $\alpha$:
  - $\diamond \alpha = 2$ : $Z$ is a Brownian motion scaled by $\sqrt{2}$ ;
  - $\diamond \alpha < 2$ : $Z$ is a pure jump process with heavy tails. More as $\alpha$ close to 1, more likely $Z_t$ takes values far from median ;
  - $\diamond$ comparison with Poisson process: $Z$ has an infinite number of (small) jumps over any time interval, allowing it to capture the extreme activity.
Similar properties with CIR model

**Boundary condition:**

The point 0 is an inaccessible boundary if and only if $2ab \geq \sigma^2$. In particular, a pure jump $\alpha$-CIR process with $ab > 0$ never reaches 0 since $\sigma = 0$.

**Branching property:**

$r$ can be decomposed as $r = r^{(1)} + r^{(2)}$ where for $i = 1, 2$, $r^{(i)}$ is an $\alpha$-CIR($a, b^{(i)}, \sigma, \sigma_Z, \alpha$) process such that $r_0 = r^{(1)}_0 + r^{(2)}_0$ and $b = b^{(1)} + b^{(2)}$. This property is a direct consequence of

- linearity of integrals
- homogeneity of measures
Equivalence of two representations

We choose the Lévy measure to be

$$\mu(d\zeta) = -\frac{1\{\zeta>0\}d\zeta}{\cos(\pi\alpha/2)\Gamma(-\alpha)\zeta^{1+\alpha}}, \quad 1 < \alpha < 2,$$

Then the root representation (2) and the integral representation (1) are equivalent in the following sense:

- The solutions of the two equations have the same probability law.
- On an extended probability space, they are equal almost surely.
Link to Hawkes process

- When $\sigma = 0$ and $\mu(d\zeta) = \delta_1(dz)$, then $r$ is given by

$$r_t = r_0 + abt - \int_0^t (a + \sigma Z)r_s ds + \sigma Z \int_0^t \int_0^{r_s} N(ds, du)$$  \hspace{1cm} (4)

which is the intensity of Hawkes process $\int_0^t \int_0^{r_s} N(ds, du)$, $N$ being the Poisson random measure with intensity $ds du$.

- Consider a sequence $\{r_t^{(n)} , t \geq 0\}$ defined by (4) with parameters $(a/n, nb, \sigma Z)$. Then

$$r_{nt}^{(n)} / n \xrightarrow{\mathcal{L}} Y_t \quad \text{in } D(\mathbb{R}_+),$$

where $D(\mathbb{R}_+)$ is the Skorokhod space of càdlàg processes and

$$Y_t = \int_0^t a(b - Y_s) ds + \sigma Z \int_0^t \int_0^{Y_s} W(ds, du).$$

Locally equivalent CIR process with jumps

- Consider the $\alpha$-CIR process with initial value $r_0$ and introduce

$$\lambda_t = r_0 + \int_0^t a (b - \lambda_s) \, ds + \sigma \int_0^t \int_0^\lambda_s W(ds, du)$$

$$+ \sigma_Z \int_0^t \int_0^{r_0} \int_{\mathbb{R}^+} \zeta \tilde{N}(ds, du, d\zeta)$$

where the processes $W$ and $\tilde{N}$ are the same as in (2).

- the above CIR process with jumps can be written as

$$d\lambda_t = r_0 + a (b - \lambda_t) \, dt + \sigma \sqrt{\lambda_t} dB_t + \sigma_Z \sqrt{r_0} dZ_t,$$

- The implicit negative drifts lead to a linear decay for $\lambda_t$ while an exponential decay for $r_t$: when $\sigma_Z$ increases, the decreasing drift plays a more important role in $\alpha$-CIR than in equivalent CIR with jumps.
Comparison between $\alpha$-CIR and CIR with $\alpha$-stable jumps (continued)

- Separating small and large jumps in CIR with jumps, we get

$$
\lambda_t = r_0 + \int_0^t a \left( b - \frac{\sigma Z r_0 \Theta(\alpha, y)}{a} - \lambda_s \right) ds + \sigma \int_0^t \int_0^{\lambda_s} W(ds, du) + \sigma_Z \int_0^t \int_0^{r_s} \zeta \tilde{N}(ds, du, d\zeta) + \sigma_Z \int_0^t \int_0^{r_s} \int_y^\infty \zeta \tilde{N}(ds, du, d\zeta)
$$

where

$$
\Theta(\alpha, y) = \frac{2}{\pi} \alpha \Gamma(\alpha - 1) \frac{\sin(\pi \alpha / 2)}{y^{\alpha - 1}}.
$$

- In a similar way, the $\alpha$-CIR process can be written as

$$
r_t = r_0 + \int_0^t \tilde{a}(\alpha, y) \left( \tilde{b}(\alpha, y) - r_s \right) ds + \sigma \int_0^t \int_0^{r_s} W(ds, du) + \sigma_Z \int_0^t \int_0^{r_s} \zeta \tilde{N}(ds, du, d\zeta) + \sigma_Z \int_0^t \int_0^{r_s} \int_y^\infty \zeta \tilde{N}(ds, du, d\zeta)
$$

where

$$
\tilde{a}(\alpha, y) = a + \sigma_Z \Theta(\alpha, y), \quad \tilde{b}(\alpha, y) = \frac{ab}{a + \sigma_Z \Theta(\alpha, y)}
$$
Simulation of processes $Z$ and $r$ with different $\alpha$

**Figure:** Three parameters of $\alpha$: 2 (blue), 1.5 (green) and 1.2 (black)
Continuous state branching process with immigration (CBI)

CBI (Kawazu & Watanabe 1971) of branching mechanism $\Psi(\cdot)$ and immigration rate $\Phi(\cdot)$: Markov process $X$ with state space $\mathbb{R}_+$ verifying

$$
\mathbb{E}_x \left[ e^{-pX_t} \right] = \exp \left[ -x\nu(t, p) - \int_0^t \Phi(\nu(s, p)) \, ds \right],
$$

where $\nu : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ satisfies

$$
\frac{\partial \nu(t, p)}{\partial t} = -\Psi(\nu(t, p)), \quad \nu(0, p) = p
$$

and $\Psi$ and $\Phi$ are functions on $\mathbb{R}_+$ given by

$$
\Psi(q) = \beta q + \frac{1}{2} \sigma^2 q^2 + \int_0^\infty (e^{-qu} - 1 + qu) \pi(du),
$$

$$
\Phi(q) = \gamma q + \int_0^\infty (1 - e^{-qu}) \nu(du),
$$

with $\sigma, \gamma \geq 0$, $\beta \in \mathbb{R}$ and $\pi, \nu$ being two Lévy measures such that $\int_0^{\infty} (u \wedge u^2) \pi(du) < \infty$ and $\int_0^{\infty} (1 \wedge u) \nu(du) < \infty$. 
Link with the CBI processes

Let $r$ be an $\alpha$-CIR $(a, b, \sigma, \sigma_Z, \alpha)$ process. Then $r$ is a CBI with

branching mechanism: $\Psi(q) = aq + \frac{\sigma^2}{2} q^2 - \frac{\sigma_Z^\alpha}{\cos(\pi \alpha/2)} q^{\alpha}$ (6)

immigration rate: $\Phi(q) = abq$. (7)

Consequences:

- As $t \to +\infty$, $r_t$ has a limite distribution $r_\infty$, given by

  $\mathbb{E}[e^{-pr_\infty}] = \exp \left\{ - \int_0^p \frac{\Phi(q)}{\Psi(q)} dq \right\}, \quad p \geq 0.$

- Laplace transform

  $\mathbb{E}\left[ e^{-\xi r_t - p \int_0^t r_s ds} \right] = \exp \left( - r_0 v(t, \xi, p) - \int_0^t \Phi(v(s, \xi, p)) ds \right),$

  with $\partial_t v(t, \xi, p) = -\Psi(v(t, \xi, p)) + p, \quad v(0, \xi, p) = \xi.$

- Let $r^{(\alpha)}$ be $\alpha$-CIR$(a, b, \sigma, \sigma_Z, \alpha)$ process, $\alpha \in (1, 2]$. Then $r^{(\alpha)} \xrightarrow{\mathcal{L}} r^{(2)}$ in $D(\mathbb{R}_+)$ as $\alpha \to 2$. 
Application to bond pricing

For simplicity, we assume that the short rate $r$ is given by an $\alpha$-CIR($a, b, \sigma, \sigma_Z, \mu, \alpha$) model under $\mathbb{Q}$.

- Zero-coupon bond price:

$$B(t, T) = \exp \left( - r_t v(T - t) - a b \int_0^{T-t} v(s) ds \right),$$

where $v(\cdot)$ is given by

$$\frac{\partial v(t)}{\partial t} = 1 - \Psi(v(t)), \quad v(0) = 0,$$

with $\Psi(q) = aq + \frac{\sigma^2}{2} q^2 - \frac{\sigma_Z^2}{\cos(\pi \alpha/2)} q^{\alpha}$. 

- We have

$$v(t) = f^{-1}(t) \quad \text{where} \quad f(t) = \int_0^t \frac{dx}{1 - \Psi(x)} \quad (8)$$
Proposition

The function $v(\cdot)$ is increasing with respect to $\alpha \in (1, 2]$. In particular, the bond price $B(0, T)$ is decreasing with respect to $\alpha$.

- $\alpha$ characterizes the tail fatness: when $\alpha$ decreases, it is more likely to take values far away from median and have large jumps.
- Generalized Blumenthal-Getoor index (e.g. Aït-Sahalia and Jacod, 2009)
  \[ \inf\{\beta > 0 : \sum_{0\leq s\leq T} \Delta r_s^\beta < \infty, \text{ a.s.}\} = \alpha. \]

- The above proposition shows that the $\alpha$-CIR model is suitable to describe the phenomenon of low interest rate trend with jumps.
Comparaison between $\alpha$-CIR and CIR with $\alpha$-stable jumps

\begin{figure}[h]
\centering
\begin{subfigure}{0.45\textwidth}
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption*{\textbf{Figure}: Bond price is decreasing w.r.t. $\alpha$, curve CIR (in red) corresponds to $\sigma_Z = 0$}
\end{subfigure}
\begin{subfigure}{0.45\textwidth}
\centering
\includegraphics[width=\textwidth]{figure2.png}
\end{subfigure}
\end{figure}
Yield curve

\[ Y(t, \theta) = -\frac{1}{\theta} \ln B(t, \theta) = r_t \frac{v(\theta)}{\theta} + \frac{ab}{\theta} \int_0^\theta v(s)ds \]

According to Keller-Ressel and Steiner (08), define \( x_0 > 0 \) as the unique solution of \( \Psi_\alpha(x) = 1 \)

**long term yield** \( Y(t, \theta) = abx_0 \)

**Normal Yield curve** \( Y(t, \theta) \) is increasing if \( r_t < ab/\Psi'_\alpha(x_0) \)

**Inverse Yield curve** \( Y(t, \theta) \) is decreasing if \( r_t > b \)

**Humped Yield curve** \( Y(t, \theta) \) has one maximum and no minimum if \( ab/\Psi'_\alpha(x_0) < r_t < b \)

Related results for forward curve.
Jump behavior

- The jumps, especially the large jumps capture the significant changes in the interest rate and may imply the downgrade risk of credit quality.
- Fix $y > 0$ and define the first time that the jump of $r$ is large than $\sigma Z y$, i.e.
  $$\tau_y = \inf\{t > 0 : \Delta r_t > \sigma Z y\}.$$ 
- Consider the truncated process $r^{(y)}$ as
  $$r_t^{(y)} = r_0 + \int_0^t \tilde{a}(\alpha, y)(\tilde{b}((\alpha, y) - r_s) ds + \sigma \int_0^t \int_0^{r_s} W(ds, du)$$
  $$+ \sigma Z \int_0^t \int_0^{r_s} \int_0^y \tilde{N}(ds, du, d\zeta).$$
- It is also a CBI process which coincides with $r$ up to $\tau_y$, and with the branching mechanism given by
  $$\Psi^{(y)} = \Psi + \sigma^\alpha Z \int_y^\infty (1 - e^{-q\zeta}) \mu(d\zeta).$$
Probability law of the first large jump

We have

\[ P(\tau_y > t) = \exp \left( -l(y, t)r_0 - ab \int_0^t l(y, s) ds \right) \]

where \( l(y, t) \) is the unique solution of

\[ \frac{dl}{dt}(y, t) = \sigma_Z^\alpha \int_y^\infty \mu(d\zeta) - \Psi(y)(l(y, t)), \]

with initial condition \( l(y, 0) = 0 \).

Equivalent form :

\[ P(\tau_y > t) = \mathbb{E} \left[ \exp \left\{ -\sigma_Z^\alpha \left( \int_y^\infty \mu(d\zeta) \right) \left( \int_0^t r_s^{(y)} ds \right) \right\} \right]. \]

which is a bond price written on the auxiliary rate \( r^{(y)} \) weighted by the measure \( \mu \) restricted on \((y, \infty)\).
Danke Schön / Merci / Thanks for your attention!