Branching diffusion representation for semilinear PDEs and Monte-Carlo approximation

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Outline

1. Introduction
2. Branching diffusion representation for semilinear PDEs
   - A first class of semilinear PDEs
   - A second class of semilinear PDEs
   - The integrability
3. Monte-Carlo method
   - The complexity
   - Numerical examples

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Feynmann-Kac formula

• Let $W.$ be a standard $d-$dimensional Brownian motion, $f : [0, T] \times \mathbb{R}^d \to \mathbb{R}$ and $g : \mathbb{R}^d \to \mathbb{R}$ be continuous and of polynomial growth.

- The heat equation on $[0, T] \times \mathbb{R}^d$:

$$\left(\partial_t u + \frac{1}{2}\Delta u + f\right)(t, x) = 0, \quad u(T, \cdot) = g(\cdot). \tag{1}$$

- A probabilistic representation

$$u(t, x) = \mathbb{E}\left[g(x + W_{T-t}) + \int_t^T f(s, x + W_{s-t})ds\right]. \tag{2}$$

Feynmann-Kac formula

Theorem (Feynmann-Kac et the reverse)

(i) *Let $u \in C^{1,2}([0,T] \times \mathbb{R}^d)$ be a classical solution of PDE (1), then u has the probabilistic representation (2).*

(ii) *Let the function $u(t,x)$ be defined by (2), then u is a "solution" of PDE (1).*

Proof. (i) By Itô formula, for $\tau_n := T \wedge \inf\{s \geq t : |W_{s-t}| \geq n\}$,

$$u(t,x) = u(\tau_n, x+W_{\tau_n - t}) - \int_t^{\tau_n} (\partial_t u + \frac{1}{2}\Delta u)(s, x+W_{s-t})ds - \int_t^{\tau_n} Du^\top \sigma dW_s$$

$$= \mathbb{E}\Big[u(\tau_n, x + W_{\tau_n - t}) + \int_t^{\tau_n} f(s, x + W_{s-t})ds\Big]$$

$$= \mathbb{E}\Big[g(x + W_{T-t}) + \int_t^{T} f(s, x + W_{s-t})ds\Big].$$

(ii) By flow property of W, $u(t,x) = \mathbb{E}[u(\tau_h, x + W_{\tau_h}) + \int_t^{\tau_h} ...]$ □

BSDE and semi-linear PDE (Pardoux-Peng)

- The semillinear parabolic PDE :

$$\partial_t u(t,x) + \frac{1}{2}\Delta u(t,x) + f(\cdot, u, Du)(t,x) = 0, \qquad u(T,\cdot) = g(\cdot).$$

- The backward SDE :

$$Y_t \;=\; g(X_T) + \int_t^T f(s, W_s, Y_s, Z_s)ds + \int_t^T Z_s dW_s.$$

Theorem

(i) Let $u \in C^{1,2}([0,T] \times \mathbb{R}^d)$ be a solution of the semilinear PDE, then $(Y_t, Z_t) := (u(t, X_t), Du(t, X_t))$ provides a solution of the the BSDE,

(ii) Define $u(t,x) := Y_t^{t,x}$, then u is a "solution" of semilinear PDE.

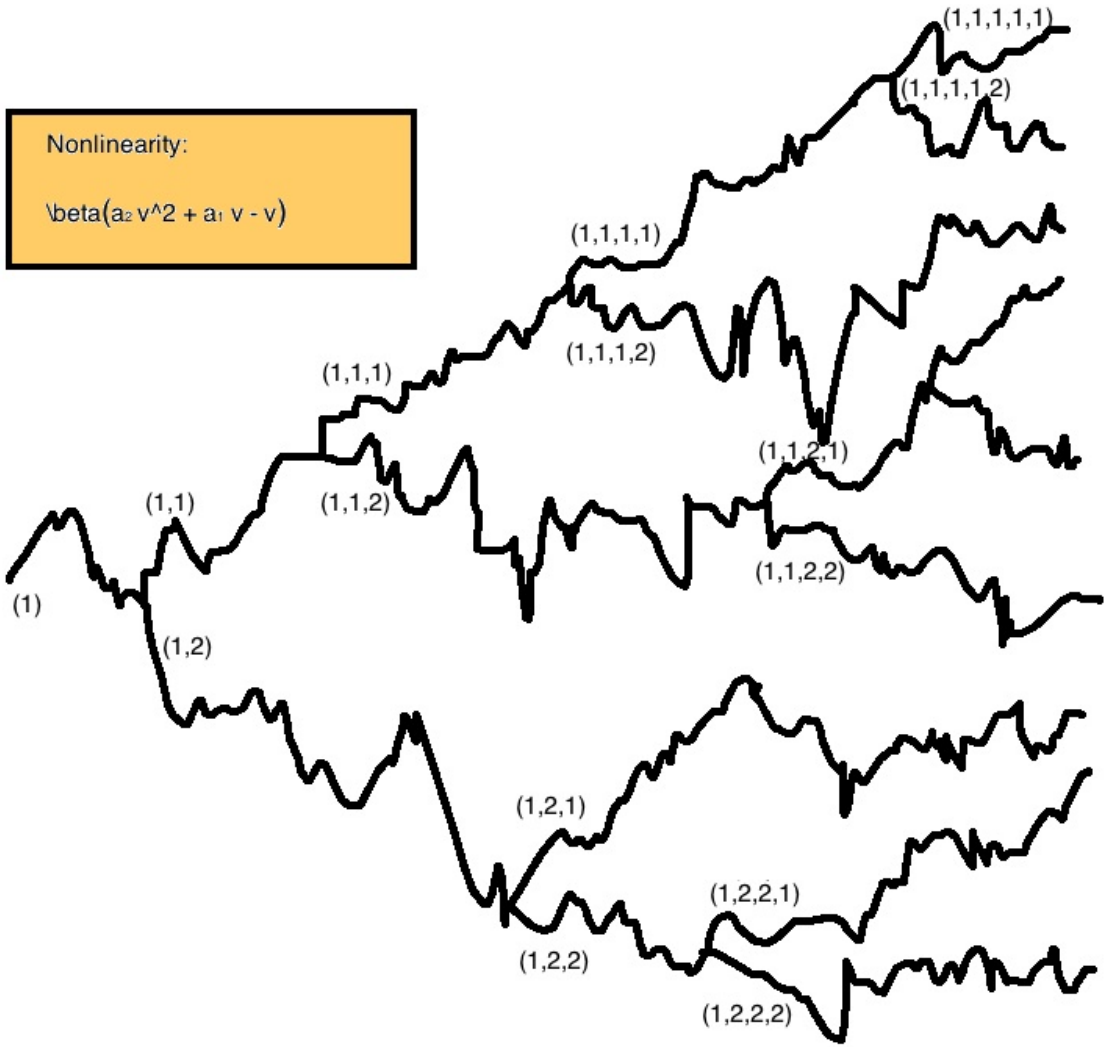
BSDE and semi-linear PDE

• BSDE and applications in finance El Karoui-Peng-Quenez (1997), etc.

• Numerical schemes of BSDEs : Bouchard-Touzi (2004), Zhang (2004), Gobet-Lemor-Warin, Briand-Labart, etc.

• 2nd order BSDE and fully nonlinear PDE : Soner-Touzi-Zhang(2012), Fahim-Touzi-Warin, Possamaï-Tan, etc.

Branching diffusion process



Branching diffusion process and semi-linear PDE

$$\mathcal{K}_t := \{\text{All particles alive at time } t\}, \quad \overline{\mathcal{K}}_T := \cup_{t\le T}\mathcal{K}_t.$$

• [Skorokhod, Watanebe, McKean, etc.] Representation of KPP equation

$$\partial_t u + \frac{1}{2}\Delta u + \beta\Big(\sum_{\ell\in\mathbb{N}} p_\ell u^\ell - u\Big) = 0, \quad u(T,\cdot) = g(\cdot),$$

by branching Brownian motion $\mathbb{E}\Big[\prod_{k\in\mathcal{K}_T} g(W_T^k)\Big]$.

• [Henry-Labordère, Henry-Labordère-Tan-Touzi] Extension and Monte-Carlo method

$$\partial_t u + \frac{1}{2}\Delta u + \beta\Big(\sum_{\ell\in\mathbb{N}} p_\ell a_\ell u^\ell - u\Big) = 0, \quad u(T,\cdot) = g(\cdot),$$

by $$\mathbb{E}\Big[\Big(\prod_{k\in\overline{\mathcal{K}}_T\setminus\mathcal{K}_T} a_{I_k}(T_k, W_{T_k}^k)\Big)\Big(\prod_{k\in\mathcal{K}_T} g(W_T^k)\Big)\Big].$$

Objective

- Extension

$$\sum_{\ell \in \mathbb{N}} p_\ell a_\ell u^\ell \quad \longrightarrow \quad \sum_{\ell=(\ell_0,\ell_1,\cdots,\ell_m)} p_\ell a_\ell \; u^{\ell_0} \; \Pi_{i=1}^m (b_i \cdot Du)^{\ell_i}.$$

- Forward numerical algorithm for semilinear PDE (Numerical examples up to dimension $d = 20$).

- Difficulties : Integrability and square integrability.

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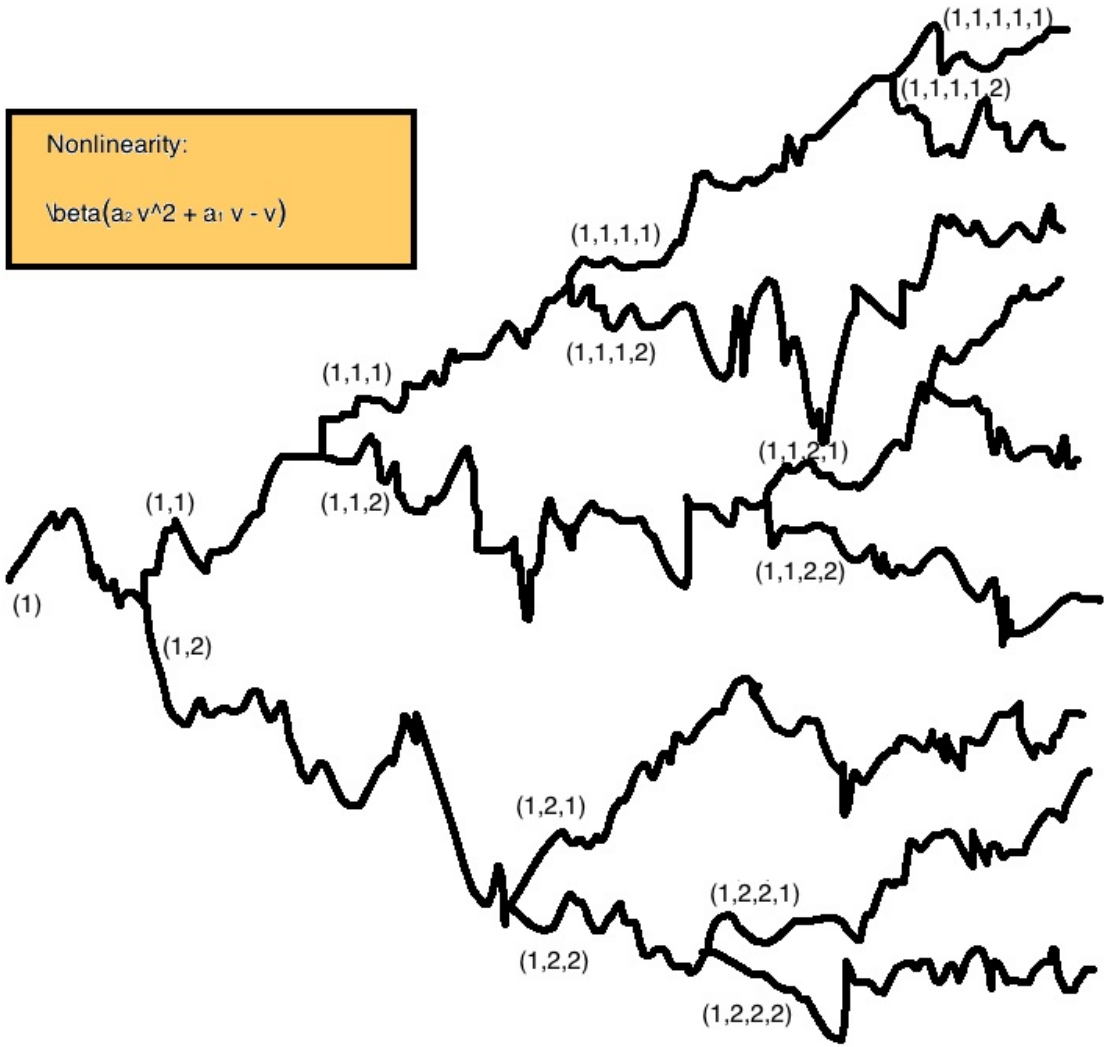
A branching Brownian motion



A branching Brownian motion

A branching Brownian motion $(W^k)_{k\in\overline{\mathcal{K}}_T}$:

- $\mathcal{K}_t$ the collection of particles alive at time t, $\overline{\mathcal{K}}_T := \cup_{t\le T}\mathcal{K}_t$.
- $\mathcal{K}_t^n \subset \mathcal{K}_t$ (resp. $\overline{\mathcal{K}}_T^n$) the collection of particles of generation n.
- A particle $k \in \overline{\mathcal{K}}_T$ defaults at time $T_k \wedge T$.
- For a particle $k \in \overline{\mathcal{K}}_T$, denote by $k-$ its parent particle, then the particle starts its life at time T_{k-} and ends its life at time $T_k \wedge T$.
- $\Delta T_k := T_k - T_{k-}$ follows a distribution of density ρ (e.g. $\mathcal{E}(\beta)$ with $\rho(t) = \beta e^{-\beta t}\mathbb{1}_{t\ge 0}$), denote $\overline{F}(T) := \int_T^\infty \rho(s)ds$.
- At default time $T_k < T$, the particle k branches into I_k offspring particles, where $\mathbb{P}(I_k = \ell) = p_\ell$.
- For $k \in \overline{\mathcal{K}}_T$, $W^k_{T_{k-}} = W^{k-}_{T_{k-}}$.

A KPP equation and branching Brownian motion

- KPP equation : Let $u \in C^{1,2}$ be a solution to

$$\partial_t u + \frac{1}{2}\Delta u + \sum_{\ell \in \mathbb{N}} a_\ell u^\ell = 0, \quad u(T,\cdot) = g(\cdot).$$

- Using Feynmann-Kac

$$\begin{aligned} u(0,0) &= \mathbb{E}\Big[g(W_T) + \int_0^T \sum_\ell \big(a_\ell u^\ell\big)(s, W_s) ds\Big] \\ &= \mathbb{E}\Big[\frac{g(W_T)}{\overline{F}(T)}\overline{F}(T) + \int_0^T \sum_\ell \big(a_\ell u^\ell\big)(s, W_s)\frac{1}{\rho(s)}\rho(s) ds\Big] \\ &= \mathbb{E}\Big[\frac{g(W^{(1)}_T)}{\overline{F}(T)}\mathbb{1}_{\{T_{(1)} \geq T\}} + \frac{a_{I_{(1)}}(T_{(1)}, W^{(1)}_{T_{(1)}})}{p_{I_{(1)}}\rho(T_{(1)})} u^{I_{(1)}}_{T_{(1)}} \mathbb{1}_{\{T_{(1)} < T\}}\Big]. \end{aligned}$$

A KPP equation and branching Brownian motion ...

- Let us introduce

$$\psi_1 := \Big[\prod_{k\in\mathcal{K}_T^1}\frac{g(W_T^k)}{\overline{F}(\Delta T_k)}\Big]\Big[\prod_{k\in\overline{\mathcal{K}}_T^1\setminus\mathcal{K}_T^1}\frac{a_{I_k}(T_k,W_{T_k}^k)}{p_{I_k}\rho(T_k)}\Big]\Big[\prod_{k\in\overline{\mathcal{K}}_T^2}u_{T_k-}\Big].$$

Then

$$\begin{aligned} u(0,0) &= \mathbb{E}\Big[\frac{g(W_T^{(1)})}{\overline{F}(T)}\mathbb{1}_{\{T_{(1)}\geq T\}} + \frac{a_{I_{(1)}}(T_{(1)},W_{T_{(1)}}^{(1)})}{p_{I_{(1)}}\rho(T_{(1)})}u_{T_{(1)}}^{I_{(1)}}\mathbb{1}_{\{T_{(1)}<T\}}\Big] \\ &= \mathbb{E}\big[\psi_1\big]. \end{aligned}$$

A KPP equation and branching Brownian motion ...

- Recall

$$\psi_1 := \Big[\prod_{k\in\mathcal{K}_T^1}\frac{g(W_T^k)}{\overline{F}(\Delta T_k)}\Big]\Big[\prod_{k\in\overline{\mathcal{K}}_T^1\setminus\mathcal{K}_T^1}\frac{a_{I_k}(T_k,W_{T_k}^k)}{p_{I_k}\rho(T_k)}\Big]\Big[\prod_{k\in\overline{\mathcal{K}}_T^2}u_{T_k-}\Big]$$

and $$u(0,0) \;=\; \mathbb{E}\big[\psi_1\big].$$

- By iteration, one has $u(0,0)=\mathbb{E}\big[\psi_n\big]=\mathbb{E}\big[\lim_{n\to\infty}\psi_n\big]=\mathbb{E}\big[\psi\big]$, where

$$\psi_n := \Big[\prod_{k\in\cup_{j=1}^n\mathcal{K}_T^j}\frac{g(W_T^k)}{\overline{F}(\Delta T_k)}\Big]\Big[\prod_{k\in\cup_{j=1}^n\overline{\mathcal{K}}_T^j\setminus\mathcal{K}_T^j}\frac{a_{I_k}(T_k,W_{T_k}^k)}{p_{I_k}\rho(T_k)}\Big]\Big[\prod_{k\in\overline{\mathcal{K}}_T^{n+1}}u_{T_k-}\Big],$$

$$\psi := \Big[\prod_{k\in\mathcal{K}_T}\frac{g(W_T^k)}{\overline{F}(\Delta T_k)}\Big]\Big[\prod_{k\in\overline{\mathcal{K}}_T\setminus\mathcal{K}_T}\frac{a_{I_k}(T_k,W_{T_k}^k)}{p_{I_k}\rho(T_k)}\Big].$$

A second class of seminlinear equation

- A second class of semilinear PDEs : Let $u \in C^{1,2}$ be a solution to

$$\partial_t u + \frac{1}{2}\Delta u + \sum_{\ell=(\ell_0,\ell_1)} a_\ell u^{\ell_0}(b\cdot Du)^{\ell_1} = 0, \quad u(T,\cdot) = g(\cdot).$$

- Using Feynmann-Kac (let $\mathbb{P}[I_{(1)} = \ell = (\ell_0,\ell_1)] = p_\ell$)

$$\begin{aligned} u(0,0) =& \mathbb{E}\Big[g(W_T) + \int_0^T \sum_\ell \big(a_\ell u^{\ell_0}(b\cdot Du)^{\ell_1}\big)(s,W_s)ds\Big] \\ =& \mathbb{E}\Big[\frac{g(W_T^{(1)})}{\overline{F}(T)}\mathbb{1}_{\{T_{(1)}\geq T\}} + \frac{a_{I_{(1)}}(T_{(1)},W)}{p_{I_{(1)}}\rho(T_{(1)})}u^{I_{(1)},0}(b\cdot Du)^{I_{(1)},1}_{T_{(1)}}\mathbb{1}_{\{T_{(1)}<T\}}\Big] \\ =& \mathbb{E}[\psi_1], \end{aligned}$$

where $\psi_1 := \Big[\prod_{k\in\mathcal{K}_T^1}\frac{g(W_T^k)}{\overline{F}(\Delta T_k)}\Big]\Big[\prod_{k\in\overline{\mathcal{K}}_T^1\setminus\mathcal{K}_T^1}\frac{a_{I_k}(T_k,W_{T_k}^k)}{p_{I_k}\rho(T_k)}\Big]\Big[\prod_{k\in\overline{\mathcal{K}}_T^2} u_{T_{k-}} \text{ or } (b\cdot Du)\Big]$.

Seminlinear equation and branching Brownian motion

- Modification on Branching process :
  - $\mathbb{P}[I_k = \ell = (\ell_0, \ell_1)] = p_\ell$,
  - At default time, the particle k branches into $|I_k|$ (independent) particles, among which $I_{k,0}$ particles are marked by 0, and $I_{k,1}$ particles are marked by 1.
  - Denote by θ_k the mark of k (initial particle is marked by 0).

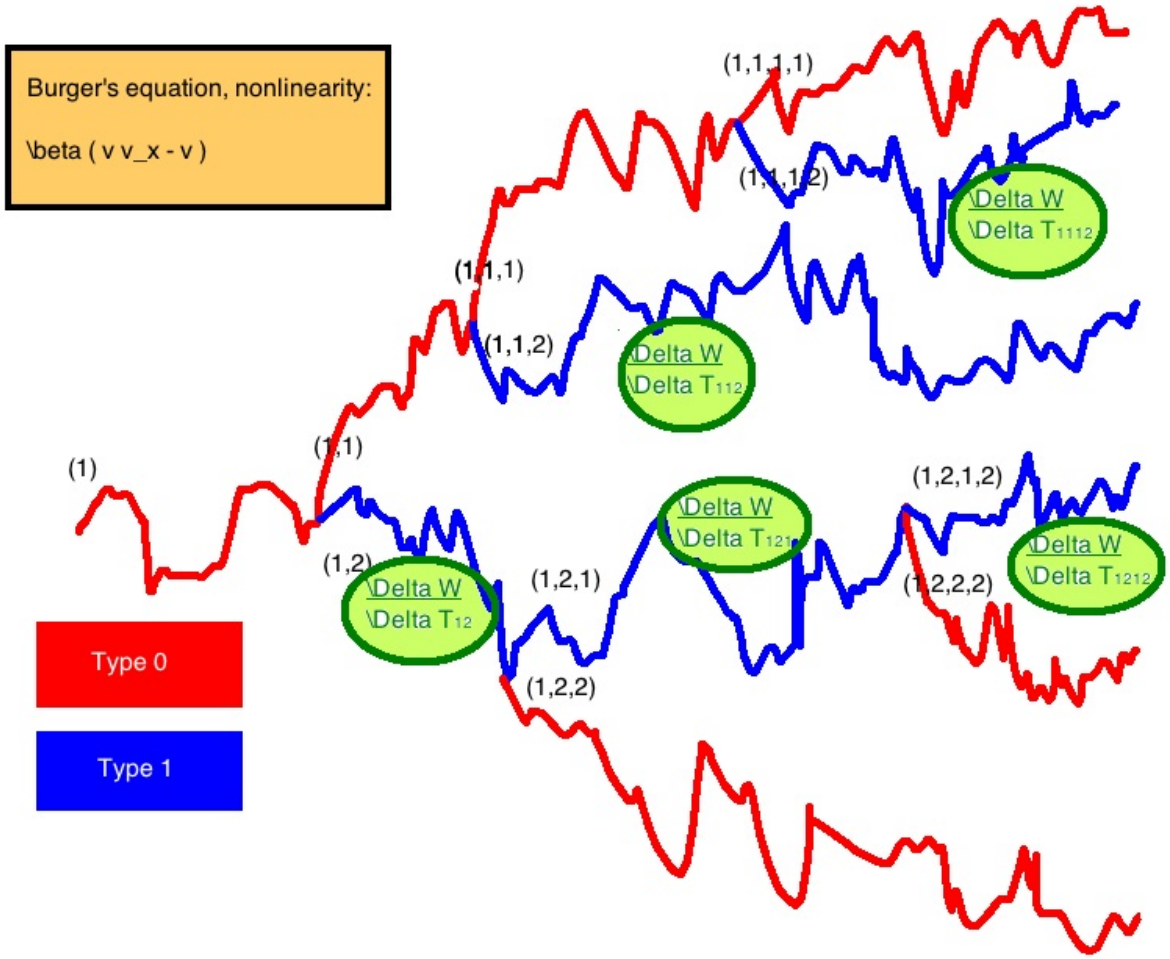
- Automatic differentiation, let $v(x) = \mathbb{E}[\phi(x + \Delta W)]$, then by integral by part formula,

$$Dv(x) \;=\; \mathbb{E}\Big[\phi(x + \Delta W)\frac{\Delta W}{\Delta T}\Big]$$

- Then

$$(b \cdot Du)(0,0) \;=\; \mathbb{E}\Big[\psi_1 \; b(0,0) \cdot \frac{\Delta W_{(1)}}{\Delta T_{(1)}}\Big].$$

A branching Brownian motion



Seminlinear equation and branching Brownian motion

- For every $k \in \overline{\mathcal{K}}_T$, introduce

$$\mathcal{W}_k := \mathbb{1}_{\{\theta_k=0\}} + b(T_{k-}, W_{T_{k-}}) \cdot \frac{\Delta W_k}{\Delta T_k} \mathbb{1}_{\{\theta_k \neq 0\}}.$$

Then $u(0,0) = \mathbb{E}[\psi_n] = \mathbb{E}[\psi]$, where

$$\psi_n := \Big[\prod_{k \in \cup_{j=1}^n \mathcal{K}_T^j} \frac{g(W_T^k)\mathcal{W}_k}{\overline{F}(\Delta T_k)} \Big] \Big[\prod_{k \in \cup_{j=1}^n \overline{\mathcal{K}}_T^j \setminus \mathcal{K}_T^j} \frac{a_{I_k}(T_k, W_{T_k}^k)\mathcal{W}_k}{p_{I_k}\rho(T_k)} \Big]$$
$$\Big[\prod_{k \in \overline{\mathcal{K}}_T^{n+1}} (u_{T_{k-}} \mathbb{1}_{\{\theta_k=0\}} + (b \cdot Du)_{T_{k-}} \mathbb{1}_{\{\theta_k=1\}} \Big],$$

and

$$\psi := \Big[\prod_{k \in \mathcal{K}_T} \frac{g(W_T^k)\mathcal{W}_k}{\overline{F}(\Delta T_k)} \Big] \Big[\prod_{k \in \overline{\mathcal{K}}_T \setminus \mathcal{K}_T} \frac{a_{I_k}(T_k, W_{T_k}^k)\mathcal{W}_k}{p_{I_k}\rho(T_k)} \Big].$$

Seminlinear equation and branching Brownian motion

- The variable ψ is defined with initial condition $(0,0)$ on $[0,T]$, by considering initial condition (t,x) on $[t,T]$, we define $\psi_n^{t,x}$, $\psi^{t,x}$.

- Recall the semilnear PDE

$$\partial_t u + \frac{1}{2}\Delta u + \sum_{\ell=(\ell_0,\ell_1)} a_\ell u^{\ell_0}(b\cdot Du)^{\ell_1} = 0, \quad u(T,\cdot) = g(\cdot).$$

Theorem

(i), *Assume that* $u \in C^{1,2}$ *is a solution to semilinear PDE, and* $(\psi_n)_{n\geq 1}$ *is u.i. Then* $u(t,x) = \mathbb{E}[\psi^{t,x}]$.

(ii), *Assume that* $(\psi^{t,x})_{(t,x)\in[0,T]\times\mathbb{R}^d}$ *and* $(\psi^{t,x}\Delta W^{t,x}_{(0)}/\Delta T^t_{(0)})_{(t,x)\in[0,T]\times\mathbb{R}^d}$ *are u.i. Then* $u(t,x) := \mathbb{E}[\psi^{t,x}]$ *is continuous, and* $Du(t,x)$ *exist and continuous. Moreover,* u *is a viscosity solution of the semilinear PDE.*

The integrability and square integrability

- Remark : One can consider fully nonlinear PDE of form

$$\partial_t u + \frac{1}{2}\Delta u + \sum a_\ell u^{\ell_0}(Du)^{\ell_1}(D^2 u)^{\ell_2} = 0.$$

But the variable ψ is not integrable.

- The linear case (unbiased simulation of SDE) :
Henry-Labordère-Tan-Touzi, Doumbia-Oudjane-Warin.

$$\partial_t u + \frac{1}{2}\Delta u + b \cdot Du = 0.$$

The integrability and square integrability

Let us introduce the constants $C_{1,q}$ and $C_{2,q}$ such that

$$|g|_\infty^q \vee \|\Delta W_k\|^q \|b_i \cdot \frac{\Delta W_k}{\Delta T_k}\|^q \leq C_{1,q}, \quad (\sqrt{\Delta T_k})^q \|b_i \cdot \frac{\Delta W_k}{\Delta T_k}\|^q \leq C_{2,q},$$

and

$$\chi^q := \Big[\prod_{k\in\mathcal{K}_T} C_{1,q}\Big(\frac{1}{\overline{F}(\Delta T_k)}\Big)^q\Big]\Big[\prod_{k\in\overline{\mathcal{K}}_T} C_{2,q}\frac{c_{I_k}}{p_{I_k}}\Big(\frac{1}{\sqrt{\Delta T_k}\rho(\Delta T_k)}\Big)^q\Big]$$

then

$$\mathbb{E}[|\psi|^q] \leq \mathbb{E}[\chi^q]$$

An explicit sufficient condition

- The ODE

$$\eta(T) = \widehat{C}_{1,q}, \quad \eta'(t) + \sum_{\ell} \widehat{C}_{2,q} |c_\ell|_\infty |\eta(t)|^{|\ell|} = 0,$$

has a finite solution on $[0, T]$ if and only if

$$\int_{\widehat{C}_{1,q}}^{\infty} \frac{dx}{\widehat{C}_{2,q} \sum |c_\ell|_\infty |x|^{|\ell|}} > T.$$

Let $\widehat{\chi}^q := \Big[\prod_{k \in \mathcal{K}_T} \widehat{C}_{1,q} \frac{1}{\overline{F}(\Delta T_k)}\Big] \Big[\prod_{k \in \overline{\mathcal{K}}_T} \widehat{C}_{2,q} \frac{c_{I_k}}{p_{I_k}} \frac{1}{\rho(\Delta T_k)}\Big]$,

then $\mathbb{E}[\widehat{\chi}^q] = \eta(0) < \infty$.

- To be compared with

$$\chi^q := \Big[\prod_{k \in \mathcal{K}_T} C_{1,q} \Big(\frac{1}{\overline{F}(\Delta T_k)}\Big)^q\Big] \Big[\prod_{k \in \overline{\mathcal{K}}_T} C_{2,q} \frac{c_{I_k}}{p_{I_k}} \Big(\frac{1}{\sqrt{\Delta T_k}\rho(\Delta T_k)}\Big)^q\Big]$$

An explicit sufficient condition

- Let us define

$$\widehat{C}_{1,q} := \frac{C_{1,q}}{\overline{F}(T)^{q-1}} \qquad \widehat{C}_{2,q} := C_{2,q} \sup_{\ell,t} \left(\frac{|c_\ell|_\infty}{p_\ell} \frac{t^{-\frac{q}{2(q-1)}}}{\rho(t)} \right)^{q-1},$$

so that $\chi^q \le \widehat{\chi}^q$.

Theorem

Assume for some $q > 1$, one has

$$\int_{\widehat{C}_{1,q}}^{\infty} \frac{dx}{\widehat{C}_{2,q} \sum |c_\ell|_\infty |x|^{|\ell|}} > T.$$

Then $(\psi^{t,x})_{(t,x)}$ and $(\psi^{t,x} \Delta W^{t,x}_{(0)} / \Delta T^t_{(0)})_{(t,x)}$ are u.i.

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The complexity of the algorithm

• Average number of particles in one simulation $m(t) := \mathbb{E}[\#\overline{\mathcal{K}}_t]$ satisfies, for $n_0 := \sum_\ell |\ell| p_\ell$,

$$m(t) = 1 + n_0 \int_0^t m(t-s)\rho(s)ds.$$

When $\rho \sim \Gamma(\kappa, \theta)$, one has

$$m(T) \;=\; \gamma(\kappa, T/\theta) \sum_{k=0}^{\infty} \frac{n_0^k}{\Gamma(k\kappa)}.$$

• Number of v.a. simulated for one particle

$$d + 1 + 1$$

• Number of computation for one particle

$$Cd$$

Numerical example

- Define

$$u(t,x) = \cos(x_1 + \cdots + x_d)\exp(\alpha(T-t))$$

is solution of semilinear PDE

$$\partial_t u + \frac{1}{2}\Delta u + cu(b_1 \cdot Du) + b_0 \;=\; 0.$$

- For numerical implementation, we choose

$$\alpha = 0.2, \quad c = 0.15, \quad b_1 = (1 + \frac{1}{d}, 1 + \frac{2}{d}, \cdots, 2).$$

A numerical example of dimension $d = 20$

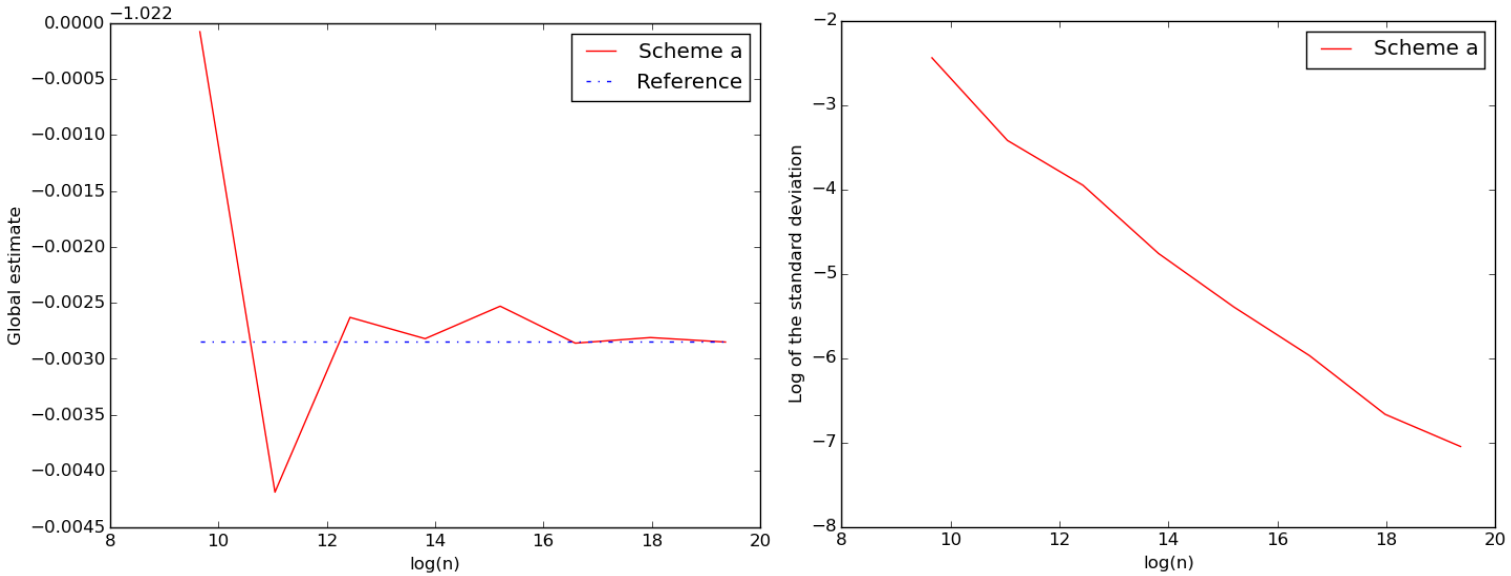

Figure : Estimation and standard deviation observed in dimension $d = 20$ depending on the log of the number of simulation used.

More Monte-Carlo techniques

- Choice of ρ, $(p_\ell)_{\ell \in L}$, the expression of nonlinearity ...
- Importance sampling, particles method ...
- Parallel calcul.
- Toward to large nonlinearity and long maturity.