Stochastic Control of Path-Dependent Systems, and Stochastic Differential Games

Nizar Touzi

Ecole Polytechnique, France

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Outline

1. Stochastic control of non-Markov diffusions
   - Smooth processes
   - The verification argument
   - Characterization by means of viscosity solutions
2. Application to stochastic differential games
   - Zero-sum stochastic differential games
   - Principal-Agent problem
   - The General Result
3. Example : optimal electricity tarification

Paths space and non-anticipative process

- $\Omega = \{\omega \in C^0([0,T],\mathbb{R}^d), \omega_0 = 0\}$, $\Lambda := [0,T] \times \Omega$
- Canonical process $X_t(\omega) := \omega(t)$, for all $t \in [0,T]$
- $\mathbb{F} = \{\mathcal{F}_t\}$ the corresponding filtration, i.e. $\mathcal{F}_t = \sigma(X_s, s \leq t)$
- $u : [0,T] \times \Omega \longrightarrow \mathbb{R}$ non-anticipative if $u(t,\omega) = u\big(t,(\omega_s)_{s\leq t}\big)$

Probability measures on the paths space

- $\mathbb{P}_0$: Wiener measure on Ω, so that X is a $\mathbb{P}_0-$Brownian motion
- $\mathbb{P} = \mathbb{P}^{\alpha,\beta}$ probability measure on Ω such that

$$X_t = \int_0^s \alpha_s^{\mathbb{P}} ds + \int_0^s \beta_s^{\mathbb{P}} dW_t^{\mathbb{P}}, \quad \mathbb{P} - \text{a.s.}$$

for some adapted processes $\alpha^{\mathbb{P}}$, $\beta^{\mathbb{P}}$

Smooth processes

- $\phi : [0,T] \times \Omega \longrightarrow \mathbb{R} \in C^{1,2}(\Lambda)$ if
  - in $C^0(\Lambda)$ (in particular, non-anticipative)
  - $\exists$ processes $\theta, Z, \Gamma \in C^0(\Lambda)$ valued in $\mathbb{R}$, $\mathbb{R}^d$, $\mathcal{S}_d(\mathbb{R})$, s.t.

$$d\phi_t = \theta_t dt + Z_t \cdot dX_t + \frac{1}{2}\Gamma_t : d\langle X \rangle_t, \quad \mathbb{P}-\text{a.s. for all } \mathbb{P} = \mathbb{P}^{\alpha,\beta}$$

where $C^0(\Lambda)$ is the set of $d-$continuous processes :

$$d\big((t,\omega),(t',\omega')\big) := |t-t'| + \big\|\omega_{\wedge t} - \omega'_{\wedge t'}\big\|_\infty$$

Denote $\partial_t \phi_t := \theta_t$, $\partial_\omega \phi_t := Z_t$, $\partial^2_{\omega\omega}\phi_t := \Gamma_t$

Stochastic control of path-dependent diffusions

- $\mathcal{U}$: Control proc. $\nu = \{\nu_t\}_{t\geq 0}$ $\mathbb{F}-$prog meas valued in $U \subset \mathbb{R}^k$

- Controlled state process : any weak solution $\mathbb{P}$ of the SDE valued in $\mathbb{R}^d$ (bounded coefficients b, σ) :

$$dX_t \quad = \quad b(t, X_\cdot, \nu_t^{\mathbb{P}})dt + \sigma(t, X_\cdot, \nu_t^{\mathbb{P}})dW^{\mathbb{P}}{}_t, \quad \mathbb{P}-\text{a.s.}$$

$\mathcal{P} := \{\mathbb{P}$ solution of SDE for some $\nu \in \mathcal{U}\}$: admissible controls

- Control problem :

$$V(t,\omega) \quad := \quad \sup_{\nu\in\mathcal{U}} \mathbb{E}\left[\xi^{t,\omega}(X_\cdot)\right]$$

where $\xi(x) = \xi\left(x_{\wedge T}\right)$, $\mathcal{F}_T-$measurable, and

$$\xi^{t,\omega} := \xi\left(\omega \otimes_t x\right), \quad (\omega \otimes_t x)_s := \mathbf{1}_{s\leq t}\omega_s + \mathbf{1}_{s>t}(\omega_t + x_{s-t})$$

Path-dependent Hamiltonian

We may also consider the (appearently) more general formulation

$$V_0^A(t,\omega) := \sup_{\mathbb{P}} \mathbb{E}^{\mathbb{P}}\Big[\underbrace{e^{-\int_0^T k_s^{t,\omega}(\nu_s^{\mathbb{P}})ds}}_{\text{discount factor}} \xi^{t,\omega} - \int_0^T e^{-\int_0^t k_s^{t,\omega}(\nu_s^{\mathbb{P}})} \underbrace{c_t^{t,\omega}(\nu_t^{\mathbb{P}})}_{\text{cost of effort}} dt \Big]$$

We restrict to $k = c = 0$ for simplicity of this presentation

Path-dependent Hamiltonian

Hamiltonian :

$$H(t,\omega,z,\gamma) \quad := \quad \sup_{u\in U}\left\{ b_t(\omega,u)\cdot z + \frac{1}{2}\sigma_t\sigma_t^\top(\omega,u) : \gamma \right\}$$

for all $(t,\omega)\in[0,T]\times\Omega$ and $(z,\gamma)\in\mathbb{R}^d\times\mathcal{S}_{\mathbb{R}}(d)$

Consider the Path-dependent HJB equation

$$\text{(HJB)} \qquad \begin{array}{l} \partial_t v + H_t\big(\omega,\partial_\omega v,\partial^2_{\omega\omega} v\big) = 0, \quad t<T,\ \omega\in\Omega \\ v_T = \xi \ \text{ on } \Omega \end{array}$$

Verification argument under $C^{1,2}(\Lambda)$ regularity

Claim : $v \in C^{1,2}(\Lambda)$ solution of HJB, then $v_0 = V_0$.
Indeed, $\forall\, \mathbb{P} \in \mathcal{P}$:

$$
\begin{aligned}
v_0 &= \mathbb{E}^{\mathbb{P}}\big[v_T\big] + \mathbb{E}^{\mathbb{P}}\big[v_0 - v_T\big] \\
&= \mathbb{E}^{\mathbb{P}}\big[\xi\big] - \mathbb{E}^{\mathbb{P}}\Big[\int_0^T dv_t\Big] \\
&= \mathbb{E}^{\mathbb{P}}\big[\xi\big] - \mathbb{E}^{\mathbb{P}}\Big[\int_0^T \partial_t v_t dt + \partial_\omega v_t \cdot dX_t + \frac{1}{2}\partial^2_{\omega\omega} v_t : d\langle X\rangle_t\Big] \\
&= \mathbb{E}^{\mathbb{P}}\big[\xi\big] - \mathbb{E}^{\mathbb{P}}\Big[\int_0^T \underbrace{\Big(-H_t(\partial_\omega v_t, \partial^2_{\omega\omega} v_t) + \partial_\omega v_t \cdot b_t^{\nu^{\mathbb{P}}} + \frac{1}{2}\partial^2_{\omega\omega} v_t : \sigma_t^{\nu^{\mathbb{P}}} \sigma_t^{\nu^{\mathbb{P}}\top}\Big)}_{\leq\, 0} dt\Big]
\end{aligned}
$$

Hence $v_0 \geq V$, and equality satisfied by $\hat{\nu}$ maximier of H

Verification argument under Sobolev regularity I

Continuity is not needed in previous argument...

integrability under each $\mathbb{P} \in \mathcal{P}$ is sufficient

as long as all integrals are well-defined

$$dv_t = -H_t(\partial_\omega v_t, \partial^2_{\omega\omega} v_t)dt + \partial_\omega v_t \cdot dX_t + \frac{1}{2}\partial^2_{\omega\omega} v_t : d\langle X\rangle_t, \ \mathcal{P} - \text{q.s.}$$

where $\mathcal{P} - \text{q.s.} \equiv \mathbb{P} - \text{a.s. for all } \mathbb{P} \in \mathcal{P}$

This decomposition, with $v_T = \xi$, implies the representation

$$\xi = v_0 + \int_0^T -H_t(\partial_\omega v_t, \partial^2_{\omega\omega} v_t)dt + \partial_\omega v_t \cdot dX_t + \frac{1}{2}\partial^2_{\omega\omega} v_t : d\langle X\rangle_t, \ \mathcal{P} - \text{q.s.}$$

Verification argument under Sobolev regularity II

In fact, we need less... Introduce

$$H_t^*(\omega, z, a) := \sup_\gamma \left\{ \frac{1}{2}\gamma : a - H_t(\omega, z, \gamma) \right\}$$

Then, denoting $\hat{a}_t := \frac{d\langle X\rangle_t}{dt}$, we only need

$$v_T = \xi \quad \text{and} \quad dv_t = H_t^*(\partial_\omega v_t, \hat{a}_t)dt + \partial_\omega v_t \cdot dX_t - dK_t, \ \mathcal{P} - \text{q.s.}$$

for some process

$$K \quad \text{non-decreasing with} \quad K_0 = 0 \quad \text{and} \quad \inf_{\mathbb{P}\in\mathcal{P}} \mathbb{E}^{\mathbb{P}}[K_T] = 0$$

This representation relaxes the condition $dK_t \ll dt$, $\mathcal{P}-$q.s.

The case of uncontrolled diffusion

Consider the case

$$(t,\omega) \longmapsto \sigma_t(\omega) \text{ independent of the control variable}$$

Then

- By Girsanov theorem, all measures in $\mathcal{P}$ are equivalent
- $H_t^*(\omega, z, \hat{a}) = -H_t(\omega, z, 0) =: F_t(\omega, z)$, and $K = 0$, $\mathcal{P}-$q.s.
  Then, representation reduces to : for some $\mathbb{P} \in \mathcal{P}$

$$v_T = \xi \quad \text{and} \quad dv_t = -F_t(\partial_\omega v_t)dt + \partial_\omega v_t \cdot dX_t, \quad \mathbb{P}-\text{a.s.}$$

or, in integral form

$$\xi \;=\; v_0 - \int_0^T F_t(\partial_\omega v_t)dt + \int_0^T \partial_\omega v_t \cdot dX_t, \quad \mathbb{P}-\text{a.s.}$$

Representation results with dominating measure

Notation : $\|Z\|^2_{\mathbb{H}^2(\mathbb{P})} := \mathbb{E}^{\mathbb{P}}\big[\int_0^T Z_t \cdot d\langle X\rangle_t Z_t\big]$

• Predictable Representation Property : for $\xi \in \mathbb{L}^2(\mathbb{P}_0)$, there is unique $Z \in \mathbb{H}^2(\mathbb{P}_0)$ s.t.

$$\xi \;=\; \mathbb{E}^{\mathbb{P}_0}[\xi] + \int_0^T Z_t \cdot dX_t, \quad \mathbb{P}_0 - \text{a.s.}$$

• Backward SDE (Pardoux & Peng '91) : assume F Lip uniformly in (t,ω). Then, for $\xi \in \mathbb{L}^2(\mathbb{P}_0)$, there is unique $Z \in \mathbb{H}^2(\mathbb{P}_0)$ s.t.

$$\xi \;=\; Y_0 - \int_0^T F_t(Z_t)dt + \int_0^T Z_t \cdot dX_t, \quad \mathbb{P}_0 - \text{a.s.}$$

Representation result without dominating measure

Second order Backward SDE
(Soner, NT & Zhang '12, Possamaï, Tan & Zhou '15)

- $b = \sigma\mu$ for some bounded $\mu : [0, T] \times \Omega \times U \longrightarrow \mathbb{R}^d$
- H Lip in (z, γ) uniformly in (t, ω), + integrability
- $\mathcal{E}\{\sup_{t \le T} \mathcal{E}[\xi|\mathcal{F}_t]\}$, with $\mathcal{E} := \sup_{\mathbb{P}} \mathbb{E}^{\mathbb{P}}$

Then, there exists unique (Z, K) s.t.

$$v_T = \xi \quad \text{and} \quad dv_t = H_t^*(Z_t, \hat{a}_t)dt + Z_t \cdot dX_t - dK_t, \ \mathcal{P} - \text{q.s.}$$

- $\sup_{\mathbb{P}} \|Z\|_{\mathbb{H}^2(\mathbb{P})} < \infty$
- K non-decreasing, $K_0 = 0$ and $\inf_{\mathbb{P} \in \mathcal{P}} \mathbb{E}^{\mathbb{P}}[K_T] = 0$

Standard viscosity solutions [M. Crandall & P.-L. Lions '83]

$g(x, y, z, \gamma)$ nondecreasing in γ. Consider the PDE :

$$(\mathrm{E}) \qquad -g(., v, Dv, D^2v)(x) = 0, \quad x \in \mathcal{O} \quad \text{(open subset of } \mathbb{R}^d)$$

- v subsolution if $-g(., v, Dv, D^2v) \leq 0$ on $\mathcal{O}$
- v supersolution if $-g(., v, Dv, D^2v) \geq 0$ on $\mathcal{O}$

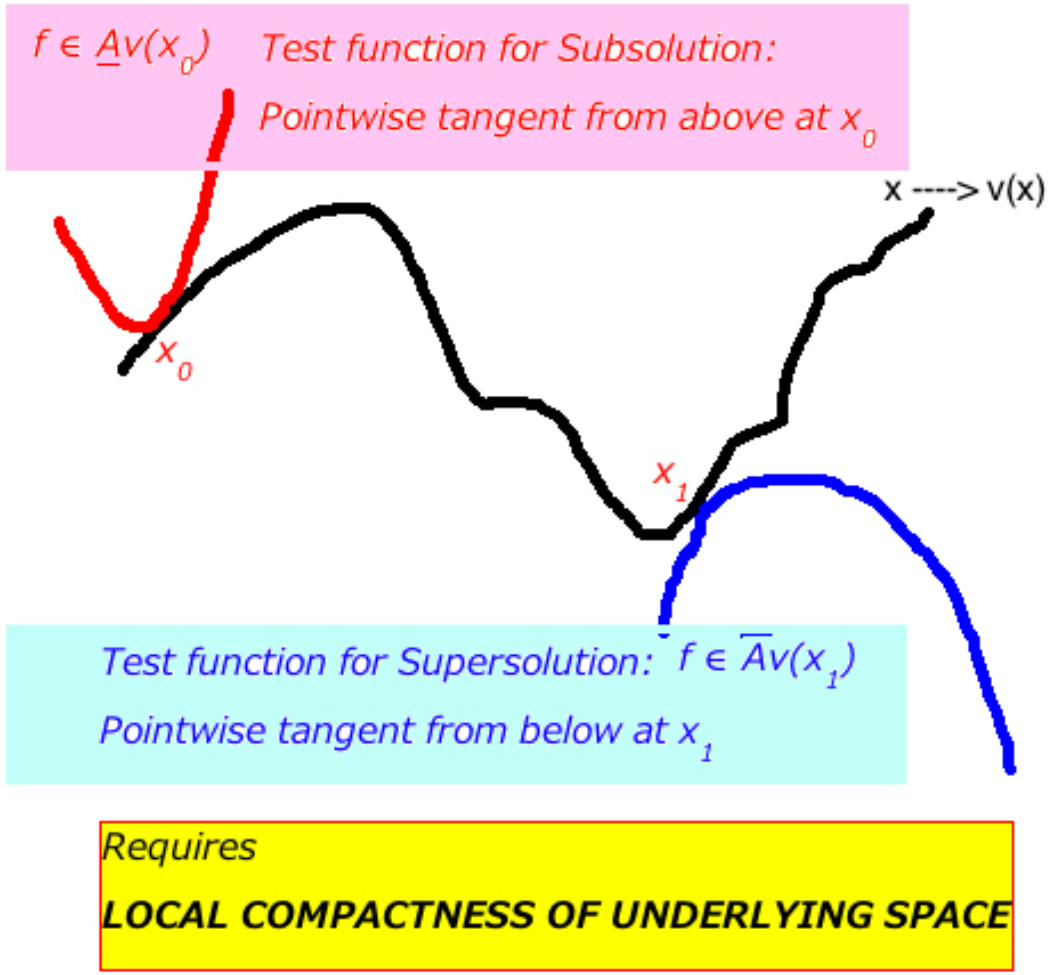

Figure – M.Crandall & PL.Lions test functions

Standard definition of viscosity solutions

Let

$$\underline{A}v(x) := \{\varphi \in C^2(\mathcal{O}) : (\varphi - v)(x) = \min_{\mathcal{O}}(\varphi - v)\}$$
$$\overline{A}v(x) := \{\varphi \in C^2(\mathcal{O}) : (\varphi - v)(x) = \max_{\mathcal{O}}(\varphi - v)\}$$

Definition $v \in \mathrm{LSC}(\mathcal{O})$ (resp. $\mathrm{USC}(\mathcal{O})$) is a viscosity subsolution (resp. supersolution) of (E) if :

$$-g\big(x, v(x), D\varphi(x), D^2\varphi(x)\big) \leq 0 \text{ (resp. } \geq 0)$$
$$\text{for all } x \in \mathcal{O} \text{ and } \varphi \in \underline{A}v(x) \text{ (resp. } \overline{A}v(x))$$

$v \in C^0(\mathcal{O})$ viscosity solution if viscosity subsol. and supersol.

Main properties of Crandall-Lions viscosity solutions

- Stability result : "five lines" proof

- Existence for value function of control problems (representation) : usually very easy

- Uniqueness : implied by comparison result. This is the difficult part of the theory in the second order case... (Crandall-Ishii Lemma)

- Existence by Perron method, given comparison

- Regularity : Caffarelli & Cabre

Intuition from the heat equation

$v(t,x) := \mathbb{E}^{\mathbb{P}_0}\left[g(B_T)\middle|B_t = x\right]$ solution of $-\partial_t v - \frac{1}{2}v_{xx} = 0$.

- Tower property :

$$\begin{aligned} v(t,x) &= \mathbb{E}^{\mathbb{P}_0}\left[\mathbb{E}^{\mathbb{P}_0}[g(B_T)|B_{t+h}]\,\middle|B_t = x\right] \\ &= \mathbb{E}^{\mathbb{P}_0}\left[v(t+h,B_{t+h})\middle|B_t = x\right] \end{aligned}$$

- Viscosity subsol : for $\varphi \in \underline{A}v(t,x)$, $v(t,x) = \varphi(t,x)$ and $v \le \varphi$

$$\Longrightarrow \quad \varphi(t,x) \le \mathbb{E}^{\mathbb{P}_0}\left[\varphi(t+h,B_{t+h})\middle|B_t = x\right]$$

$$==\Rightarrow \quad \left(-\partial_t\varphi - \frac{1}{2}\varphi_{xx}\right)(t,x) \le 0$$

Main observation : only need $\mathbb{E}^{\mathbb{P}_0}[v(.,.)] \le \mathbb{E}^{\mathbb{P}_0}[\varphi(.,.)]$

Smooth test processes

TEST FUNCTIONS TANGENT IN MEAN

Nonlinear expectation $\overline{\mathcal{E}}_L := \sup_{\mathbb{P}\in\mathcal{P}_L} \mathbb{E}^{\mathbb{P}}$ and $\underline{\mathcal{E}}_L := \inf_{\mathbb{P}\in\mathcal{P}_L} \mathbb{E}^{\mathbb{P}}$
$\mathcal{T}$: collection of stopping times, $\mathsf{H}_\varepsilon := \varepsilon \wedge \inf\{t : |B_t| \geq \varepsilon\}$

Test processes for subsolution and supersolution

$$\underline{\mathcal{A}}^L u_t(\omega) := \left\{ \varphi \in C^{1,2}(\Lambda) : \ \exists\, \varepsilon > 0,\ 0 = \min_{\tau\in\mathcal{T}} \underline{\mathcal{E}}_L \left[(\varphi - u^{t,\omega})_{\cdot\wedge\mathsf{H}_\varepsilon} \right] \right\}$$

$$\overline{\mathcal{A}}^L u_t(\omega) := \left\{ \varphi \in C^{1,2}(\Lambda) : \ \exists\, \varepsilon > 0,\ 0 = \max_{\tau\in\mathcal{T}} \overline{\mathcal{E}}_L \left[(\varphi - u^{t,\omega})_{\cdot\wedge\mathsf{H}_\varepsilon} \right] \right\}$$

More test functions $\Longrightarrow$ better chances for uniqueness
$\Longrightarrow$ more demanding existence

Definition (Ekren, NT & Zhang 2012)

$u \in C^0(\Lambda)$ is a **viscosity...**

- **subsolution** of PPDE if there exists L :

$$-\partial_t \varphi_0 - G\big(t, \omega, u_t(\omega), \partial_\omega \varphi_0, \partial^2_{\omega\omega} \varphi_0\big) \leq 0$$
$$\text{for all } (t, \omega) \in [0, T) \times \Omega \text{ and } \varphi \in \underline{\mathcal{A}}^L u(t, \omega)$$

- **supersolution** of PPDE if there exists L :

$$-\partial_t \varphi_0 - G\big(t, \omega, u_t(\omega), \partial_\omega \varphi_0, \partial^2_{\omega\omega} \varphi_0\big) \geq 0$$
$$\text{for all } (t, \omega) \in [0, T) \times \Omega \text{ and } \varphi \in \overline{\mathcal{A}}^L u(t, \omega)$$

- **solution** of PPDE if it is viscosity subsolution and supersolution

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Zero-sum stochastic differential games

- Agents A chooses control process $\alpha \in \mathcal{A}$
- Agents B chooses control process $\beta \in \mathcal{B}$
- Controlled state process $\mathbb{P}^{\alpha,\beta}$ weak solution of SDE

$$dX_t = b_t(X, \alpha_t, \beta_t)dt + \sigma_t(\alpha_t, \beta_t)dW^{\mathbb{P}^{\alpha,\beta}}, \quad \mathbb{P}^{\alpha,\beta} - \text{a.s.}$$

- Zero-sum stochastic differential game :

$$\overline{V} := \inf_{\alpha\in\mathcal{A}} \sup_{\beta\in\mathcal{B}} J(\alpha,\beta) \quad \text{and} \quad \underline{V} := \sup_{\beta\in\mathcal{B}} \inf_{\alpha\in\mathcal{A}} J(\alpha,\beta)$$

where

$$J(\alpha,\beta) := \mathbb{E}^{\mathbb{P}^{\alpha,\beta}}\left[\xi(X_{\wedge T})\right]$$

Input from path-dependent stochastic control

• Dynamic programming : very delicate, and requires to introduce the notion of strategy controls

$$a : (\omega, \beta) \in \Omega \times \mathcal{B} \longmapsto a(\omega, \alpha) \in \mathcal{A}$$
$$b : (\omega, \alpha) \in \Omega \times \mathcal{A} \longmapsto b(\omega, \alpha) \in \mathcal{B}$$

[Fleming & Souganidis '89]... not justified from the game theoretic viewpoint !

• Cardiallaguet & Rainer '12 restore the situation in Markovian case by restricting to delayed controls and using strong solutions

• Our approach : fix the control of one of the agent $\Longrightarrow$ path-dependent control problem, and use previous tools...

Work in progress with Possamaï, Soner & Zhang

Moral hazard

• Adam Smith (1723-1790) : moral hazard is a major risk in economics :

Situation where an agent may benefit from an action whose cost is supported by others

Should not count on agents' morality...

(Static) Principal-Agent Problem

- Principal does not observe Agent's effort, then contract

$$\xi = \xi(X^a)$$

- Agent determines optimal effort by

$$V_A(\xi) := \max_a \mathbb{E}\, U_A\big(\xi(X^a) - c(a)\big) \quad \Longrightarrow \quad \hat{a}(\xi)$$

- Principal chooses optimal contract by solving

$$\max_\xi \mathbb{E}\, U_P\big(X^{\hat{a}(\xi)} - \xi(X^{\hat{a}(\xi)})\big) \quad \text{under constraint} \quad V_A(\xi) \geq R$$

Non-zero sum Stackelberg game

Difficult to solve, and so restrict to affine contracts...

From static to dynamic Principal-Agent problem

This problem is more accessible in continuous time

Started by Holmström & Milgrom '85...

Nobel Prize 2016 winners : Oliver Hart and Bengt Holmström

Principal-Agent problem : general formulation

- Agent solves the control problem :

$$V_0^A(\xi) := \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}^{\mathbb{P}}\Big[K_T\xi - \int_0^T K_t c_t(\nu_t^{\mathbb{P}})dt\Big]$$

where $K_t = e^{-\int_0^t k_s^\nu ds}$ and $\mathbb{P} \in \mathcal{P}$ weak solution of Output proc :

$$dX_t \;=\; \sigma_t(X,\beta_t)\big[\lambda_t(X,\alpha_t)dt + dW_t^{\mathbb{P}}\big], \quad \mathbb{P}-\text{a.s.}$$

- Given solution $\mathbb{P}^*(\xi)$, Principal solves the optimization problem

$$V_0^P \;:=\; \sup_{\xi\in\Xi_R} \mathbb{E}^{\mathbb{P}^*(\xi)}\Big[K_T U(\ell(X) - \xi(X))\Big]$$

where Ξ_R : $\xi(X_.)$, such that $V_0^A(\xi) \geq R$

Path-dependent Hamiltonian

- Path-dependent Hamiltonian for the Agent problem :

$$H_t(\omega, y, z, \gamma) := \sup_{a,b} \left\{ \sigma_t(\omega, a)\lambda_t(\omega, b) \cdot z + \frac{1}{2}\sigma_t\sigma_t^\top(\omega, a) : \gamma - k_t(\omega, a, b)y - c_t(\omega, a, b) \right\}$$

- For $Y_0 \in \mathbb{R}$ and Z, Γ $\mathbb{F}^X -$ prog meas, define

$$dX_t = \nabla_z H_t(X, Y_t^{Z,\Gamma}, Z_t, \Gamma_t)dt + \left\{ 2\nabla_\gamma H_t(X, Y_t^{Z,\Gamma}, Z_t, \Gamma_t) \right\}^{\frac{1}{2}} dW_t$$

$$dY_t^{Z,\Gamma} = Z_t \cdot dX_t + \frac{1}{2}\Gamma_t : d\langle X \rangle_t - H_t(X, Y_t^{Z,\Gamma}, Z_t, \Gamma_t)dt$$

$$V_0(X_0, Y_0) := \sup_{Z,\Gamma} \mathbb{E}\left[U(\ell(X) - Y_T^{Z,\Gamma}) \right]$$

Main result

Theorem (Cvitanić, Possamaï & NT '15) Under Assumptions ...,

$$V_0^P = \sup_{Y_0 \geq R} V_0(X_0, Y_0)$$

Given maximizer $Y_0^\star$, the corresponding optimal controls $(Z^\star, \Gamma^\star)$ induce an optimal contract

$$\xi^\star = Y_0^\star + \int_0^T Z_t^\star \cdot dX_t + \frac{1}{2}\Gamma_t^\star : d\langle X \rangle_t - H_t(X, Y_t^{Z^\star,\Gamma^\star}, Z_t^\star, \Gamma_t^\star)dt$$

On the function V

$\bar{X} := (X, Y)$ satisfies $d\bar{X}_t = \bar{\mu}(\bar{X}_t, Z_t, \Gamma_t)dt + \bar{\sigma}(\bar{X}_t, Z_t, \Gamma_t)dW_t$:

$$dX_t = \nabla_z H_t(X, Y_t^{Z,\Gamma}, Z_t, \Gamma_t)dt + \left\{2\nabla_\gamma H_t(X, Y_t^{Z,\Gamma}, Z_t, \Gamma_t)\right\}^{\frac{1}{2}} dW_t$$
$$dY_t^{Z,\Gamma} = Z_t \cdot dX_t + \frac{1}{2}\Gamma_t : d\langle X\rangle_t - H_t(X, Y_t^{Z,\Gamma}, Z_t, \Gamma_t)dt$$

and recall $V_0(\bar{X}_0) := \sup_{Z,\Gamma} \mathbb{E}\left[U(\ell(X_T) - Y_T^{Z,\Gamma})\right]$

$V_0(\bar{X}_0) = V(0, \bar{X}_0)$; V solution of Hamilton-Jacobi-Bellman eq.

$$\partial_t V_0 + \sup_{z,\gamma}\left\{\bar{\mu}(., z, \gamma)DV_0 + \frac{1}{2}\overline{\sigma\sigma}^\top(., z, \gamma) : D^2 V_0\right]\Big\} = 0$$
$$V_0(T, x, y) = U(\ell(x) - y)$$

Sketch of the proof

Recall

$$Y_T^{Z,\Gamma} = Y_0 + \int_0^t Z_t \cdot dX_t + \frac{1}{2}\Gamma_t : d\langle X\rangle_t - H_t(X, Y_t^{Z,\Gamma}, Z_t, \Gamma_t)dt$$

Proposition $V_0^A(Y_T^{Z,\Gamma}) = Y_0$ and any maximizer of the Hamiltonian $(a^\star, b^\star)(Y, Z, \Gamma)$ induces a solution $\hat{\mathbb{P}}^{Z,\Gamma}$ of the Agent problem

$$dX_t = \nabla_z H_t(X, Y_t, Z_t, \Gamma_t)dt + \left\{2\nabla_\gamma H_t(X, Y_t, Z_t, \Gamma_t)\right\}^{\frac{1}{2}} dW_t, \quad \hat{\mathbb{P}}^{Z,\Gamma}-\text{a.s.}$$

Then a sufficient condition for our main result is

$$\text{for all } \xi \in ?? \quad \exists\,(Y_0, Z, \Gamma) \quad \text{s.t.} \quad \xi = Y_T^{Z,\Gamma},\ \mathcal{P}-q.s.$$

$$\text{OR, weaker condition} \quad \exists\,(Y_0^n, Z^n, \Gamma^n) \quad \text{s.t.} \quad \text{“}\xi \longleftarrow Y_T^{Z^n,\Gamma^n}\text{”}$$

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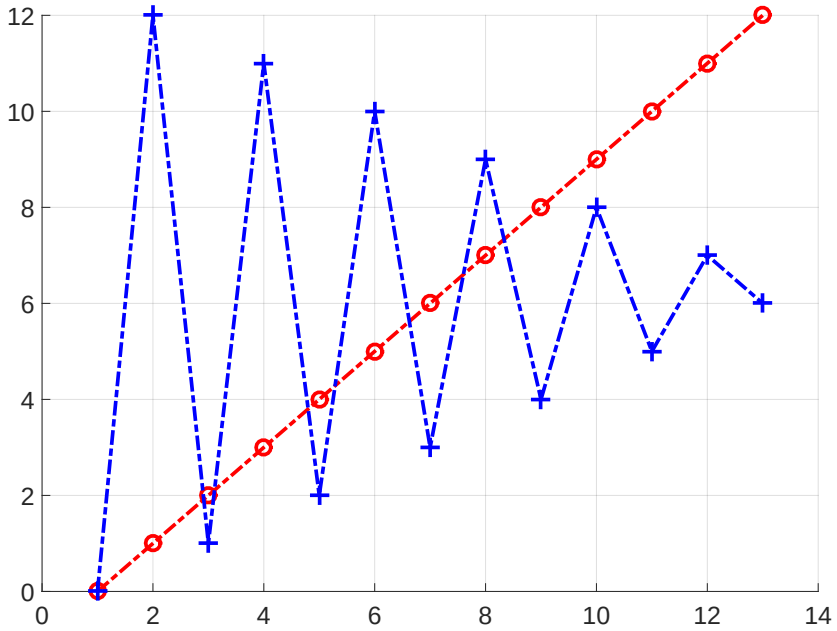
Moral hazard : electricity tarification

Electricity generation by solar / windfall technologies is volatile

$\Longrightarrow$ Although consumers demand is quite predictable, adjustment of electricity demand by classical generation means inherits important variability

$\Longrightarrow$ Path-dependent tarification

Volatility of consumption



Total consumption of X = Total consumption of X

$\langle X \rangle = 1^2 + \cdots + 1^2 = 12$ $\langle X \rangle = 12^2 + 11^2 + \cdots + 1^2 = 650$

Electricity tarification model (Aïd, Possamaï & NT)

Electric power demand in excess to a predictable reference pattern :

$$dX^{\nu}{}_t = -\alpha_t dt + \sigma\beta_t dW_t, \quad \alpha_t \geq 0, \quad \beta_t \in (0,1]$$

Consumer problem

$$\sup_{\nu=(\alpha,\beta)} \mathbb{E}\Big[-\xi + \int_0^T \big(u(X^{\nu}{}_t) - c(\nu_t)\big)dt\Big] \quad \Longrightarrow \quad \hat{\nu}(\xi)$$

$u(x) := -e^{-\eta x}$ and $c(a,b) := c_0 a^2 + c_1 b^{-2}$

Producer problem :

$$\sup_{\xi(.)} \mathbb{E}\Big[\xi(X^{\hat{\nu}(\xi)}) + \int_0^T \pi(X^{\hat{\nu}(\xi)}{}_t)dt - q\langle X^{\hat{\nu}(\xi)}\rangle_t\Big]$$

Optimal tarification

Our main result characterizes the optimal tarification as

$$\hat{\xi} = \int_0^T \hat{Z}_t \cdot dX_t + \frac{1}{2}\hat{\Gamma}_t : d\langle X\rangle_t - H(\hat{Z}_t, \hat{\Gamma}_t)dt$$

where $\langle X\rangle$ is the quadratic variation of the output X

$$H(z,\gamma) \quad := \quad \sup_{\alpha,\beta}\Big\{ -\alpha z + \frac{1}{2}\sigma^2\beta^2\gamma - c_0|\alpha|^2 - c_1|\beta|^{-2}\Big\}$$

and $\hat{Z}$, $\hat{\Gamma}$ are determined by means of a HJB equation...

Empirical results I

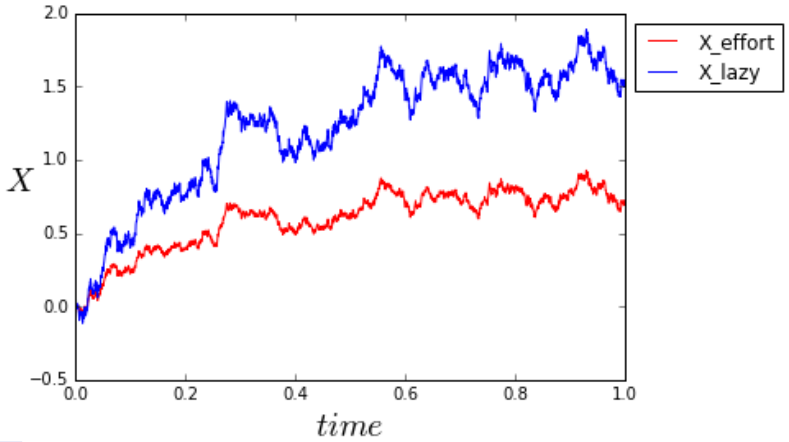


Figure – A sample path of daily consumption : incited, non incited

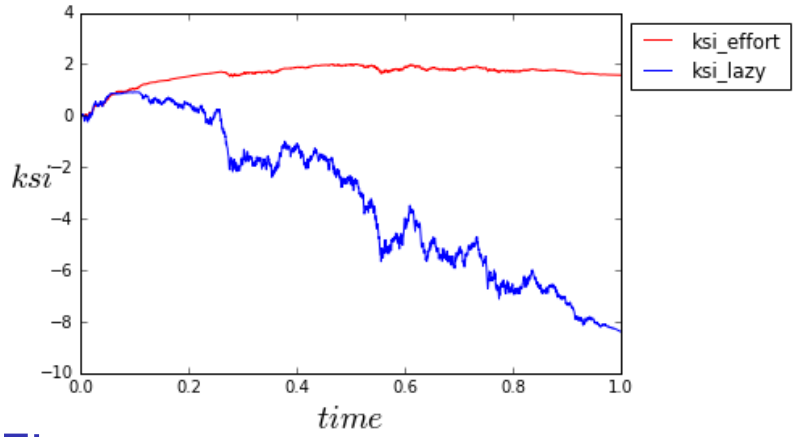


Figure – A sample path of incentive premium : incited, non incited

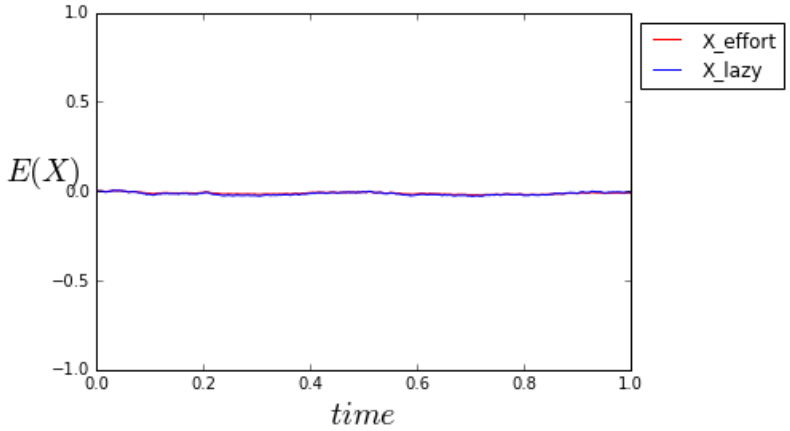


Figure – Average daily consumption : incited, non incited

Empirical results II

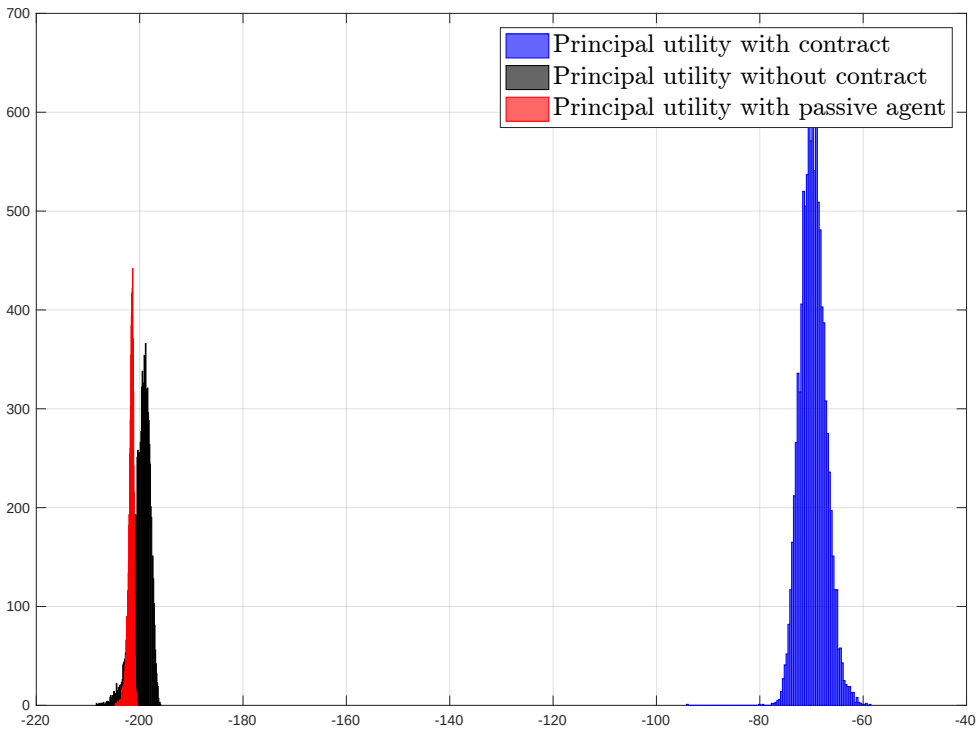


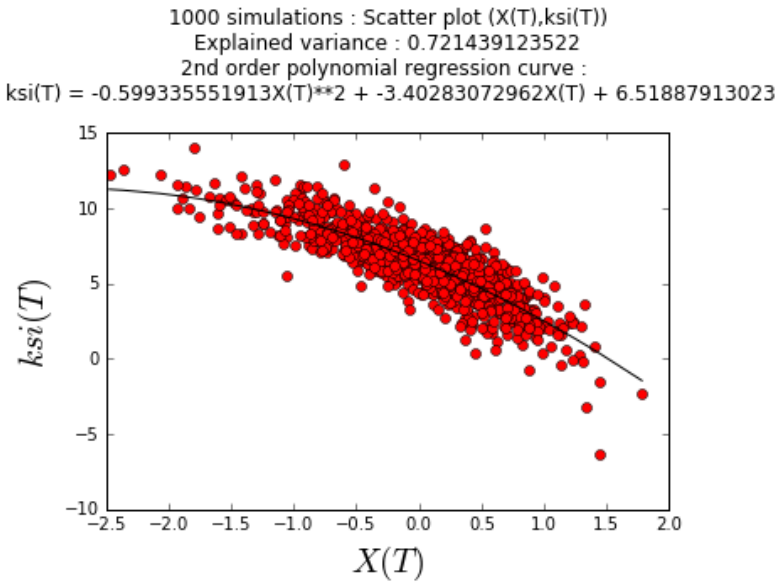
Figure – Principal's final wealth

Approximate contract

Recall

$$\hat{\xi} = \int_0^T \hat{Z}_t \cdot dX_t + \frac{1}{2}\hat{\Gamma}_t : d\langle X\rangle_t - H(\hat{Z}_t, \hat{\Gamma}_t)dt$$

$\Longrightarrow$ Simulate N paths of X, and compute the corresponding $\hat{\xi}$'s

$\Longrightarrow$ Regression of $\hat{\xi}$

- on $X_T \Longrightarrow R^2 \sim 47\%$
- on $(X_T, X_T^2) \Longrightarrow R^2 \sim 60\%$
- on $(X_T, X_T^2, X_{T/2}, X_{T/2}^2)$
  $\Longrightarrow R^2 \sim 90\%$
- on $(X_{2^{-i}T}, X_{2^{-i}T}^2)_{0\le i\le 4}$
  $\Longrightarrow R^2 \sim 99\%$