Spin Glasses

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June 30, 2010

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1 Preliminary topics

1.1 Gaussian random variables

Let $N \in \mathbb{N}$, and γ_N be the standard normal distribution on $\mathbb{R}^N$. γ_N has the density $(2\pi)^{-N/2} \exp\left[-|x|^2/2\right]$ with respect to Lebesgue measure, where $|x|$ denotes the

Euclidean norm. It is evident that γ_N is invariant under rotations, i.e. of $\phi : \mathbb{R}^N \to \mathbb{R}^N$ is an orthogonal map, then $\gamma_N \phi^{-1} = \gamma_N$. If $f : \mathbb{R}^N \to \mathbb{R}$ is Lipshitz continuous then it is integrable with respect to γ_N. We write $\gamma_N(f)$ for its expectation.

Theorem 1.1
Let f be Lipshitz continuous with

$$\|f\|_{\mathrm{Lip}} := \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|}.$$

Then for any $t > 0$

$$\gamma_N\left(\left\{x \in \mathbb{R}^N : |f(x) - \gamma_N(f)| > t\right\}\right) \leq 2\exp\left[-2t^2/\pi^2 \|f\|_{\mathrm{Lip}}^2\right].$$

(The constant in the exponent on the right hand side is too small: one can get an estimate with $2\exp\left[-t^2/2\|f\|_{\mathrm{Lip}}^2\right]$, but the proof is a bit more involved, and we will be fully satisfied with the above estimate).

Proof. It should be known that a Lipshitz continuous function is differentiable almost everywhere, with gradient ∇f satisfying

$$|\nabla f(x)| \leq \|f\|_{\mathrm{Lip}}.$$

In the proof of the theorem, we may assume that $\gamma_N(f) = 0$, as we can subtract this constant from f without influencing $\|\cdot\|_{\mathrm{Lip}}$. By Chebyshev, we get for any $t, \lambda > 0$

$$\gamma_N(f > t) \leq \mathrm{e}^{-\lambda t} \int \mathrm{e}^{\lambda f} d\gamma_N.$$

By Jensen's inequality, we have

$$\int \mathrm{e}^{-\lambda f} d\gamma_N \geq \mathrm{e}^{-\lambda \int f d\gamma_N} = 1,$$

and therefore, we have

$$\int \mathrm{e}^{\lambda f} d\gamma_N \leq \int \int \mathrm{e}^{\lambda(f(x) - f(y))} \gamma_N(dx)\, \gamma_N(dy).$$

For $x, y \in \mathbb{R}^N$, we interpolate between x and y by setting

$$x(\theta) := x \sin\theta + y \cos\theta.$$

Then

$$\begin{aligned} f(x) - f(y) &= f(x(\pi/2)) - f(x(0)) \\ &= \int_0^{\pi/2} \frac{d}{d\theta} f(x(\theta))\, d\theta = \int_0^{\pi/2} \left\langle \nabla f(x(\theta)), x'(\theta) \right\rangle d\theta. \end{aligned}$$

Using Jensen again, we get

$$\begin{aligned}
&\int\int e^{\lambda(f(x)-f(y))}\gamma_N(dx)\,\gamma_N(dy) \\
= &\int\int \exp\left[\frac{\lambda\pi}{2}\frac{2}{\pi}\int_0^{\pi/2}\left\langle \nabla f(x(\theta)), x'(\theta)\right\rangle d\theta\right]\gamma_N(dx)\,\gamma_N(dy) \\
\leq &\frac{2}{\pi}\int_0^{\pi/2}\int\int \exp\left[\frac{\lambda\pi}{2}\left\langle \nabla f(x(\theta)), x'(\theta)\right\rangle\right]\gamma_N(dx)\,\gamma_N(dy)\,d\theta.
\end{aligned}$$

For any fixed θ, $(x(\theta), x'(\theta))$ is just an orthogonal transformation of (x, y) in $\mathbb{R}^{2N}$, and $\gamma_N \otimes \gamma_N$ is the standard Gaussian measure on $\mathbb{R}^{2N}$. Therefore, by rotational invariance of the Gauss measure,

$$\int\int \exp\left[\frac{\lambda\pi}{2}\left\langle \nabla f(x(\theta)), x'(\theta)\right\rangle\right]\gamma_N(dx)\,\gamma_N(dy)$$

is independent of θ, and therefore

$$\begin{aligned}
\int\int e^{\lambda(f(x)-f(y))}\gamma_N(dx)\,\gamma_N(dy) &\leq \int\int \exp\left[\frac{\lambda\pi}{2}\left\langle \nabla f(x), y\right\rangle\right]\gamma_N(dx)\,\gamma_N(dy) \\
&= \int \exp\left[\frac{\lambda^2\pi^2}{8}\left|\nabla f(x)\right|^2\right]\gamma_N(dx) \\
&\leq \exp\left[\frac{\lambda^2\pi^2}{8}\left\|f\right\|_{\mathrm{Lip}}^2\right].
\end{aligned}$$

Therefore, we have for any $\lambda > 0$

$$\gamma_N(f > t) \leq \exp\left[-\lambda t + \frac{\lambda^2\pi^2}{8}\left\|f\right\|_{\mathrm{Lip}}^2\right].$$

Optimizing over λ means taking $\lambda = 4t/\pi^2\left\|f\right\|_{\mathrm{Lip}}^2$ which gives

$$\gamma_N(f > t) \leq \exp\left[-2t^2/\pi^2\left\|f\right\|_{\mathrm{Lip}}^2\right].$$

We get the same estimate for $-f$, and therefore

$$\gamma_N(|f| > t) \leq 2\exp\left[-2t^2/\pi^2\left\|f\right\|_{\mathrm{Lip}}^2\right].$$

■

The second result we need from Gaussian variables is Wick's identity

Theorem 1.2
Let $(X_1, \ldots, X_d)$ be a centered Gaussian random vector with covariance matrix $\Sigma = (\gamma_{ij})$, and let $\Phi : \mathbb{R}^d \to \mathbb{R}$ be a continuously differentiable function satisfying

$$\left|\Phi(x_1, \ldots, x_d)\right| \leq C\exp\left[C\,|x|\right]$$

for some $C > 0$. Then

$$E\left(X_i \Phi\left(X_1, \ldots, X_d\right)\right) = \sum_j \gamma_{ij} E \frac{\partial \Phi}{\partial x_j}\left(X_1, \ldots, X_d\right).$$

Proof. We first treat the case where the X_i are i.i.d. standard Gaussians. Then the statement is simply

$$E\left(X_i \Phi\left(X_1, \ldots, X_d\right)\right) = E \frac{\partial \Phi}{\partial x_j}\left(X_1, \ldots, X_d\right),$$

and it suffices to consider the case $d = 1$. For that special case, it is just partial integration

$$\begin{aligned} E\left(X \Phi\left(X\right)\right) &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-x^2/2} x \Phi\left(x\right) dx \\ &= -\frac{1}{\sqrt{2\pi}} \mathrm{e}^{-x^2/2} \Phi\left(x\right)\Big|_{x=-\infty}^{\infty} + \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-x^2/2} \Phi'\left(x\right) dx, \end{aligned}$$

and the first term vanishes by the growth condition on Φ.

For the general case, we represent the X's through a linear transformation of i.i.d. Gaussians ξ_i :

$$X_i = \sum_{j=1}^{d} a_{ij} \xi_j,$$

where the matrix $A = \left(a_{ij}\right)$ satisfies $AA^T = \Sigma$. Then

$$E\left(X_i \Phi\left(X_1, \ldots, X_d\right)\right) = \sum_j a_{ij} E \xi_j \Phi\left(A\boldsymbol{\xi}\right),$$

$\boldsymbol{\xi}$ as a column vector, and

$$\begin{aligned} E \xi_j \Phi\left(A\boldsymbol{\xi}\right) &= \sum_l a_{lj} E \frac{\partial \Phi}{\partial x_l}\left(A\boldsymbol{\xi}\right) \\ &= \sum_l a_{lj} E \frac{\partial \Phi}{\partial x_l}\left(X\right). \end{aligned}$$

This proves the claim. ■

1.2 Point processes

The point processes we consider are all either on $\mathbb{R}$ or $\mathbb{R}^+$, but we can give the basic definitions for arbitrary locally compact separable metric spaces X. The Borel-σ-field is denoted by $\mathcal{X}$. A measure μ on $(X, \mathcal{X})$ is called Radon measure if $\mu(K) < \infty$ for any compact $K \subset X$. We write R_X for the set of Radon measures on $(X, \mathcal{X})$. We can equip R_X with the topology of vague convergence which is generated by the evaluation mappings $\mu \longmapsto \int f d\mu$, $f \in C_0(X)$, where $C_0(X)$ denotes the set of continuous functions $X \to \mathbb{R}$ of compact support.

It is known that on R_X there exists a metric ρ which is complete, and such that R_X has a countable dense subset, which is a metric for vague convergence. One says that (R_X, ρ) is a Polish space. The Borel-field $\mathcal{R}_X$ for this metric space is also the σ-field on R_X generated by the mappings $\mu \longmapsto \mu(A)$, $A \in \mathcal{X}$. These facts are easily checked.

We consider probability measure on $(R_X, \mathcal{R}_X)$. There is then the notion of weak convergence: A sequence $\{Q_n\}_{n\in\mathbb{N}}$ of probability measures on $(R_X, \mathcal{R}_X)$ is said to converge weakly to a probability measure Q if

$$\lim_{n\to\infty} \int F(\mu)\, Q_n(d\mu) = \int F(\mu)\, Q(d\mu)$$

for any bounded continuous function $F : R_X \to \mathbb{R}$.

A convenient tool for the investigation of weak convergence is the Laplace functional. Let $\phi \in C_0^+(X)$. These are the non-negative functions in $C_0(X)$. If $Q \in R_X$, the Laplace functional L_Q on $C_0(X)$ is defined by

$$L_Q(\phi) := \int \exp\left[-\int \phi\, d\mu\right] Q(d\mu).$$

One has the following characterization properties which we state without proof.

Proposition 1.3

a) If $L_Q(\phi) = L_{Q'}(\phi)$ for all ϕ, then $Q = Q'$.

b) If $\{Q_n\}$ is a sequence of probability measures on $(R_X, \mathcal{R}_X)$, and Q is a probability measure, then $\{Q_n\}$ converges weakly to Q if and only if

$$\lim_{n\to\infty} L_{Q_n}(\phi) = L_Q(\phi)$$

holds for all $\phi \in C_0^+(X)$.

If $A \subset X$ is measurable and has compact closure, then $R_X \ni \mu \to \mu(A)$ is measurable. If Q is a probability measure on $(R_X, \mathcal{R}_X)$, we write

$$\overline{Q}(A) := \int \mu(A)\, Q(d\mu). \tag{1.1}$$

In principle, this may be infinite even if A has compact closure, but we will only be interested in cases where $\overline{Q}$ is a Radon measure. We call $\overline{Q}$ the intensity measure of Q. Let $F : X \to [0, \infty)$ be a measurable mapping. Then $R_X \ni \mu \to \int F d\mu \in [0, \infty]$ is a measurable mapping.

Lemma 1.4

$$\int \left(\int F d\mu\right) Q(d\mu) = \int F d\overline{Q}.$$

In particular, if $\int F d\overline{Q} < \infty$, then $\int F d\mu < \infty$, Q-a.s.

Proof. Left as an exercise to the reader. ■

Of interest for us are only point measures on X, i.e. measures of the form

$$\sum_{i \in I} \delta_{x_i},$$

where $\{x_i\}$ is a finite or countable sequence in X which has the property that $\sum_i 1_K(x_i) < \infty$ for any compact subset $K \subset X$. The set of measures of this form is denoted by $R_{p,X}$. It is easy to see that this is a closed subset of R_X, and therefore a Borel set.

Definition 1.5

A random variable Ξ defined on some probability space $(\Omega, \mathcal{F}, P)$ which takes values in $(R_X, \mathcal{R}_X)$ and satisfies $P(\Xi \in R_{p,X}) = 1$ is called a ***point process****.*

It is not difficult to see that one can realize such a point process through a finite or infinite sequence $\{\xi_k\}$ or X-valued random variables: $\Xi = \sum_k \delta_{\xi_k}$. The ordering of the random variables is irrelevant for the point process. If $X = \mathbb{R}$ and the point process has points with a largest element, which are all distinct, then one can choose a fixed ordering of the points by ordering them downwards $\xi_1 > \xi_2 > \ldots$. This is sometimes convenient, although it is not really relevant.

Definition 1.6

Let μ be a Radon measure on $(X, \mathcal{X})$. Then a point process Ξ is called a ***Poisson point process with intensity measure*** *μ if the following two conditions are satisfied*

- *If $A \subset X$ has compact closure then $\Xi(A)$ is Poisson distributed with parameter $\mu(A)$.*
- *If $A_1, \ldots, A_n$ are pairwise disjoint sets, then $\Xi(A_1), \ldots, \Xi(A_n)$ are independent random variables.*

We say that Ξ is a PPP (μ) if it is a Poisson point process with intensity measure μ. If course, we have then $\mu = \overline{Q}$ where Q is the law of Ξ.

For a point process, we write L_Ξ for the Laplace functional of its distribution:

$$L_\Xi(\phi) = \int \exp\left[-\int \phi \, d\mu\right] P\Xi^{-1}(d\mu) = E \exp\left[-\int \phi \, d\Xi\right].$$

Proposition 1.7

If Ξ is a PPP (μ) *then for all $\phi \in C_0^+(X)$*

$$L_\Xi(\phi) = \exp\left[-\int \left(1 - \mathrm{e}^{-\phi(x)}\right) \mu(dx)\right].$$

Proof. Let $\phi \in C_0^+(X)$. Given $\varepsilon > 0$, we can find finitely many $A_1, \ldots, A_n \in \mathcal{X}$ with compact closure, and nonnegative numbers $a_1, \ldots, a_n$ such that

$$\left\| \phi - \sum\nolimits_i a_i 1_{A_i} \right\|_\infty \leq \varepsilon.$$

Replacing ϕ by the simple function $s = \sum_i a_i 1_{A_i}$, we get

$$\begin{aligned} E \exp\left[-\int s d\Xi\right] &= E \exp\left[-\sum\nolimits_i a_i \Xi(A_i)\right] \\ &= \prod\nolimits_{i=1}^n E \exp\left[-a_i \Xi(A_i)\right], \end{aligned}$$

as the $\Xi(A_i)$ are independent. As they are Poisson with parameter $\mu(A_i)$ we get

$$\begin{aligned} E \exp\left[-a_i \Xi(A_i)\right] &= \mathrm{e}^{-\mu(A_i)} \sum_{k=0}^{\infty} \frac{\mu(A_i)^k}{k!} \mathrm{e}^{-k a_i} \\ &= \exp\left[-\mu(A_i)\left(\mathrm{e}^{-a_i} - 1\right)\right], \end{aligned}$$

i.e.

$$\begin{aligned} E \exp\left[-\int s d\Xi\right] &= \exp\left[-\sum\nolimits_i \mu(A_i)\left(\mathrm{e}^{-a_i} - 1\right)\right] \\ &= \exp\left[-\int\left(1 - \mathrm{e}^{-s(x)}\right)\mu(dx)\right]. \end{aligned}$$

The result now follows by a simple approximation procedure. ■

To construct a PPP (μ) we simply have to construct a probability measure Q on $(R_X, \mathcal{R}_X)$ with

$$L_Q(\phi) = \exp\left[-\int\left(1 - \mathrm{e}^{-\phi(x)}\right)\mu(dx)\right]$$

for all $\phi \in C_0^+(X)$.

Theorem 1.8
For any Radon measure μ on $(X, \mathcal{X})$, a PPP (μ) *exists.*

Proof. We prove this first in the case that $\mu(X) < \infty$. Then $\hat{\mu} := \mu/\mu(X)$ is a probability measure on $(X, \mathcal{X})$. We choose a sequence $\xi_1, \xi_2, \ldots$ of independent random variables which are distributed according the $\hat{\mu}$. Furthermore, independently of these, we choose an independent Poisson distributed random variable N with parameter $\mu(X)$. Then we define

$$\Xi := \sum_{i=1}^{N} \delta_{\xi_i}.$$

We claim that this is a PPP (μ). It is evident that it is a point process, so we compute its Laplace functional

$$\begin{aligned}
L_{\Xi}(\phi) &= E\exp\left[-\int \phi \, d\Xi\right] = E\exp\left[-\sum\nolimits_{i=1}^{N}\phi(\xi_i)\right] \\
&= \sum_{k=1}^{\infty}\frac{\mu(X)^k}{k!}\mathrm{e}^{-\mu(X)}E\exp\left[-\sum\nolimits_{i=1}^{k}\phi(\xi_i)\right] \\
&= \sum_{k=1}^{\infty}\frac{\mu(X)^k}{k!}\mathrm{e}^{-\mu(X)}\left\{E\exp\left[-\phi(\xi)\right]\right\}^k \\
&= \sum_{k=1}^{\infty}\frac{\mu(X)^k}{k!}\mathrm{e}^{-\mu(X)}\left\{\int \exp\left[-\phi(x)\right]\frac{\mu(dx)}{\mu(X)}\right\}^k \\
&= \sum_{k=1}^{\infty}\frac{1}{k!}\mathrm{e}^{-\mu(X)}\left\{\int \exp\left[-\phi(x)\right]\mu(dx)\right\}^k \\
&= \exp\left[-\int\left(1-\mathrm{e}^{-\phi(x)}\right)\mu(dx)\right].
\end{aligned}$$

In the case $\mu(X) = \infty$, we chop X into countably many pairwise disjoint, measurable and relatively compact sets $X_1, X_2, \ldots$. Then $\mu(X_i) < \infty$, and we can define independent PPP (μ_i)'s $\Xi_1, \Xi_2, \ldots$, where $\mu_i(A) := \mu(A \cap X_i)$. Then $\sum_i \Xi_i$ does the job. I leave that to the reader to check. ■

We are interested only in the case where X is $\mathbb{R}$ or $\mathbb{R}^d$ or an open subset of these spaces, and where μ has a density with respect to Lebesgue measure. If g is such a density, we say that a PPP (μ) is a Poisson point process with density g, and sometimes write PPP $(t \to g(t))$.

If X, X' are two separable, locally compact metric spaces, and $f : X \to X'$ is a continuous mapping, then f defines a mapping from measures μ on X to measures μf^{-1} on X'. However, if μ is Radon, then not necessarily, μf^{-1} is Radon. We therefore assume that f has the property that $f^{-1}(K)$ is compact in X whenever $K \subset X'$ is compact. Then $\mu \to \mu f^{-1}$ maps $R_{p,X}$ into $R_{p,X'}$.

Lemma 1.9
Under this condition, if Ξ is a PPP (μ), then Ξf^{-1} is a PPP $\left(\mu f^{-1}\right)$.

Proof. We compute the Laplace functional. Let $\phi \in C_0^+ (X')$. Then

$$\begin{aligned}
L_{\Xi f^{-1}} (\phi) &= \int \exp \left[- \int \phi \, d\nu \right] P\Xi^{-1} f^{-1} (d\nu) \\
&= \int \exp \left[- \int \phi \, d \left(\nu f^{-1} \right) \right] P\Xi^{-1} (d\nu) \\
&= \int \exp \left[- \int (\phi \circ f) \ d\nu \right] P\Xi^{-1} (d\nu) \\
&= \exp \left[- \int \left(1 - \mathrm{e}^{-\phi \circ f(x)} \right) \mu (dx) \right] \\
&= \exp \left[- \int \left(1 - \mathrm{e}^{-\phi(x)} \right) \mu f^{-1} (dx) \right].
\end{aligned}$$

■

1.3 The basic jargon of statistical mechanics

We investigate certain probability measures on *finite* but *large* sets Σ. Typically, one is interested in properties of these measures when the number of elements in this set goes to ∞. We therefore let Σ depend on a parameter $N \in \mathbb{N}$, and we write Σ_N. A typical choices for Σ_N is $\{-1, 1\}^N$. The elements $\sigma \in \Sigma_N$ are called **spin configurations**, and in case Σ_N is a product $\Sigma_N = S^N$, S a finite set, e.g. $\{-1, 1\}$, the components of σ are called the **individual spins**, or just the **spins**.

The measures P we consider have positive mass on *every* element $\sigma \in \Sigma_N$. We therefore can write

$$\exp \left[-H (\sigma) \right], \tag{1.2}$$

where $H : \Sigma_N \to \mathbb{R}$. The minus sign is only for historical reasons (from physics). It will be very convenient (for reasons which will become clear later), *not* to assume that $\sum_\sigma \exp \left[-H (\sigma) \right] = 1$, and to do the normalization separately. We therefore just assume that H is a real valued function $\Sigma_N \to \mathbb{R}$, and then we define a probability measure $\mathcal{G}$ on Σ_N by

$$\mathcal{G} (\sigma) = \frac{1}{Z} \exp \left[-H (\sigma) \right], \tag{1.3}$$

where

$$Z := \sum_\sigma \exp \left[-H (\sigma) \right].$$

$\mathcal{G}$ is evidently a probability measure on Σ_N, for every choice of H. At this stage, there is of course no real difference between (1.2) and (1.3), as we can always replace H by $H + \log Z$, and then we have a representation of the type (1.2).

The real use of keeping Z separate is that H often depends on certain extra parameters, for instance the "temperature" or the "inverse temperature". The latter is just a positive parameter which is always denoted by β. Typically, this parameter is a multiplicative one, and so we replace H by βH. Then of course also Z depends on β, and we write Z_β instead. Furthermore, $\mathcal{G}$ also depends on β, and we have

$$\mathcal{G}_\beta(\sigma) := \frac{1}{Z_\beta} \exp\left[-\beta H(\sigma)\right].$$

Now, we should also remember, that there is the parameter N which governs the size of the system, and we therefore write H as depending on N, i.e. H_N. Then everything is N-dependent, and we have

$$\mathcal{G}_{\beta,N}(\sigma) := \frac{1}{Z_{\beta,N}} \exp\left[-\beta H_N(\sigma)\right].$$

This is called the **Gibbs measure** with **Hamiltonian** H_N and **inverse temperature** β on Σ_N. $Z_{\beta,N}$ is called the **partition function**. The expression "function" is a bit strange as it is just a constant, but this constant depends on β, and maybe on other parameters. This dependence is the reason one calls it "function". The finite N **free energy** is defined by

$$F_{\beta,N} = \frac{1}{N} \log Z_{\beta,N}.$$

Often, there is a limit of this quantity, as $N \to \infty$:

$$\begin{aligned} f(\beta) \quad : \quad &= \lim_{N\to\infty} F_{\beta,N} \\ &= \lim_{N\to\infty} \frac{1}{N} \log \sum_{\sigma\in\Sigma_N} \exp\left[-\beta H_N(\sigma)\right]. \end{aligned}$$

The existence of this limit has of course to be proved, and is not true in generality.

The importance of f is based on the fact that the physically important quantities can be expressed through it. For instance, if one takes the derivative in β, provided it exists, and it is interchangeable with the $N \to \infty$ limit, one gets

$$\frac{df(\beta)}{d\beta} = -\lim_{N\to\infty} \frac{1}{Z_{\beta,N}} \sum_\sigma \frac{H_N(\sigma)}{N} \exp\left[-\beta H_N(\sigma)\right],$$

which is the Gibbs average of the energy per site in the $N \to \infty$ limit.

The simplest such model is when the spins are all independent, i.e. when

$$H(\sigma) = \sum_{i=1}^N h(\sigma_i), \ \ h : S \to \mathbb{R}.$$

Then, of course,

$$F_N = \log \sum_{\sigma\in S} \mathrm{e}^{-\beta h(\sigma)} = f(\beta).$$

1.4 The Curie-Weiss model

The next, slightly more interesting, case is the **Curie-Weiss model**, which is the simplest model exhibiting what in physics jargon is called a phase transition. It has $\Sigma_N := \{-1, 1\}^N$, and

$$H_N(\sigma) := -\frac{1}{2N} \sum_{i,j=1}^{N} \sigma_i \sigma_j = -\frac{1}{2N} \left(\sum_{i=1}^{N} \sigma_i \right)^2, \ \sigma = (\sigma_1, \ldots, \sigma_N)$$

The diagonal term $\sum_{i=j}$ is just 1, and this cancels out with the normalization and does not influence the Gibbs measure. We could therefore as well just take the sum $\sum_{i \neq j}$ which is often done. The key point is that this Hamiltonian reflects an interaction of any individual spin σ_i with the average of the other spins $\Sigma_{j:j\neq i}\sigma_j / (N-1)$. The total "interaction energy" is then

$$\frac{1}{2} \sum_i \sigma_i \frac{\Sigma_{j:j\neq i}\sigma_j}{N-1} = \frac{1}{2} \frac{1}{N-1} \sum_{i \neq j} \sigma_i \sigma_j.$$

That there is $N-1$ instead of N is of no importance for large N.

Occasionally, one also has a so-called external field which give the σ_i a global tilt. Then the Hamiltonian is

$$H_N(\sigma) := -\frac{1}{2N} \sum_{i,j=1}^{N} \sigma_i \sigma_j - h \sum_{i=1}^{N} \sigma_i.$$

$h \in \mathbb{R}$ is an additional parameter. The Curie-Weiss Gibbs measure is therefore

$$\mathcal{G}_{\beta,h,N}(\sigma) := \frac{1}{Z_{\beta,h,N}} \exp\left[\frac{\beta}{2N} \sum_{i,j=1}^{N} \sigma_i \sigma_j + \beta h \sum_{i=1}^{N} \sigma_i \right],$$

where

$$Z_{\beta,h,N} := \sum_{\sigma \in \Sigma_N} \exp\left[\frac{\beta}{2N} \sum_{i,j=1}^{N} \sigma_i \sigma_j + \beta h \sum_{i=1}^{N} \sigma_i \right]. \tag{1.4}$$

This model can easily be analyzed by Stirling's formula. The point is that the Hamiltonian is a function of $\bar{\sigma}_N := \frac{1}{N} \sum_{i=1}^{N} \sigma_i$:

$$H_N(\sigma) = -N \left(\frac{1}{2} \bar{\sigma}_N^2 + h \bar{\sigma}_N \right),$$

and Z can be written as expectation under standard coin tossing:

$$Z_N = 2^N E_N^{\mathrm{CT}} \exp\left[-N\beta \left(\bar{\sigma}_N^2 / 2 + h \bar{\sigma}_N \right) \right].$$

The coin tossing expectation E^{CT} can be computed in terms of Stirlings formula up to any precision one likes. The rough large deviation behavior, in the usual large deviation jargon, is

$$P_N^{\mathrm{CT}}\left(\bar{\sigma}_N \sim x\right) \sim \exp\left[-NI\left(x\right)\right]$$

with the entropy function

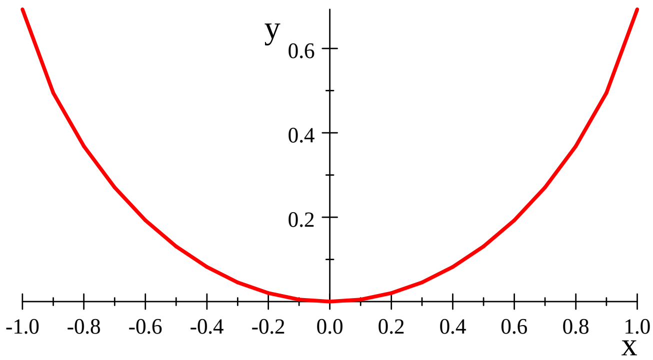
$$I\left(x\right) = \begin{cases} \frac{1+x}{2}\log\left(1+x\right) + \frac{1-x}{2}\log\left(1-x\right) & \text{if } x \in [-1,1] \\ \infty & \text{if } x \notin [-1,1] \end{cases}.$$

Therefore,

$$f\left(\beta,h\right) = \lim_{N\to\infty}\frac{1}{N}\log Z_N = \log 2 + \sup_x\left[\frac{\beta}{2}x^2 + \beta hx - I\left(x\right)\right].$$

For those who are not familiar with these type of arguments, I leave it as an exercise to derive it from Stirling's formula.

The function $x \to I\left(x\right) = \frac{1+x}{2}\log\left(1+x\right) + \frac{1-x}{2}\log\left(1-x\right)$ looks as follows

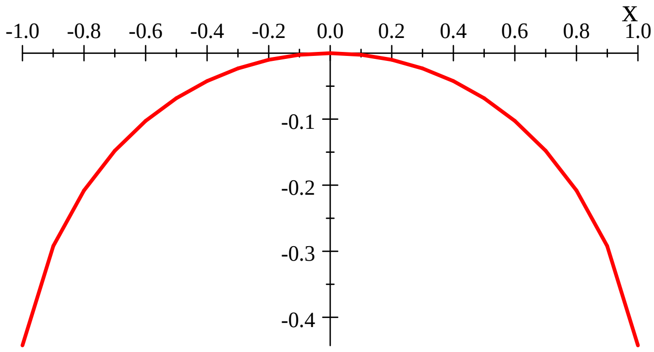


It is of course even. Furthermore, the function is continuous on the full interval $[-1,1]$ with

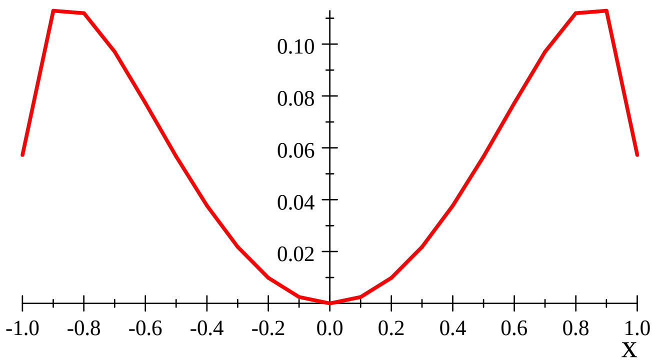
$$\lim_{x\to\pm1} I\left(x\right) = \log 2,$$

but the tangent diverges as $x \to \pm1$. The behavior of the Curie-Weiss model is determined by the function $x \to g_{\beta,h}\left(x\right) := \frac{\beta}{2}x^2 + \beta hx - I\left(x\right)$. This depends heavily on β and h.

Case $h = 0$: In this case the above function is even, but there is a crucial difference depending on whether $\beta \leq 1$ or $\beta > 1$. Below there are plots for $\beta = 1/2$, and $\beta = 3/2$.



$g_{\beta,h}$ for $h = 0, \beta = 1/2$



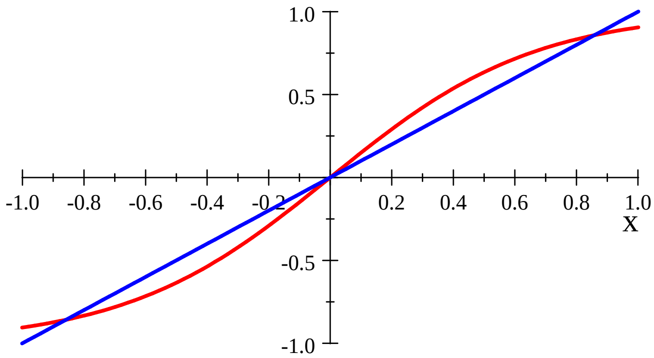
$g_{\beta,h}$ for $h = 0, \beta = 3/2$

The crucial difference is coming from the second derivative:

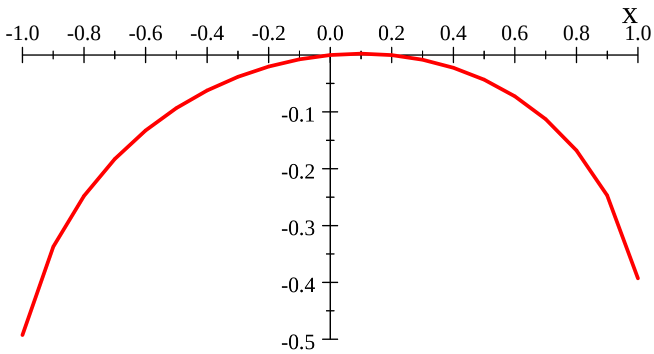
$$\frac{dg}{dx} = \beta x - \left(\frac{1}{2}\log(1+x) - \frac{1}{2}\log(1-x)\right),$$

$$\frac{d^2g}{dx^2} = \beta - \frac{1}{2(1+x)} - \frac{1}{2(1-x)}.$$

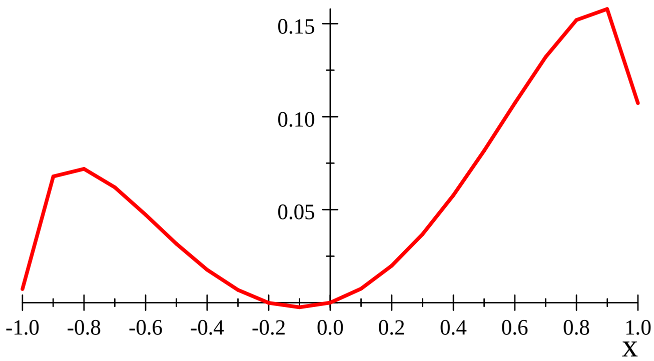
For $\beta < 1$, the second derivative is negative everywhere, and therefore the function is strictly concave with a unique maximum at 0. This remains true for $\beta = 1$, where the second derivative is 0 at 0, but negative for $x \neq 0$. However, for $\beta > 1$, the second derivative is positive at 0, and negative for x sufficiently close to ± 1. Therefore, 0 is a local minimum, and the maxima of the function are elsewhere. Setting the first derivative 0, one gets the equation $m = \tanh(\beta m)$ for the maximum m. For $\beta \leq 1$, there is just the solution 0 for this equation, but for $\beta > 1$, there are 2 other solutions $\pm m_\beta$. Below, there are the two curves $x \to \tanh(\beta x)$ for $\beta = 3/2$ (in red), and $x \to x$ (in blue).



Case $h \neq 0$: In that case, 0 is never a maximum. The curve still depends on the value of β, but it has always one unique global maximum, although it may have a local maximum besides that. Below are two examples, both with $h = 1/20$, and the first with $\beta = 1/2$, while the second with $\beta = 3/2$:



$g_{\beta,h}$ for $h = 1/20$, $\beta = 1/2$



$g_{\beta,h}$ for $h = 1/20$, $\beta = 3/2$

It is easily checked that there is a unique $m_{\beta,h} \neq 0, \pm 1$, such that

$$g_{\beta,h}\left(m_{\beta,h}\right) = \max_{x \in [-1,1]} g_{\beta,h}\left(x\right).$$

This value $m_{\beta,h}$ satisfies the mean-field equation

$$m = \tanh\left(\beta h + \beta m\right), \tag{1.5}$$

as is easily checked.

We summarize the basic results:

Theorem 1.10

a) *Let $Z_{\beta,h,N}$ be the partition function of the Curie-Weiss model as defined in (1.4). Then*

$$f\left(\beta,h\right) = \lim_{N\to\infty}\frac{1}{N}\log Z_{\beta,h,N}$$

exists is given by

$$f\left(\beta,h\right) = \sup_{x\in[-1,1]} g\left(\beta,h\right) + \log 2.$$

b) *If $h \neq 0$, then $\bar{\sigma}_N$ converges in $P_{\beta,h,N}$-probability to $m_{\beta,h}$, i.e. for any $\varepsilon > 0$ one has*

$$\lim_{N\to\infty}\mathcal{G}_{\beta,h,N}\left(\left|\frac{S_N}{N} - m_{\beta,h}\right| \geq \varepsilon\right) = 0.$$

c) *If $h = 0$ and $\beta \leq 1$, then $\bar{\sigma}_N$ converges in $P_{\beta,0,N}$-probability to 0. If $\beta > 1$, then the $P_{\beta,0,N}$-law of S_N/N converges to*

$$\frac{1}{2}\delta_{m_\beta} + \frac{1}{2}\delta_{-m_\beta}.$$

This means that for any $0 < \varepsilon < m_\beta$, one has

$$\begin{aligned}\lim_{N\to\infty}\mathcal{G}_{\beta,0,N}\left(\left|\frac{S_N}{N} - m_\beta\right| \leq \varepsilon\right) &= 1/2,\\ \lim_{N\to\infty}\mathcal{G}_{\beta,0,N}\left(\left|\frac{S_N}{N} + m_\beta\right| \leq \varepsilon\right) &= 1/2.\end{aligned}$$

Proof. Left as an exercise. All the statements follow easily from Stirling's formula. ■

$m_{\beta,h}$ is the mean magnetization $\bar{\sigma}_N$ in the $N \to \infty$ limit under the Gibbs measure. In physics literature, the equation (1.5) is usually derived via a "cavity" argument. For that, one argues that m should be the Gibbs expectation for a single spin. By symmetry, it doesn't matter which one takes, so we take the last one:

$$m \approx E_{\beta,h,N}\left(\sigma_N\right) = \frac{\sum_\sigma \sigma_N \exp\left[-\beta\sigma_N\frac{1}{N}\sum_{j=1}^{N-1}\sigma_j - \beta h\sigma_N - \beta H_{N-1}\left(\sigma^{(N-1)}\right)\right]}{\sum_\sigma \exp\left[-\beta\sigma_N\frac{1}{N}\sum_{j=1}^{N-1}\sigma_j - \beta h\sigma_N - \beta H_{N-1}\left(\sigma^{(N-1)}\right)\right]},$$

where $\sigma^{(N-1)} = \left(\sigma_1,\ldots,\sigma_{N-1}\right)$, and H_{N-1} is the Hamiltonian on the first $N-1$ spins. Summing first σ_N out in this expression, and the other ones afterwards, one gets

$$E_{\beta,h,N}\left(\sigma_N\right) = \frac{E_{\beta,h,N-1}\sinh\left(\beta\frac{1}{N}\sum_{j=1}^{N-1}\sigma_j + \beta h\right)}{E_{\beta,h,N-1}\cosh\left(\beta\frac{1}{N}\sum_{j=1}^{N-1}\sigma_j + \beta h\right)}.$$

Under the Gibbs measure on the first $N-1$ spin variables, one should have

$$\frac{1}{N}\sum_{j=1}^{N-1}\sigma_j \approx \frac{1}{N-1}\sum_{j=1}^{N-1}\sigma_j \approx m,$$

the last approximation by disregarding possible fluctuations around the mean. By this chain of arguments, one gets

$$E_{\beta,h,N}\left(\sigma_N\right) \approx \tanh\left(\beta h+\beta m\right),$$

which leads in the $N\to\infty$ limit to (1.5). In spin glass theory, there are similar equations, the TAP equations, which however are much more delicate to discuss and prove.

It is not difficult to get more information than in Theorem 1.10 out with some refinements of the arguments. For instance one can prove that in the "one-phase region", i.e. either $h\neq 0$ or $h=0$ and $\beta\leq 1$, the spins under $\mathcal{G}_{\beta,h,N}$ behave like i.i.d. spins with possibly tilted mean. To be precise, for $m\in(-1,1)$ consider Bernoulli measure with mean, i.e. $p_m(1) := (1+m)/2$, $p_m(-1) = 1-p_m(1) = (1-m)/2$.

Proposition 1.11
Under the above conditions, one has for any $K\in\mathbb{N}$:

$$\lim_{N\to\infty}\mathcal{G}_{\beta,h,N}\left(\sigma_1=i_1,\ldots,\sigma_K=i_K\right)=\prod\nolimits_{j=1}^{K}p_m\left(i_j\right),$$

where $m=0$ for $h=0,\beta\leq 1$, and $m=m_{\beta,h}$ for $h\neq 0$.

Proof. The easiest way to prove it is based on the well known fact that drawing from an urn with red and black balls, without replacement, is approximately the same as drawing with replacement, provided that the number of drawings is small compared to the total number of balls in the urn. To be precise: Consider $\sigma_1,\ldots,\sigma_N$ under the Bernoulli measure P^{CT}, then

$$\left|P^{\mathrm{CT}}\left(\sigma_1=i_1,\ldots,\sigma_K=i_K\mid\bar{\sigma}_N\right)-\prod\nolimits_{j=1}^{K}p_{\bar{\sigma}_N}\left(i_j\right)\right|\leq\varepsilon\left(K,N\right) \tag{1.6}$$

uniformly in $\bar{\sigma}_N,\ i_1,\ldots,i_K$, where $\lim_{N\to\infty}\varepsilon\left(K,N\right)=0$ for any K.

If we write E^{Gibbs} for the expectation under the Gibbs distribution, and if F is any function $\Sigma_N\to\mathbb{R}$, then, as the Hamiltonian depends only on $\bar{\sigma}_N$, one has

$$\begin{aligned}
E^{\mathrm{Gibbs}}\left(F\right) &= \frac{E^{\mathrm{CT}}\left(F\exp\left[-\beta H\right]\right)}{E^{\mathrm{CT}}\left(\exp\left[-\beta H\right]\right)}=\frac{E^{\mathrm{CT}}\left(E^{\mathrm{CT}}\left(F\mid\bar{\sigma}_N\right)\exp\left[-\beta H\left(\bar{\sigma}_N\right)\right]\right)}{E^{\mathrm{CT}}\left(\exp\left[-\beta H\left(\bar{\sigma}_N\right)\right]\right)}\\
&= E^{\mathrm{Gibbs}}\left(E^{\mathrm{CT}}\left(F\mid\bar{\sigma}_N\right)\right).
\end{aligned}$$

Applying that to

$$F\left(\sigma\right)=I\left(\sigma_1=i_1,\ldots,\sigma_K=i_K\right),$$

and using (1.6), and Theorem 1.10, the claim follows. ■

The proposition states that in the one-phase region, the Gibbs-measure under the $N \to \infty$ limit is simply coin tossing with possibly tilted mean. There is a similar statement also in the two-phase region, i.e. $\beta > 1$, $h = 0$. In that case the Gibbs distribution converges to a mixture of two tilted Bernoulli measures. Here is the statement, the proof is left as an exercise:

Proposition 1.12
Assume $\beta > 1$ and $h = 0$, and let m_β be the positive solution of $m = \tanh(\beta m)$. Then for any K, and any $i_1, \ldots, i_K \in \{-1, 1\}$:

$$\lim_{N\to\infty} \mathcal{G}_{\beta,0,N}\left(\sigma_1 = i_1, \ldots, \sigma_K = i_K\right) = \frac{1}{2}\prod\nolimits_{j=1}^{K} p_{m_\beta}\left(i_j\right) + \frac{1}{2}\prod\nolimits_{j=1}^{K} p_{-m_\beta}\left(i_j\right).$$

These properties of the Curie-Weiss model are coming under the name "symmetry breaking". The coin tossing measures are the so-called "pure states". If $h \neq 0$ or $h = 0$ and $\beta \leq 1$, the Gibbs measure converges to a pure state. In the case $h = 0$, $\beta > 1$, the Gibbs measure converges to a mixture of two symmetric pure states. As in that case, the relevant pure states, namely coin tossing with mean m_β and $-m_\beta$ are not symmetric under sign change, one says that the model "breaks" the symmetry, although, of course, the limiting measure is still symmetric.

Even the very simplest mean field spin glasses have a much more complicated symmetry breaking which is mathematically still not fully understood.

2 The Sherrington-Kirkpatrick model, high temperature

2.1 The SK-model and other spin glass models

A spin glass is simply a Gibbs measure with a random Hamiltonian. This means that for any $\sigma \in \Sigma_N$, the $H_N(\sigma)$ is a random variable defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. In order to emphasize this, we sometimes write $H_{N,\omega}(\sigma)$, $\omega \in \Omega$. We will use $\mathbb{P}$ exclusively for the probability measure on the space describing the "disorder".

The SK-model is the "spin glass version" of the Curie-Weiss model. The set of spin configurations is again $\Sigma_N = \{-1, 1\}^N$, and every spin variable is interacting with every other spin variable. However the interaction strength is random. This means that one has a set of random variables J_{ij}, $1 \leq i < j \leq N$, defined on $(\Omega, \mathcal{F}, \mathbb{P})$, and a Hamiltonian H which depends on $\omega \in \Omega$:

$$H_{N,\omega}(\sigma) = -\sum_{1\leq i<j\leq N} J_{ij}(\omega)\,\sigma_i\sigma_j. \tag{2.1}$$

The simplest assumption one can make is that the J_{ij} are independent centered random variables, and even Gaussian ones (which helps a lot later, although it is not really of importance). In the Curie-Weiss model, the interaction strengths between pairs of spins was of order $1/N$. Here the situation is different. The basic property one would like to have is that the total "influence" of the spins σ_j, $j \neq i$, on σ_i is of order 1. This requires that

$$\sum_{j:j>i} J_{ij}\sigma_j + \sum_{j:j<i} J_{ji}\sigma_j$$

is of order one, and therefore, the variance of the J_{ij} should be $1/\sqrt{N}$. We therefore write the Hamiltonian as

$$-H_{N,\omega}(\sigma) = \frac{1}{\sqrt{N}} \sum_{1\leq i<j\leq N} g_{ij}(\omega)\,\sigma_i\sigma_j,$$

where the random variables g_{ij}, $i<j$, are i.i.d. standard Gaussian random variables.

The classical short range spin glass is the **Edwards-Anderson-model**. In this case, one starts with a finite subset $\Lambda \subset\subset \mathbb{Z}^d$, for instance $\Lambda_n \stackrel{\text{def}}{=} \{-n,\ldots,n\}^d$, and $\Sigma_n \stackrel{\text{def}}{=} \{-1,1\}^{\Lambda_n}$. The Hamiltonian is then via a short range interaction:

$$-H(\sigma) \stackrel{\text{def}}{=} \frac{1}{2} \sum_{i,j\in\Lambda:|i-j|=1} g_{ij}\sigma_i\sigma_j.$$

For this model, there is essentially no mathematical theory.

The crucial feature of all spin glasses is the presence of frustrations: If one considers three sites i,j,k with spin variables $\sigma_i,\sigma_j,\sigma_k$ it can happen (and actually often happens) that from the interactions, σ_i,σ_j and σ_i,σ_k would "like" to have the same sign, but σ_j,σ_k would like to have opposite sign. This makes it extraordinary difficult to discuss ground states, i.e. configurations σ with minimal energy $H(\sigma)$, a task which is trivial in classical ferromagnetic (or antiferromagnetic) Ising type models. In fact, the Pirogov-Sinai theory is based on the assumptions that there is a simple set of ground states, so there is no change to apply such theories to spin glasses. Also, classical correlation inequalities like the FKG inequality are never valid in spin glasses.

For any fixed σ, this is simply a Gaussian random variable, as it is a linear combination of i.i.d. Gaussians. Furthermore, the family $\{H_N(\sigma)\}_{\sigma\in\Sigma_N}$ is a 2^N-dimensional Gaussian random vector. Its distribution is determined by its covariances. These are easy to compute

$$\begin{aligned}
\mathbb{E}\left(H_N(\sigma)\,H_N(\sigma')\right) &= \frac{1}{N}\mathbb{E}\sum_{1\leq i<j\leq N} g_{ij}\sigma_i\sigma_j \sum_{1\leq i<j\leq N} g_{ij}\sigma'_i\sigma'_j \\
&= \frac{1}{N}\sum_{1\leq i<j\leq N}\sum_{1\leq r<s\leq N} \sigma_i\sigma_j\sigma'_r\sigma'_s\mathbb{E}g_{ij}g_{rs} \\
&= \frac{1}{N}\sum_{1\leq i<j\leq N} \sigma_i\sigma_j\sigma'_i\sigma'_j \qquad (2.2)\\
&= \frac{1}{2N}\sum_{i,j=1}^{N} \sigma_i\sigma_j\sigma'_i\sigma'_j - \frac{1}{2} \\
&= \frac{N}{2}\left(\frac{1}{N}\sum_{i=1}^{N}\sigma_i\sigma'_i\right)^2 - \frac{1}{2}.
\end{aligned}$$

$$R_N(\sigma,\sigma') := \frac{1}{N}\sum_{i=1}^{N}\sigma_i\sigma_i' \tag{2.3}$$

is denoted as the **overlap** of the two spin configurations. We therefore see that the covariances are given in terms of these overlaps

$$\mathbb{E}\left(H_N(\sigma)\,H_N(\sigma')\right) = \frac{N}{2}R_N^2(\sigma,\sigma') - \frac{1}{2}.$$

In particular, the variances are of order $N/2$.

Occasionally, it is convenient to have the g_{ij} defined also for $i > j$, and we set $g_{ij} = g_{ji}$. One also includes an external field in the form of a sum

$$-h\sum_i \sigma_i$$

which is included in the Hamiltonian. In principle one could also consider *random* external fields in the form of

$$-\sum_i h_i\sigma_i$$

where the h_i are also random variables.

The important point is that one now considers the partition function, and the Gibbs measure for *fixed* but "typical" ω, i.e. one considers the random variable

$$Z_{\beta,h,N,\omega} := \sum_\sigma \exp\left[\frac{\beta}{\sqrt{N}}\sum_{1\le i<j\le N} g_{ij}(\omega)\,\sigma_i\sigma_j + h\sum_i \sigma_i\right].$$

In principle, one should multiply h by β, which is usually done in physics literature, but mathematically it is useless and awkward. As before, we write

$$F_{\beta,h,N,\omega} = \frac{1}{N}\log Z_{\beta,h,N,\omega}.$$

The Gibbs measure is then

$$\mathcal{G}_{\beta,h,N,\omega}(\sigma) = \frac{1}{Z_{\beta,h,N,\omega}}\exp\left[\frac{\beta}{\sqrt{N}}\sum_{1\le i<j\le N} g_{ij}(\omega)\,\sigma_i\sigma_j + h\sum_i \sigma_i\right].$$

The first question one might ask is why this model is of any interest, besides the evident physics background[1]. One answer is that it is connected with problems probabilists had always been interested in, namely distributions of maxima of family of random variables. Consider the family $\{H_N(\sigma)\}_{\sigma\in\Sigma_N}$, $H_N(\sigma)$ given in (2.1), and we are interested in

$$M := \max_\sigma\left(-H_N(\sigma)\right).$$

[1] There is in fact an ongoing sharp controversy in the physics literature whether the SK-models and the methods to "solve" it are of relevance to more realistic models of spin glasses.

The minus sign is of course of no importance and we leave out the external field. It is not difficult to see that this maximum is of order N : One direction is trivial

$$\begin{aligned}\mathbb{P}\left(\max_\sigma\left(-H_N(\sigma)\right)\geq tN\right) &\leq 2^N\max_\sigma\mathbb{P}\left(\left(-H_N(\sigma)\right)\geq tN\right)\\ &\leq 2^N\max_\sigma\mathbb{P}\left(\frac{(-H_N(\sigma))}{\sqrt{\operatorname{var}(H_N(\sigma))}}\geq\frac{tN}{\sqrt{\operatorname{var}(H_N(\sigma))}}\right)\\ &\leq 2^N\mathbb{P}\left(\frac{(-H_N(\sigma))}{\sqrt{\operatorname{var}(H_N(\sigma))}}\geq\sqrt{2}t\sqrt{N}\right)\\ &\leq 2^N\frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{2}t\sqrt{N}}\exp\left[-t^2N\right],\end{aligned}$$

simply because $H_N(\sigma)/\sqrt{\operatorname{var}(H_N(\sigma))}$ is standard normally distributed, and the well-known inequality

$$\int_x^\infty \mathrm{e}^{-y^2/2}dy\leq\frac{1}{x}\mathrm{e}^{-x^2/2}$$

for $x>0$. We therefore see that for $t>\sqrt{\log 2}$, we have

$$\sum_N\mathbb{P}\left(\max_\sigma\left(-H_N(\sigma)\right)\geq tN\right)<\infty,$$

and from Borel-Cantelli, we get

$$\limsup_{N\to\infty}\frac{1}{N}\max_\sigma\left(-H_N(\sigma)\right)\leq\sqrt{\log 2},\ \mathbb{P}-\text{a.s.}$$

A bit more delicate is the following

Exercise 2.1
Prove that for some $c>0$, one has

$$\liminf_{N\to\infty}\frac{1}{N}\max_\sigma\left(-H_N(\sigma)\right)\geq c,\ \mathbb{P}-\text{a.s.}$$

Although it is not overly difficult to prove that such a $c>0$ exists, it is *extremely* difficult to get the "correct" constant, i.e. $\alpha>0$ with

$$\lim_{N\to\infty}\frac{1}{N}\max_\sigma\left(-H_N(\sigma)\right)=\alpha.$$

One of the aims of spin glass theory is to provide a method to achieve that. Another question is to discuss fluctuations around the maximum which, of course, is even more challenging.

The discussion of the suprema of Gaussian random fields (and related questions like continuity properties) has been a big theme in probability theory since the seventies of the last century, with important results by Dudley, Fernique, Adler, Talagrand and others. These results are very general, but it is usually impossible to get the correct constant in the results above. Therefore probabilists like Talagrand were totally struck by the fact that physicists had a theory which *was* able to determine the constant.

The constant α is not an easy object, but given in terms of a variational problem which itself is quite complicated. For SK, there is no explicit expression, but the variational formula gives a way to easily compute it numerically. More striking however was the fact that there was no real proof, but "only" a couple of hair-raising arguments, like the notorious replica trick.

The connection with the free energy is that the ground state energy is connected with the $\beta \to \infty$ limit:

$$-\frac{1}{N}\min_{\sigma} H_N(\sigma) = \lim_{\beta\to\infty} \frac{1}{N\beta} \log \sum_{\sigma} \exp\left[-\beta H_N(\sigma)\right].$$

If an interchange of the $\beta \to \infty$ and the $N \to \infty$ is justified, then one would have

$$\alpha = \lim_{\beta\to\infty} \frac{f(\beta)}{\beta},$$

where

$$f(\beta) := \lim_{N\to\infty} \frac{1}{N} \log \sum_{\sigma} \exp\left[-\beta H_N(\sigma)\right].$$

We will prove the existence of $f(\beta)$ in the next section, and the fact that it is non-random which is not totally obvious.

There are similar models of interest in combinatorics, for instance in combinatorial optimization. One such case is the optimal assignment problem. In the simplest case one has twice N objects, say N girls and N boys. For every girl i and boy j, there is a mutual "satisfaction" of matching i with j, say U_{ij}. The problem is to find a perfect matching, i.e. an assignment of girls to the boys such that the sum of the satisfactions is maximal. Mathematically formulated, one is looking at

$$S_N = \max_{\pi} \sum_{i=1}^{N} U_{i\pi(i)},$$

the maximum running over all permutations of N elements. We assume now that the U_{ij} are i.i.d. uniformly distributed on $[0,1]$. Mathematically, it is the same whether we are maximizing the satisfaction or minimizing it. The latter is formally slightly more convenient. Of course, we could try to find a matching such that for any i, $\pi(i)$ is chosen that $U_{i\pi(i)} = \min_j U_{ij}$, but a moments reflection shows that this will not work as there may be different girls i which would choose the same boy, something which is forbidden. It however turns out that $\sum_i \min_j U_{ij}$ is not so far off from S_N. A simple computation gives that

$$\mathbb{E}\min_{j} U_{ij} = \frac{1}{N} + o\left(\frac{1}{N}\right),$$

and therefore

$$\mathbb{E}\sum_i \min_j U_{ij} = 1 + o(1).$$

It is a mathematical proved result, that

$$\lim_{N\to\infty} \mathbb{E}S_N = \frac{\pi^2}{6}. \tag{2.4}$$

This was first derived in the physics literature by regarding it as spin glass problem. One introduces a finite temperature model by taking $\beta > 0$, and setting

$$F_{\beta,N} = \frac{1}{N}\log\sum_\pi \exp\left[-\beta N \sum\nolimits_i U_{i\pi(i)}\right],$$

one lets $N \to \infty$, and then divide it by β, and lets $\beta \to \infty$. The outcome from spin glass computation[2] was that the limit is indeed $\pi^2/6$. However, this was by no means a mathematically rigorous proof. A proof of (1.2) was first given by David Aldous in 2001[3]. From the spin glass theory viewpoint however, the problem is not very interesting and rather "trivial", as it does not exhibit the so-called "replica symmetry breaking" like the SK model.

As a last model (there are many others), we consider the so-called perceptron. This is connected with neural networks in artificial intelligence, but we only discuss the mathematical aspects. The problem is to consider M randomly chosen half spaces in $\mathbb{R}^N$, call them $U_1, \ldots, U_M$. By rotational invariance of the standard Gaussian distribution, we can describe them as

$$U_k = \{\mathbf{x} : \mathbf{x}\cdot\mathbf{g}_k \geq 0\},$$

where $\mathbf{g}_k = (g_{k1}, \ldots, g_{kN})$, and the g_{ij} are i.i.d. standard Gaussian random variables.

The problem is to find out for which M, the intersection of $\Sigma_N = \{-1,1\}^N$ with all the half spaces is empty. One computation is easy:

$$\mathbb{E}\left|\Sigma_N \cap \bigcap\nolimits_{k=1}^M U_k\right| = 2^{N-M}.$$

Therefore, if $M = [\alpha N]$, $N \to \infty$, and $\alpha > 1$, then this expectation is exponentially small in N, and this implies by Borel-Cantelli, that $\Sigma_N \cap \bigcap_{k=1}^M U_k = \emptyset$ for large N, $\mathbb{P}$-a.s. On the other hand, if $\alpha < 1$, then the expected number is growing exponentially in N, but some reflection shows that this does not necessary mean that $\Sigma_N \cap \bigcap_{k=1}^M U_k \neq \emptyset$ with large probability. In fact, it is now prove by Talagrand that there exists a number $\alpha_0 < 1$ such that this set is empty for large N, provided $\alpha > \alpha_0$, and non-empty, for $\alpha < \alpha_0$. Again one formulates it first as a finite temperature problem by looking at the partition function

$$\sum_\sigma \exp\left[\beta \sum_k 1_{\{\sigma\notin U_k\}}\right]$$

[2] Mézard, M., Parisi, G.: *Replicas and optimization.* J. Physique Lett **46** (1985)

[3] Aldous, D.: *The* $\zeta(2)$ *limit in the random assignment problem.* Random Structures and Algorithms **18**, 381-418 (2001)

and hopes that one get some information about the original problem when taking β large.

For the SK-model, the variational formula for the free energy, the Parisi formula, is now proved, and therefore also the variational formula for the ground state energy. For many other models like the perceptron, the situation is however much less satisfactory, and one is probably still *very* far from a full understanding of spin glasses. The reason for the success with the SK-model is to a considerably extend hidden in some very special structures the SK has, and which other models don't. Also for the SK, despite the fact that the Parisi formula is proved, there are many fundamental unsolved problems, mostly about the behavior of the Gibbs measure itself.

2.2 First properties of the free energy of the SK model

The partition function is called "quenched" in the physics jargon. In contrast, one has the so-called "annealed" partition function which is just obtained from taking the $\mathbb{E}$-expectation

$$\begin{aligned}
Z_{\beta,h,N}^{\mathrm{ann}} \quad &:= \mathbb{E}Z_{\beta,h,N} \\
&= \sum_{\sigma}\mathbb{E}\exp\left[\frac{\beta}{\sqrt{N}}\sum_{1\leq i<j\leq N}g_{ij}\sigma_i\sigma_j + h\sum_i\sigma_i\right] \\
&= \sum_{\sigma}\exp\left[h\sum_i\sigma_i\right]\mathbb{E}\prod_{i<j}\exp\left[\frac{\beta}{\sqrt{N}}g_{ij}\sigma_i\sigma_j\right] \\
&= \sum_{\sigma}\exp\left[h\sum_i\sigma_i\right]\prod_{i<j}\exp\left[\frac{\beta^2}{2N}\right] \\
&= 2^N\cosh(h)^N\exp\left[\frac{(N-1)\beta^2}{4}\right].
\end{aligned}$$

The annealed free energy in the $N\to\infty$ limit is

$$f^{\mathrm{ann}}(\beta,h) = \lim_{N\to\infty}\frac{1}{N}\log Z_{\beta,h,N}^{\mathrm{ann}} = \frac{\beta^2}{4} + \log\cosh(h) + \log 2.$$

A much more complicated affair is the prove that the quenched free energy exists.

Theorem 2.2

a)

$$f(\beta,h) = \lim_{N\to\infty}\frac{1}{N}\mathbb{E}\log Z_{\beta,h,N} \in \mathbb{R}$$

exists.

b)

$$\lim_{N\to\infty}\frac{1}{N}\log Z_{\beta,h,N} = f(\beta,h), \ \mathbb{P}-\text{a.s.}$$

(This property is usually called "self-averaging" of the free energy).

c) $f(\beta,h)$ *is a convex function of* $(\beta,h)\in\mathbb{R}^+\times\mathbb{R}$.

d)

$$f(\beta,h)\leq f^{\mathrm{ann}}(\beta,h),\ \forall\beta,h$$

Proof. The proof of a) is due to Guerra and Toninelli.[4]

Let $N_1,N_2\in\mathbb{N}$, and $N:=N_1+N_2$. We choose independent standard Gaussians g_{ij},g'_{ij},g''_{ij}, and define for $t\in[0,1]$ the Hamiltonian $H_t(\sigma)$ which depends on the g,g',g'' :

$$\begin{aligned} -H_t(\sigma) \quad : &= \beta\sqrt{\frac{t}{N}}\sum_{1\leq i<j\leq N}g_{ij}\sigma_i\sigma_j+\beta\sqrt{\frac{1-t}{N_1}}\sum_{1\leq i<j\leq N_1}g'_{ij}\sigma_i\sigma_j \qquad (2.5)\\ &+\beta\sqrt{\frac{1-t}{N_2}}\sum_{N_1<i<j\leq N}g''_{ij}\sigma_i\sigma_j+h\sum_{i=1}^{N}\sigma_i. \end{aligned}$$

(We incorporate β into the Hamiltonian). We will need the derivative with respect to t :

$$\begin{aligned} -\frac{dH_t(\sigma)}{dt} \quad &= \quad \frac{\beta}{2}\left\{\sqrt{\frac{1}{tN}}\sum_{1\leq i<j\leq N}g_{ij}\sigma_i\sigma_j\right. \qquad (2.6)\\ &\left.-\sqrt{\frac{1}{(1-t)N_1}}\sum_{1\leq i<j\leq N_1}g'_{ij}\sigma_i\sigma_j-\sqrt{\frac{1}{(1-t)N_2}}\sum_{N_1<i<j\leq N}g''_{ij}\sigma_i\sigma_j\right\} \end{aligned}$$

Then we define the partition function

$$Z(t):=\sum_{\sigma\in\Sigma_N}\exp\left[-H_t(\sigma)\right],$$

and the Gibbs measure $\mathcal{G}_t(\sigma):=\exp\left[-H_t(\sigma)\right]/Z(t)$, with expectation $\mathcal{E}_t$. (Please always remember that these are *quenched* expectations, i.e. they still depend on ω). Evidently, $Z(1)$ is the partition function Z_N we are looking after (depending on the random variables g), and $Z(0)$ is the product of two of our partition function $Z_{N_1}(g')\,Z_{N_2}(g'')$, where the important point is that the factors are independent, as they depend on independent random variables.

Differentiating with respect to t, we get

$$\begin{aligned} \frac{d}{dt}\frac{1}{N}\mathbb{E}\log Z(t) \quad &= \quad \frac{1}{N}\mathbb{E}\frac{1}{Z(t)}\frac{dZ(t)}{dt}\\ &= \quad \frac{1}{N}\sum_{\sigma\in\Sigma_N}\mathbb{E}\frac{\exp\left[-H_t(\sigma)\right]}{Z(t)}\frac{dH_t(\sigma)}{dt}. \end{aligned}$$

For the derivative we implement the expression (2.6) getting

$$\frac{d}{dt}\frac{1}{N}\mathbb{E}\log Z(t)=S_1-S_2-S_3,$$

[4] Guerra, F. and Toninelli F.L.: *The thermodynamic limit in mean field spin glass models.* Comm. Math. Phys. **230**, 71-79 (2002)

where

$$S_1 := \frac{\beta}{2\sqrt{t}N^{3/2}} \sum_{\sigma \in \Sigma_N} \sum_{1 \le i < j \le N} \sigma_i \sigma_j \mathbb{E} g_{ij} \frac{\exp\left[H_t(\sigma)\right]}{Z(t)},$$

and S_2, S_3 are similar terms from the second and third summand of (2.6). As all the g, g', g'' are i.i.d., the Gaussian partial integration gives

$$\begin{aligned}
\mathbb{E} g_{ij} \frac{\exp\left[-H_t(\sigma)\right]}{Z(t)} &= \mathbb{E} \frac{\partial}{\partial g_{ij}} \frac{\exp\left[-H_t(\sigma)\right]}{Z(t)} \\
&= \mathbb{E} \frac{\frac{\partial}{\partial g_{ij}} \exp\left[-H_t(\sigma)\right]}{Z(t)} - \mathbb{E} \frac{\exp\left[-H_t(\sigma)\right]}{Z(t)^2} \frac{\partial Z(t)}{\partial g_{ij}} \\
&= \mathbb{E} \frac{\exp\left[-H_t(\sigma)\right]}{Z(t)} \frac{\partial H_t(\sigma)}{\partial g_{ij}} - \mathbb{E} \frac{\exp\left[-H_t(\sigma)\right]}{Z(t)^2} \frac{\partial Z(t)}{\partial g_{ij}}.
\end{aligned}$$

$$-\frac{\partial H_t(\sigma)}{\partial g_{ij}} = \beta \sqrt{\frac{t}{N}} \sigma_i \sigma_j,$$

$$\frac{\partial Z(t)}{\partial g_{ij}} = \sum_{\sigma \in \Sigma_N} \exp\left[-H_t(\sigma)\right] \frac{\partial\left(-H_t(\sigma)\right)}{\partial g_{ij}} = \beta \sqrt{\frac{t}{N}} \sum_{\sigma \in \Sigma_N} \sigma_i \sigma_j \exp\left[-H_t(\sigma)\right].$$

We therefore get

$$S_1 = \frac{\beta^2}{2N^2} \sum_{\sigma \in \Sigma_N} \sum_{1 \le i < j \le N} \sigma_i \sigma_j \left[\mathbb{E} \frac{\exp\left[-H_t(\sigma)\right]}{Z(t)} \sigma_i \sigma_j - \mathbb{E} \frac{\exp\left[-H_t(\sigma)\right]}{Z(t)^2} \sum_{\sigma' \in \Sigma_N} \sigma'_i \sigma'_j \exp\left[-H_t\left(\sigma'\right)\right] \right],$$

where we have renamed the summation in the last term into σ' to distinguish it from the first summation. In the first part, remark that the σ_i's just appear in squares, which is 1. Therefore, we get

$$\begin{aligned}
S_1 &= \frac{\beta^2}{2N^2} \sum_{1 \le i < j \le N} \mathbb{E} \sum_{\sigma \in \Sigma_N} \frac{\exp\left[-H_t(\sigma)\right]}{Z(t)} \\
&\quad - \frac{\beta^2}{2N^2} \sum_{\sigma \in \Sigma_N} \sum_{1 \le i < j \le N} \sigma_i \sigma_j \mathbb{E} \frac{\exp\left[-H_t(\sigma)\right]}{Z(t)^2} \sum_{\sigma' \in \Sigma_N} \sigma'_i \sigma'_j \exp\left[-H_t\left(\sigma'\right)\right] \\
&= \frac{\beta^2}{2N^2} \frac{N(N-1)}{2} - \frac{\beta^2}{2N^2} \sum_{1 \le i < j \le N} \mathbb{E} \sum_{\sigma, \sigma' \in \Sigma_N} \sigma_i \sigma_j \frac{\exp\left[-H_t(\sigma)\right]}{Z(t)} \sigma'_i \sigma'_j \frac{\exp\left[-H_t(\sigma')\right]}{Z(t)} \\
&= \frac{\beta^2}{2N^2} \frac{N(N-1)}{2} - \frac{\beta^2}{2N^2} \sum_{1 \le i < j \le N} \mathbb{E}\left[\mathcal{E}_t\left(\sigma_i \sigma_j\right)\right]^2.
\end{aligned}$$

We can sum over all i, j, and get

$$S_1 = \frac{\beta^2}{4N^2} \sum_{i,j=1}^{N} \left(1 - \mathbb{E}\left[\mathcal{E}_t\left(\sigma_i \sigma_j\right)\right]^2\right).$$

By a similar computation, one gets

$$\begin{aligned} S_2 &= \frac{\beta^2}{4NN_1} \sum_{i,j=1}^{N_1} \left(1 - \mathbb{E}\left[\mathcal{E}_t\left(\sigma_i\sigma_j\right)\right]^2\right) \\ S_3 &= \frac{\beta^2}{4NN_2} \sum_{i,j=N_1+1}^{N} \left(1 - \mathbb{E}\left[\mathcal{E}_t\left(\sigma_i\sigma_j\right)\right]^2\right). \end{aligned}$$

We can replace $\left[\mathcal{E}_t\left(\sigma_i\sigma_j\right)\right]^2$ by taking expectations of two independent copies of σ : σ, σ' - so called "replicas" - under the product measure $\mathcal{G}_t^{\otimes 2}$ on Σ_N^2. σ, σ' are then just the two projections $\Sigma_N^2 \to \Sigma_N$.

$$\left[\mathcal{E}_t\left(\sigma_i\sigma_j\right)\right]^2 = \mathcal{E}_t^{\otimes 2}\left(\sigma_i\sigma_j\sigma_i'\sigma_j'\right).$$

Then

$$\mathbb{E}\left[\mathcal{E}_t\left(\sigma_i\sigma_j\right)\right]^2 = \int \mathcal{E}_{t,\omega}^{\otimes 2}\left(\sigma_i\sigma_j\sigma_i'\sigma_j'\right)\mathbb{P}\left(d\omega\right).$$

The integrated measure $\int \mathcal{E}_{t,\omega}^{\otimes 2}\,\mathbb{P}\left(d\omega\right)$ is a measure on Σ_N^2, and we write expectations with respect to this measure by $\nu_t^{(2)}$. With these reformulations, we have

$$\begin{aligned} S_1 &= \frac{\beta^2}{4N^2} \sum_{i,j=1}^{N} \left(1 - \nu_t^{(2)}\left(\sigma_i\sigma_j\sigma_i'\sigma_j'\right)\right) \qquad (2.7) \\ &= \frac{\beta^2}{4}\left(1 - \nu_t^{(2)}\left(R_N\left(\sigma,\sigma'\right)^2\right)\right). \end{aligned}$$

Here $R_N\left(\sigma,\sigma'\right)$ is the overlap of σ and σ', defined in (2.3). Similarly, define

$$R^{(1)} := \frac{1}{N_1}\sum\nolimits_{i=1}^{N_1} \sigma_i\sigma_i', \ R^{(2)} := \frac{1}{N_2}\sum\nolimits_{i=N_1+1}^{N} \sigma_i\sigma_i',$$

so that

$$R = \frac{N_1}{N}R_1 + \frac{N_2}{N}R_2. \qquad (2.8)$$

Plugging that into the computation for S_2 and S_3, we get

$$\frac{d}{dt}\frac{1}{N}\mathbb{E}\log Z\left(t\right) = S_1 - S_2 - S_3 = -\frac{\beta^2}{4}\nu_t^{(2)}\left(R^2 - \frac{N_1}{N}R_1^2 - \frac{N_2}{N}R_2^2\right).$$

From (2.8), one gets

$$R^2 \le \frac{N_1}{N}R_1^2 + \frac{N_2}{N}R_2^2$$

and therefore

$$\frac{d}{dt}\frac{1}{N}\mathbb{E}\log Z\left(t\right) \ge 0.$$

From that we conclude

$$\begin{aligned}\frac{1}{N}\mathbb{E}\log Z(1) &\geq \frac{1}{N}\mathbb{E}\log Z(0)\\ \mathbb{E}\frac{1}{N}\log Z_N &\geq \frac{N_1}{N}\mathbb{E}\frac{1}{N_1}\log Z_{N_1}+\frac{N_2}{N}\mathbb{E}\frac{1}{N_2}\log Z_{N_2}.\end{aligned}$$

This is a superadditivity property of the sequence of real numbers $\mathbb{E}\frac{1}{N}\log Z_N$. It is well known and very easy that from this property it follows that

$$f(\beta,h)=\lim_{N\to\infty}\mathbb{E}\frac{1}{N}\log Z_N$$

exists, and equals

$$\sup_N \mathbb{E}\frac{1}{N}\log Z_N.$$

In order to prove a), it only remains to show that this supremum is finite, but this follows from Jensen's inequality

$$\mathbb{E}\frac{1}{N}\log Z_N \leq \frac{1}{N}\log \mathbb{E}Z_N,$$

and the supremum of the latter is finite by the annealed computation we had done before. We in fact have the following annealed bound:

$$f(\beta,h)\leq \frac{\beta^2}{4}+\log\cosh(h)+\log 2.$$

b) This follows by Theorem 1.1 applied to the functions $\varphi:\mathbb{R}^{N(N-1)/2}\to\mathbb{R}$ given by

$$\varphi(x)=\log\sum_\sigma \exp\left[\frac{\beta}{\sqrt{N}}\sum_{1\leq i<j\leq N}x_{ij}\sigma_i\sigma_j+h\sum_{i=1}^N\sigma_i\right].$$

Clearly

$$\begin{aligned}\left|\sum_{1\leq i<j\leq N}x_{ij}\sigma_i\sigma_j-\sum_{1\leq i<j\leq N}y_{ij}\sigma_i\sigma_j\right| &= \left|\sum_{1\leq i<j\leq N}(x_{ij}-y_{ij})\sigma_i\sigma_j\right|\\ &\leq \sqrt{\frac{N(N-1)}{2}}\sqrt{\sum_{1\leq i<j\leq N}(x_{ij}-y_{ij})^2}\\ &= \|x-y\|\sqrt{\frac{N(N-1)}{2}}\leq\frac{N}{\sqrt{2}}\|x-y\|\end{aligned}$$

by the Schwarz inequality, where $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^{N(N-1)/2}$. Therefore, with

$$\psi(x,\sigma)\stackrel{\text{def}}{=}\frac{\beta}{\sqrt{N}}\sum_{1\leq i<j\leq N}x_{ij}\sigma_i\sigma_j+h\sum_{i=1}^N\sigma_i$$

we obtain

$$\exp\left[\psi\left(y,\sigma\right)\right]\exp\left[-\frac{\beta\sqrt{N}}{\sqrt{2}}\left\|x-y\right\|\right]\leq\exp\left[\psi\left(x,\sigma\right)\right]\leq\exp\left[\psi\left(y,\sigma\right)\right]\exp\left[\frac{\beta\sqrt{N}}{\sqrt{2}}\left\|x-y\right\|\right],$$

and therefore

$$\left|\varphi\left(x\right)-\varphi\left(y\right)\right|\leq\frac{\beta\sqrt{N}}{\sqrt{2}}\left\|x-y\right\|,$$

i.e. φ is Lipshitz with

$$\left\|\varphi\right\|_{\mathrm{Lip}}\leq\frac{\beta\sqrt{N}}{\sqrt{2}}.$$

From Theorem 1.1 we obtain

$$\mathbb{P}\left(\left|\frac{1}{N}\log Z_N-\frac{1}{N}\mathbb{E}\log Z_N\right|\geq N^{-1/4}\right)\leq 2\exp\left[-\frac{4\sqrt{N}}{\pi^2\beta^2}\right]. \tag{2.9}$$

As

$$\sum_N\exp\left[-\frac{4\sqrt{N}}{\pi^2\beta^2}\right]<\infty,$$

it follows by the Borel-Cantelli Lemma that with $\mathbb{P}$-probability one, only for finitely many N, one has

$$\left|\frac{1}{N}\log Z_N-\frac{1}{N}\mathbb{E}\log Z_N\right|\geq N^{-1/4},$$

and therefore

$$\lim_{N\to\infty}\frac{1}{N}\log Z_{\beta,h,N}=\lim_{N\to\infty}\frac{1}{N}\mathbb{E}\log Z_{\beta,h,N}=f\left(\beta,h\right),\ \mathbb{P}-\text{a.s.}$$

c): Let $\beta,\beta'>0$, and $h,h'\in\mathbb{R}$, and $\lambda\in[0,1]$. Put $\beta\left(\lambda\right):=\lambda\beta+\left(1-\lambda\right)\beta',\ h\left(\lambda\right):=\lambda h+\left(1-\lambda\right)h'$. Then

$$\begin{aligned}
Z_{\beta(\lambda),h(\lambda),N} &= \sum_\sigma\exp\left[\frac{\lambda\beta+\left(1-\lambda\right)\beta'}{\sqrt{N}}\sum_{i<j}g_{ij}\sigma_i\sigma_j+\big(\lambda h+\left(1-\lambda\right)h'\big)\sum_i\sigma_i\right]\\
&= \sum_\sigma\left\{\exp\left[\frac{\beta}{\sqrt{N}}\sum_{i<j}g_{ij}\sigma_i\sigma_j+h\sum_i\sigma_i\right]\right\}^{\lambda}\\
&\quad\times\left\{\exp\left[\frac{\beta'}{\sqrt{N}}\sum_{i<j}g_{ij}\sigma_i\sigma_j+h'\sum_i\sigma_i\right]\right\}^{1-\lambda}\\
&\leq \left\{\sum_\sigma\exp\left[\frac{\beta}{\sqrt{N}}\sum_{i<j}g_{ij}\sigma_i\sigma_j+h\sum_i\sigma_i\right]\right\}^{\lambda}\\
&\quad\times\left\{\sum_\sigma\exp\left[\frac{\beta'}{\sqrt{N}}\sum_{i<j}g_{ij}\sigma_i\sigma_j+h'\sum_i\sigma_i\right]\right\}^{1-\lambda}
\end{aligned}$$

by the Hölder inequality. Therefore

$$\frac{1}{N}\log Z_{\beta(\lambda),h(\lambda),N} \leq \lambda \frac{1}{N}\log Z_{\beta,h,N} + (1-\lambda)\frac{1}{N}\log Z_{\beta',h',N}.$$

Going the the $N \to \infty$ limit, we get

$$f(\beta(\lambda), h(\lambda)) \leq \lambda f(\beta, h) + (1-\lambda) f(\beta', h').$$

d) is obvious from Jensen's inequality. ■

Exercise 2.3
Replace the Gaussian variables g_{ij} in the Hamiltonian by i.i.d. symmetric Bernoulli variables ξ_{ij} taking values ± 1. Write $Z_{\beta,h,N}^{\text{Bernoulli}}$ for the corresponding partition function. Prove that

$$\lim_{N\to\infty} \frac{1}{N}\mathbb{E}\log Z_{\beta,h,N}^{\text{Bernoulli}} = f(\beta, h)$$

for all β, h, where the expectation on the left hand side is with respect to the Bernoulli-variables ξ_{ij}, and the right hand side is the SK free energy.
Hint: *Interpolate in a suitable way between the SK-Hamiltonian and the Bernoulli one, and try to control the derivative.*

In physics jargon one calls the property b) the *self-averaging* of the free energy. This means that for $N \to \infty$, it does not keep any randomness. It will be shown later, that many properties of the SK-model shouldn't be self-averaging (although this is not really proved mathematically).

Another question is whether the free energy $f(\beta, h)$ equals the annealed free energy. If this happens, one typically says that the disorder is irrelevant. It will turn out that this is never the case if $h \neq 0$, but it is true if $h = 0$ and β is small. This was first proved by Aizenman, Lebowitz and Ruelle[5]

Theorem 2.4
Assume $h = 0$ and $\beta \leq 1$. Then

$$f(\beta, 0) = \frac{\beta^2}{4} + \log 2.$$

Proof. The proof is based on the so-called "second moment method". We compute

[5] Aizenman, M., Lebowitz, J. and Ruelle, D. *Some rigorous results on the Sherrington-Kirpatrick model.* Comm. Math. Phys. **112,** 3-20 (1987)

$\mathbb{E}Z^2$:

$$\begin{aligned}
\mathbb{E}Z_{\beta,0,N}^2 &= \sum_{\sigma,\tau}\mathbb{E}\exp\left[\frac{\beta}{\sqrt{N}}\sum_{1\le i<j\le N} g_{ij}\left(\sigma_i\sigma_j+\tau_i\tau_j\right)\right] \\
&= \sum_{\sigma,\tau}\exp\left[\frac{\beta^2}{2N}\sum_{1\le i<j\le N}\left(\sigma_i\sigma_j+\tau_i\tau_j\right)^2\right] \\
&= \sum_{\sigma,\tau}\exp\left[\frac{\beta^2}{N}\sum_{1\le i<j\le N}\left(1+\sigma_i\sigma_j\tau_i\tau_j\right)\right] \\
&= 2^{2N}\exp\left[\frac{\beta^2\left(N-1\right)}{2}\right]2^{-2N}\sum_{\sigma,\tau}\exp\left[\frac{\beta^2}{N}\sum_{1\le i<j\le N}\sigma_i\sigma_j\tau_i\tau_j\right] \\
&= 2^{2N}\exp\left[\frac{\beta^2\left(N-1\right)}{2}\right]2^{-2N}\sum_{\sigma,\tau}\exp\left[\frac{\beta^2}{2N}\left(\sum\nolimits_i\sigma_i\tau_i\right)^2-\frac{\beta^2}{2}\right] \\
&= \mathrm{e}^{-\beta^2/2}2^{2N}\exp\left[\frac{\beta^2\left(N-1\right)}{2}\right]2^{-2N}\sum_{\sigma,\tau}\exp\left[\frac{\beta^2}{2N}\left(\sum\nolimits_i\sigma_i\tau_i\right)^2\right].
\end{aligned}$$

The σ,τ-sum with the 2^{-2N} in front is just an expectation over two independent coin tossing sequence, and then $\sigma_i\tau_i$ under this measure has just the same distribution as a single coin tossing. Therefore

$$2^{-2N}\sum_{\sigma,\tau}\exp\left[\frac{\beta^2}{2N}\left(\sum\nolimits_i\sigma_i\tau_i\right)^2\right]=2^{-N}\sum_{\sigma}\exp\left[\frac{\beta^2}{2N}\left(\sum\nolimits_i\sigma_i\right)^2\right].$$

This is exactly the partition function of the Curie-Weiss model with an additional 2^{-N} in front, and β replaced by $\beta^2/2$. Therefore

$$\lim_{N\to\infty}\frac{1}{N}\log 2^{-N}\sum_{\sigma}\exp\left[\frac{\beta^2}{2N}\left(\sum\nolimits_i\sigma_i\right)^2\right]=\sup_{x\in[-1,1]}\left[\frac{\beta^2}{2}x^2-y\left(x\right)\right]=0$$

for $\beta^2/2\le 1/2$, i.e. $\beta\le 1$. It is in fact not difficult to prove (using the trick with the Gaussian variable at the end of Section **??** for instance) that for $\beta<1$, one has

$$\sup_N 2^{-N}\sum_{\sigma}\exp\left[\frac{\beta^2}{2N}\left(\sum\nolimits_i\sigma_i\right)^2\right]\le C\left(\beta\right)<\infty.$$

It can be proved by carefully evaluating Stirling's formula. In the next section, we however need a similar result in a more complicated situation which can no longer be handled by explicit computation, so I present here an argument which will work there, too.

The trick is to remove the square by an extra Gaussian integration, relying on the fact that

$$\mathrm{e}^{a^2/2} = E\left(\mathrm{e}^{aZ}\right)$$

for a standard Gaussian variable Z, which is evident by completing squares in the exponent

$$E\left(\mathrm{e}^{aZ}\right) = \frac{1}{\sqrt{2\pi}} \int \exp\left[az - z^2/2\right] dz.$$

Therefore, we have[6]

$$\exp\left[\frac{\beta^2}{2N}\left(\sum\nolimits_i \sigma_i\right)^2\right] = E\left(\exp\left[\frac{\beta}{\sqrt{N}} Z \sum\nolimits_i \sigma_i\right]\right).$$

The σ-summation can now easily be done individually on the σ_i, leading to

$$\begin{aligned} 2^{-N} \sum_\sigma \exp\left[\frac{\beta^2}{2N}\left(\sum\nolimits_i \sigma_i\right)^2\right] &= E \cosh^N\left(\frac{\beta}{\sqrt{N}} Z\right) \\ &= E \exp\left[N \log\cosh\left(\frac{\beta}{\sqrt{N}} Z\right)\right]. \end{aligned}$$

Now,

$$\frac{d \log\cosh(x)}{dx} = 1 - \tanh^2(x) \leq 1,$$

and so

$$\log\cosh(x) \leq x^2/2,$$

$$E \exp\left[N \log\cosh\left(\frac{\beta}{\sqrt{N}} Z\right)\right] \leq E \exp\left[\frac{\beta^2}{2} Z^2\right] =: C(\beta) < \infty,$$

if $\beta < 1$. Therefore, we have for $\beta < 1$

$$\begin{aligned} \mathbb{E} Z_{\beta,0,N}^2 &\leq C(\beta)\, 2^{2N} \exp\left[\frac{\beta^2(N-1)}{2}\right] \\ &= C(\beta)\left(\mathbb{E} Z_{\beta,0,N}\right)^2. \end{aligned}$$

Let $A_N := \{Z_N \geq \mathbb{E} Z_N/2\}$. Then

$$\mathbb{E} Z_N = \mathbb{E}\left(Z_N; A_N^c\right) + \mathbb{E}\left(Z_N; A_N\right) \leq \frac{\mathbb{E} Z_N}{2} + \sqrt{\mathbb{E}\left(Z_N^2\right) \mathbb{P}\left(A_N\right)},$$

and therefore

$$\mathbb{P}(A_N) \geq \frac{\left(\mathbb{E} Z_N\right)^2}{4\mathbb{E}\left(Z_N^2\right)} \geq C(\beta) > 0,$$

[6] The trick is widely used in physics, and sometimes is called "Hubbard-Stratonovich transformation". The physicist Res Jost (1918-1990) used the call it the "Babylonian trick", because the Babylonians invented the method of completing squares.

i.e.

$$\mathbb{P}\left(\frac{1}{N}\log Z_N \geq \frac{1}{N}\log \mathbb{E}Z_N - \frac{\log 2}{N}\right) \geq C(\beta).$$

Combining with (2.9), we see that

$$f(\beta, 0) = \lim_{N\to\infty} \frac{1}{N}\mathbb{E}\log Z_N \geq \lim_{N\to\infty} \frac{1}{N}\log \mathbb{E}Z_N = \frac{\beta^2}{4} + \log 2$$

for $\beta < 1$. Using Theorem 2.2 d), we conclude that $f(\beta, 0) = \beta^2/4 + \log 2$ for $\beta < 1$. The same holds true for $\beta = 1$ because of the convexity of f which implies continuity, as f is bounded. ∎

We will see in Section 2.3 that $f(\beta, 0) < \beta^2/4$ for $\beta > 1$.

2.3 The high temperature phase of the SK-model.

2.3.1 A toy computation

In order to motivate the form of the limiting free energy in the high-temperature phase, we first look back at the Curie-Weiss model with an external field, i.e. where we have the Hamiltonian (packing β, h inside the Hamiltonian)

$$-H(\sigma) = \frac{\beta}{N}\sum_{1\leq i<j\leq N}\sigma_i\sigma_j + h\sum_{i=1}^N \sigma_i.$$

We assume $h > 0$. There we proved that under the Gibbs measure, the σ_i are essentially i.i.d. with a mean m given by the mean-field equation (1.5)

$$m = \tanh(h + \beta m).$$

(We drop β in front of h in accordance with our habit in the SK-model).

We argue now that in the SK-model, the σ_i are under the Gibbs distribution still approximately independent, at least for β small, but there is certainly no reason why the means of the spins should be the same, or even non-random. In fact, what finally turns out is that the means

$$m_i \stackrel{\text{def}}{=} \mathcal{E}(\sigma_i),$$

are random variables, which itself (under the probability measure of the disorder $\mathbb{P}$) are approximately i.i.d. To justify all that takes quite some efforts, and we probably have no time to discuss this in details.

Assuming that for the moment, we make a toy computation, where we assume that there is only a random interaction of the first spin σ_1 with the others $\sigma_2, \ldots, \sigma_N$, but the spins come with a deterministic a-priori tilt, not necessarily the same for the different spins. Therefore, we look at the Hamiltonian

$$-H(\sigma) = \frac{\beta}{\sqrt{N}}\sum_{j=2}^N g_{1j}\sigma_1\sigma_j + \sum_{i=1}^N h_i\sigma_i.$$

If the interaction would not be present, then the means of the spins would simply be $\tanh(h_i)$. We compute now (in the $N \to \infty$ limit) the mean of the first spin and compare it with $\tanh(h_1)$.

$$\mathcal{E}(\sigma_1) = \frac{\sum_\sigma \sigma_1 \exp[-H(\sigma)]}{\sum_\sigma \exp[-H(\sigma)]}.$$

In the computation of the sum, we can easily sum out $\sigma_2, \ldots, \sigma_N$:

$$\sum_{\sigma_2,\ldots,\sigma_N} \exp[-H(\sigma)] = 2^{N-1} \mathrm{e}^{h_1\sigma_1} \prod_{j=2}^{N} \cosh\left(h_j + \frac{\beta}{\sqrt{N}} g_{1j}\sigma_1\right).$$

We expand cosh up to second order:

$$\begin{aligned}\cosh\left(h_j + \frac{\beta}{\sqrt{N}} g_{1j}\sigma_1\right) &= \cosh(h_j) + \frac{\beta}{\sqrt{N}} g_{1j}\sigma_1 \sinh(h_j) + \frac{\beta^2}{2N} g_{1j}^2 \cosh(h_j) + O\left(N^{-3/2}\right) \\ &= \cosh(h_j)\left(1 + \frac{\beta^2}{2N} g_{1j}^2 + \frac{\beta}{\sqrt{N}} g_{1j}\sigma_1 \tanh(h_j)\right) + O\left(N^{-3/2}\right).\end{aligned}$$

The $\cosh(h_j)$ evidently cancels out in the computation of $\mathcal{E}(\sigma_1)$, and the $O\left(N^{-3/2}\right)$ plays no rôle in the $N \to \infty$ limit. Therefore

$$\mathcal{E}(\sigma_1) \approx \frac{\sum_{\sigma_1} \sigma_1 \exp\left[h_1\sigma_1 + \sum_{j=2}^{N} \log\left(1 + \frac{\beta^2}{2N} g_{1j}^2 + \frac{\beta}{\sqrt{N}} g_{1j}\sigma_1 \tanh(h_j)\right)\right]}{\sum_{\sigma_1} \exp\left[h_1\sigma_1 + \sum_{j=2}^{N} \log\left(1 + \frac{\beta^2}{2N} g_{1j}^2 + \frac{\beta}{\sqrt{N}} g_{1j}\sigma_1 \tanh(h_j)\right)\right].}.$$

Expanding the logarithm again up to $O\left(N^{-3/2}\right)$, we see that all the terms not containing σ_1 cancel out and we finally get

$$\mathcal{E}(\sigma_1) \approx \tanh\left(h_1 + \frac{\beta}{\sqrt{N}} \sum\nolimits_{j=2}^{N} g_{1j} \tanh(h_j)\right).$$

One should no remember that $\tanh(h_j)$ is the mean of σ_j *in absence* of the interaction with the first spin. The physicists now argue that the same formula should be true, namely that

$$m_i \approx \tanh\left(h + \frac{\beta}{\sqrt{N}} \sum_{j:j\neq i} g_{ij} m_j^{(i)}\right), \tag{2.10}$$

where $m_j^{(i)}$ is computed as the mean under the Gibbs distribution where the interactions with the i-th spin are put to 0. Furthermore, we have put $g_{ij} \stackrel{\text{def}}{=} g_{ji}$ for $i > j$. For convenience, one usually sets $g_{ii} = 0$.

The $m_j^{(i)}$ can be approximated by the m_j itself through a correction of order $N^{-1/2}$. This correction cannot be neglected, even in the $N \to \infty$ limit, and it leads to an equation for the sequence of means, with what is called an Onsager correction term. This is not of importance for the moment, but here is the outcome:

$$m_i \approx \tanh\left(h + \frac{\beta}{\sqrt{N}} \sum_{j:j\neq i} g_{ij} m_j - \frac{\beta^2}{N} m_i \sum_{j=1}^{N} m_j^2\right).$$

These are the celebrated TAP-equations. If time allows, we will discuss them later.

Anyway, stepping back to (2.10), one sees that in

$$\frac{1}{\sqrt{N}} \sum_{j:j\neq i} g_{ij} m_j^{(i)}, \tag{2.11}$$

the $m_j^{(i)}$ are independent of the g_{ij}. Therefore, conditionally on the $\left\{m_j^{(i)}\right\}$, the expression is Gaussian with variance

$$\frac{1}{N} \sum_{j:j\neq i} m_j^{(i)2} \approx \frac{1}{N} \sum_{j=1}^{N} m_j^{(i)2}.$$

It is not difficult to see that here the correction of the replacement of $m_j^{(i)}$ by m_j should not be of importance. Arguing now, that, at least in the high temperature regime, the m_i are not "far from" being i.i.d. under $\mathbb{P}$, one would get that the above expression is a constant $q = q(\beta, h)$:

$$\frac{1}{N} \sum_{j:j\neq i} m_j^{(i)2} \approx \frac{1}{N} \sum_{j} m_j^2 \approx q(\beta, h).$$

The expression (2.11) should then simply be $\sqrt{q}$ times a Gaussian. (2.10) would then state that

$$m_i \approx \tanh\left(h + \beta\sqrt{q} Z_i\right) \tag{2.12}$$

with (approximately independent) standard Gaussians Z_i. Therefore, we get the self-consistency equation for q :

$$q(\beta, h) = \lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} m_j^2 = \int \tanh^2\left(h + \beta\sqrt{q} z\right) \frac{1}{\sqrt{2\pi}} \exp\left[-z^2/2\right] dz.$$

For $h > 0$, this equation for q has a unique solution:

Lemma 2.5
Let $\beta, h > 0$ be arbitrary. Then the equation

$$q = \int \tanh^2\left(h + \beta\sqrt{q} z\right) \frac{1}{\sqrt{2\pi}} \exp\left[-z^2/2\right] dz \tag{2.13}$$

has a unique solution $q(\beta, h) \geq 0$.

Proof. For notational convenience we write g for a standard Gaussian random variable. Then the right hand side of (**??**) writes as $E\tanh^2\left(h+\beta\sqrt{q}g\right)$. Define the function

$$f\left(x\right):=\frac{1}{x}E\tanh^2\left(h+\sqrt{x}g\right)$$

on $\mathbb{R}^+$.

$$\begin{aligned}
f'\left(x\right) &= -\frac{1}{x^2}E\tanh^2\left(h+\sqrt{x}g\right)+\frac{1}{x^{3/2}}E\left[\tanh\left(h+\sqrt{x}g\right)\tanh'\left(h+\sqrt{x}g\right)\right]\\
&= -\frac{1}{x^2}E\tanh^2\left(h+\sqrt{x}g\right)+\frac{1}{x^2}E\left[\sqrt{x}g\frac{\tanh\left(h+\sqrt{x}g\right)}{\cosh^2\left(h+\sqrt{x}g\right)}\right]\\
&= -\frac{1}{x^2}E\tanh^2\left(h+\sqrt{x}g\right)+\frac{1}{x^2}E\left[\left(h+\sqrt{x}g\right)\frac{\tanh\left(h+\sqrt{x}g\right)}{\cosh^2\left(h+\sqrt{x}g\right)}\right]-\frac{h}{x^2}E\left[\frac{\tanh\left(h+\sqrt{x}g\right)}{\cosh^2\left(h+\sqrt{x}g\right)}\right]
\end{aligned}$$

If we define the random variable $Y:=h+\sqrt{x}g$, we get

$$x^2f'\left(x\right)=E\left[Y\frac{\tanh Y}{\cosh^2 Y}-\tanh^2 Y\right]-hE\left[\frac{\tanh Y}{\cosh^2 Y}\right]. \tag{2.14}$$

Now, for any $y>0$ one has

$$y<\sinh y\cosh y,$$

as the reader may check himself, and therefore

$$\frac{y\tanh y}{\cosh^2 y}<\tanh^2 y.$$

As both sides are even functions, it holds true for all y, except at $y=0$ where both sides are 0, and therefore, the first summand on the right hand side of (2.14) is strictly negative. As to the second, we remark that Y is normally distributed with mean h and variance x, and therefore

$$\begin{aligned}
E\left[\frac{\tanh Y}{\cosh^2 Y}\right] &= \frac{1}{\sqrt{2\pi x}}\int\frac{\tanh y}{\cosh^2 y}\exp\left[-\frac{1}{2x}\left(y^2-2hy+h^2\right)\right]dy\\
&= \frac{1}{\sqrt{2\pi x}}\int\frac{\tanh y\sinh\left(hy/x\right)}{\cosh^2 y}\exp\left[-\frac{1}{2x}\left(y^2+h^2\right)\right]dy\geq 0
\end{aligned}$$

the second equation because $\tanh\left(y\right)$ is odd.

We conclude that f is strictly decreasing on $\mathbb{R}^+$. As $\lim_{x\to 0}f\left(x\right)=\infty$, and $\lim_{x\to\infty}f\left(x\right)=0$, we conclude that there is a unique solution of $f\left(x\right)=1$. $\blacksquare$

Assuming that the above explained picture is correct, namely that for small β, the σ_i under the Gibbs distribution are approximately independent with means m_i, where the m_i, under $\mathbb{P}$, are approximately i.i.d. with $\mathbb{E}\left(m_i^2\right)=q$, we can approximate q directly from the spin variables, but we have to take a replicated system. Therefore take σ,σ' under the product measure $\mathcal{G}_\omega^{\otimes 2}\overset{\text{def}}{=}\mathcal{G}_\omega\otimes\mathcal{G}_\omega$ with fixed disorder ω, we get that

$$\frac{1}{N}\sum_{i=1}^{N}\sigma_i\sigma_i'\approx\frac{1}{N}\sum_{i=1}^{N}m_i^2\approx q.$$

For this reason, a key ingredient of the discussion always is to prove that

$$\frac{1}{N}\sum_{i=1}^{N}\sigma_i\sigma_i'-q$$

is small under the measure $\nu^{(2)}(\sigma,\sigma')\stackrel{\text{def}}{=}\int\mathcal{G}_\omega^{\otimes 2}(\sigma,\sigma')\,\mathbb{P}(d\omega)$. This actually is not true for large β, but it is for small β.

2.3.2 Guerra's replica symmetric upper bound

The original claim by Sherrington-Kirkpatrick was that

$$f(\beta,h)=\mathrm{RS}(\beta,h)\stackrel{\text{def}}{=}\inf_{q\geq 0}\left\{\frac{(1-q)^2\beta^2}{4}+E\log\cosh\left(h+\beta\sqrt{q}Z\right)+\log 2\right\}.$$

We will later see that this is correct for small enough β, but is wrong for large β. It is readily checked that the infimum in q satisfies the fixed point equation (2.13).

Guerra's idea was to try a simple comparison of the system with SK-Hamiltonian with a simple Hamiltonian with independent spins, but the correct distribution of the tilts, namely given by (2.12). The first result was the following remarkable bound:

Theorem 2.6 (Guerra)
For all $\beta>0$, $h\in\mathbb{R}$, and any N, one has

$$\frac{1}{N}\mathbb{E}\log Z_{\beta,h,N}\leq\mathrm{RS}(\beta,h),$$

and in particular

$$f(\beta,h)\leq\mathrm{RS}(\beta,h).$$

Proof. [7]The proof is again by interpolation quite similar as in (2.5). Let for an arbitrary number $q\geq 0$, and $t\in[0,1]$

$$-H_t(\sigma)=\beta\sqrt{\frac{t}{N}}\sum_{1\leq i<j\leq N}g_{ij}\sigma_i\sigma_j+\beta\sqrt{1-t}\sum_{i=1}^{N}\sqrt{q}g_i\sigma_i+h\sum_{i=1}^{N}\sigma_i \tag{2.15}$$

where g_i is a set of standard Gaussian variables, independent of the g_{ij}'s. Remark that this is interpolating between the SK-Hamiltonian, and the above mentioned Hamiltonian with independent spins.

For the moment, we have not even to assume that q is the right one, and we can just take it arbitrary ≥ 0. We write again

$$Z_N(t)=\sum_\sigma\exp\left[-H_t(\sigma)\right],\ \mathcal{G}_t(\sigma)=\frac{\exp\left[-H_t(\sigma)\right]}{Z(t)}, \tag{2.16}$$

[7]The proof was first presented by Francesco Guerra at a conference on Vulcano in 1998.

$$\phi(t) = \frac{1}{N}\mathbb{E}\log Z_N(t). \tag{2.17}$$

Remark that

$$\begin{aligned}
\phi(0) &= \int \log\cosh\left(\beta\sqrt{q}x + h\right)\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-x^2/2}dx + \log 2, \\
\phi(1) &= \frac{1}{N}\mathbb{E}\log Z_{\beta,h,N}
\end{aligned}$$

We again compute the derivative of $\phi(t)$ with respect to t. There is only a slight variation of the computation in Section 2.2. The derivative of the Hamiltonian is

$$\frac{d(-H_t)}{dt} = \frac{\beta}{2\sqrt{tN}}\sum_{1\le i<j\le N} g_{ij}\sigma_i\sigma_j - \frac{\beta}{2\sqrt{(1-t)}}\sum_{i=1}^{N}\sqrt{q}g_i\sigma_i$$

leading to

$$\frac{d\phi}{dt} = S_1 - S_2,$$

where the computation for S_1 is exactly the same as in (2.7), giving

$$S_1 = \frac{\beta^2}{4}\left(1 - \nu_t^{(2)}\left(R_N^2\right)\right),$$

with

$$R_N\left(\sigma,\sigma'\right) = \frac{1}{N}\sum_{i=1}^{N}\sigma_i\sigma_i'.$$

The computation of S_2 is similar but the outcome is slightly different from the one in Section 2.2:

$$\begin{aligned}
S_2 &= \frac{\beta}{2N\sqrt{1-t}}\mathbb{E}\left(\sum_\sigma \frac{1}{Z(t)}\sum\nolimits_i \sqrt{q}g_i\sigma_i \exp\left[-H_t(\sigma)\right]\right) \\
&= \frac{\beta\sqrt{q}}{2N\sqrt{1-t}}\mathbb{E}\sum_\sigma\sum\nolimits_i \sigma_i\left[\left(-\frac{\exp\left[-H_t(\sigma)\right]}{Z(t)^2}\frac{\partial Z(t)}{\partial g_i}\right) + \frac{\exp\left[-H_t(\sigma)\right]}{Z(t)}\frac{\partial\left(-H_t(\sigma)\right)}{\partial g_i}\right] \\
&= \frac{\beta^2 q}{2}\mathbb{E}\left[-\sum_\sigma\sum_\tau R_N(\sigma,\tau)\frac{\exp\left[-H_t(\sigma)\right]\exp\left[-H_t(\tau)\right]}{Z(t)^2} + \sum_\sigma\frac{\exp\left[-H_t(\sigma)\right]}{Z(t)}\right] \\
&= \frac{\beta^2 q}{2}\left(1 - \nu_t^{(2)}(R_N)\right).
\end{aligned}$$

Therefore, we get

$$\begin{aligned}
\frac{d\phi}{dt} &= \frac{\beta^2}{4}\nu_t^{(2)}\left\{1 - R_N^2 - 2q\left(1 - R_N\right)\right\} \\
&= \frac{\beta^2}{4}\left\{(1-q)^2 - \nu_t^{(2)}\left[\left(R_N(\sigma,\tau) - q\right)^2\right]\right\},
\end{aligned}$$

which integrated gives

$$\phi(t) - \phi(0) = \frac{\beta^2 t}{4}(1-q)^2 - \frac{\beta^2}{4}\int_0^t \nu_s^{(2)}\left[(R_N(\sigma,\tau)-q)^2\right] ds, \tag{2.18}$$

and dropping the second summand and taking $t = 1$:

$$\phi(1) - \phi(0) \leq \frac{\beta^2}{4}(1-q)^2 .$$

This implies that for *any* N, we have

$$\frac{1}{N}\mathbb{E}\log Z_{\beta,h,N} \leq \mathrm{RS}(\beta,h).$$

■

The proof does not only give the desired result, but gives also an expression of the difference, namely

$$\mathrm{RS}(\beta,h) - \frac{1}{N}\mathbb{E}\log Z_{\beta,h,N} = \frac{\beta^2}{4}\int_0^1 \nu_t^{(2)}\left[(R_N(\sigma,\tau)-q)^2\right] dt \tag{2.19}$$

In order to prove that $f(\beta,h) = \mathrm{RS}(\beta,h)$, one therefore "only" has to show that for the optimal q (i.e. the one given by (**??**)), one has $R_N(\sigma,\tau) \simeq q$ with large $\nu_s^{(2)}$-probability, at least in the t-average. This is not true for large β, but it is true for small β, as we will prove in the next section.

It should also be remarked that Guerra's bound already proves that $f(\beta,0) < \beta^2/4$ for $\beta > 1$. Up to $\beta = 1$, the unique fixed point of (2.13) with $h = 0$ is at $q = 0$ which gives $\mathrm{RS}(\beta,0) = \beta^2/4$ for $\beta \leq 1$, but for $\beta > 1$, there is a fixed point at $q > 0$ which gives a smaller value, so $\mathrm{RS}(\beta,0) < \beta^2/4$ and Guerra's bound proves that $f(\beta,0) \neq f^{\mathrm{ann}}(\beta,0)$, as soon as $\beta > 1$. This was first proved by Comets[8] with a more complicated argument.

2.3.3 The free energy in the high temperature case, and for non-zero external field: Quadratic replica coupling

In this chapter, we prove that $\mathrm{RS}(\beta,h)$ *is* the free energy of the SK-model provided β is small enough. The line separating this high temperature region from the low temperature one is believed to be given by the famous de Almeyda-Thouless line: $f(\beta,h) = \mathrm{RS}(\beta,h)$ should be correct provided

$$\beta^2 \int \frac{1}{\cosh^4\left(h+\beta\sqrt{q}x\right)}\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-x^2/2}dx \leq 1, \tag{2.20}$$

where $q(\beta,h)$ is the solution of the (2.13). This is mathematically an open problem.

[8] Comets, F. *A spherical bound for the Sherrington-Kirkpatrick model.* Astérisque **236** (1996), 103–108.

Theorem 2.7
$f(\beta,h)=\mathrm{RS}(\beta,h)$ holds for $\beta<1$.

Unfortunately, the proof does not cover the full AT-region (2.20). Talagrand has worked much on this problem and the results in his book characterize the high temperature region completely, but not really explicitly, and without proving that it agrees with the AT-region.

The basic idea how to prove that $f(\beta,h)=\mathrm{RS}(\beta,h)$ for small β is to use the expression (2.19) and try to show that the right hand side goes to zero. There are a number of variants of the method. In principle, it would be desirable to have one which does not rely on special tricks like dropping error terms which by luck are positive. I present here an argument due to Guerra-Toninelli which still very much relies on such tricks.

From (2.18), we see that the "only" thing to prove is that

$$\int_0^1 \nu_t^{(2)}\left[\left(R_N(\sigma,\tau)-q\right)^2\right]dt$$

approaches 0 as $N\to\infty$. We start by estimating $\nu_t^{(2)}\left[\left(R_N(\sigma,\tau)-q\right)^2\right]$ using Jensen: For any positive λ, we have

$$\begin{aligned}
\frac{\lambda}{4}\nu_t^{(2)}\left(R_N\left(\sigma,\sigma'\right)-q\right)^2 &= \frac{\lambda}{4}\mathbb{E}\mathcal{E}_t^{\otimes 2}\left(R_N\left(\sigma,\sigma'\right)-q\right)^2\\
&\le \frac{1}{2N}\mathbb{E}\log\mathcal{E}_t^{\otimes 2}\exp\left[\frac{\lambda}{2}N\left(R_N\left(\sigma,\sigma'\right)-q\right)^2\right]\\
&= \frac{1}{2N}\mathbb{E}\log\frac{\sum_{\sigma,\sigma'}\exp\left[\frac{\lambda}{2}N\left(R_N\left(\sigma,\sigma'\right)-q\right)^2\right]\mathrm{e}^{-H_t(\sigma)-H_t(\sigma')'}}{\sum_{\sigma,\sigma'}\mathrm{e}^{-H_t(\sigma)-H_t(\sigma')'}}\\
&= \frac{1}{2N}\mathbb{E}\log\sum_{\sigma,\sigma'}\exp\left[\frac{\lambda}{2}N\left(R_N\left(\sigma,\sigma'\right)-q\right)^2-H_t(\sigma)-H_t\left(\sigma'\right)\right]\\
&\quad -\frac{1}{2N}\mathbb{E}\log\sum_{\sigma,\sigma'}\mathrm{e}^{-H_t(\sigma)-H_t(\sigma')'}
\end{aligned}$$

To shift the $R_N(\sigma,\sigma')-q$ term into the exponent is of course quite a brutal move, but it turns out to convenient. Remark that the second summand is simply $\phi(t)$. We write

$$Z(t,\lambda)\stackrel{\mathrm{def}}{=}\sum_{\sigma,\sigma'}\exp\left[\frac{\lambda}{2}N\left(R_N\left(\sigma,\sigma'\right)-q\right)^2-H_t(\sigma)-H_t\left(\sigma'\right)\right], \tag{2.21}$$

and

$$\psi(t,\lambda)\stackrel{\mathrm{def}}{=}\frac{1}{2N}\mathbb{E}\log Z(t,\lambda),$$

so that we get the estimate

$$\frac{\lambda}{4}\nu_t^{(2)}\left(R_N\left(\sigma,\sigma'\right)-q\right)^2\le\psi(t,\lambda)-\phi(t). \tag{2.22}$$

We should remark that we can interpret the exponent in (2.21) above as a Hamiltonian, say $\mathcal{H}_{t,\lambda}(\sigma,\sigma')$ on Σ_N^2, where the two copies are coupled together by the quadratic term. We write $\mathcal{G}_{t,\lambda}^{\mathrm{coup}}(\sigma,\sigma')$ for the corresponding Gibbs measure..q will later be chosen to be the unique solution of (2.13), but for the moment, this is of no relevance. We will drop the parameters N,β,h in the notations, as long as we do nothing with them.

We have now to compute the partial derivatives with respect to t and λ. The λ-derivative is easy:

$$\frac{\partial\psi}{\partial\lambda}=\frac{1}{4}\nu_{t,\lambda}^{\mathrm{coup}}\left[\left(R_N(\sigma,\sigma')-q\right)^2\right],$$

where $\nu_{t,\lambda}^{\mathrm{coup}}\stackrel{\mathrm{def}}{=}\mathbb{E}\mathcal{G}_{t,\lambda}^{\mathrm{coup}}$. The computation of the t-derivative is essentially the same as in the last section. Remark however that we already start with two replicas, and so the derivative produces two additional ones. The outcome is

$$\begin{aligned}\frac{\partial\psi}{\partial t} &= \frac{\beta^2}{4}\left\{1+\nu_{t,\lambda}^{\mathrm{coup}}\left[R_N(\sigma,\sigma')^2\right]-2\nu_{t,\lambda}^{\mathrm{coup}(2)}\left[R_N(\sigma,\tau)^2\right]\right\}\\ &\quad -\frac{\beta^2 q}{2}\left\{1+\nu_{t,\lambda}^{\mathrm{coup}}\left[R_N(\sigma,\sigma')\right]-2\nu_{t,\lambda}^{\mathrm{coup}(2)}\left[R_N(\sigma,\tau)\right]\right\}\\ &= \frac{\beta^2}{4}\left\{\nu_{t,\lambda}^{\mathrm{coup}}\left[\left(R_N(\sigma,\sigma')-q\right)^2\right]+(1-q)^2-2\nu_{t,\lambda}^{\mathrm{coup}(2)}\left[\left(R_N(\sigma,\tau)-q\right)^2\right]\right\}\end{aligned}$$

Here we stress a bit the notation: $R_N(\sigma,\tau)$ refers to taking the overlap between two configurations σ,τ, being however part of a duplicated system $((\sigma,\sigma'),(\tau,\tau'))$, and $\nu_{t,\lambda}^{\mathrm{coup}(2)}\left[R_N(\sigma,\tau)^2\right]$ then means

$$\nu_{t,\lambda}^{\mathrm{coup}(2)}\left[R_N(\sigma,\tau)^2\right]\stackrel{\mathrm{def}}{=}\mathbb{E}\sum_{\sigma,\sigma'\tau,\tau'}R_N(\sigma,\tau)^2\frac{1}{Z^2}\mathrm{e}^{-H^{\mathrm{coup}}(\sigma,\sigma')-H^{\mathrm{coup}}(\tau,\tau')}.$$

It should be remarked that σ,σ' are coupled through the quadratic replica coupling in the Gibbs measure, whereas (σ,σ') and (τ,τ') uncoupled.

The clever idea by Guerra and Toninelli was to choose λ dependent on t in an appropriate way. We set $\lambda=\lambda(t)=\delta\beta^2+\beta^2(1-t)$, $\delta>0$ specified later. Remark that for $t=1$, the coupling parameter λ is small but still positive, and for $t=0$, it is large. The trick of this interpolation is to relate the $t=1$ case with still a small coupling to the $t=0$ case with a larger coupling. The large coupling is of course bad for the analysis, but as it is at $t=0$ where the spin variables are independent, we will be able to handle that by tricks we essentially already have done. Anyway, plugging in the expression we have obtained for the derivatives, we get

$$\frac{d\psi(t,\lambda(t))}{dt}=\frac{\beta^2}{4}(1-q)^2-\frac{\beta^2}{2}\nu_{t,\lambda}^{\mathrm{coup}(2)}\left[R_N(\sigma,\tau)^2\right]\leq\frac{\beta^2}{4}(1-q)^2.$$

Therefore

$$\psi(t,\lambda(t))\leq\frac{\beta^2 t}{4}(1-q)^2+\psi(0,\lambda(0)),\tag{2.23}$$

and using (2.22), we have, writing

$$\Delta_N(\beta,h,q,\delta) \stackrel{\text{def}}{=} \psi(0,\lambda(0)) - \phi(0)$$

$$\begin{aligned}
\frac{\lambda(t)}{4}\nu_t^{(2)}\left(R_N(\sigma,\sigma')-q\right)^2 &\leq \frac{\beta^2 t}{4}(1-q)^2 + \psi(0,\lambda(0)) - \phi(t) \\
&= \Delta - [\phi(t)-\phi(0)] + \frac{\beta^2 t}{4}(1-q)^2 \\
&= \Delta + \frac{\beta^2}{4}\int_0^t \nu_s^{(2)}\left(R_N(\sigma,\sigma')-q\right)^2 ds,
\end{aligned}$$

the last equation, using (2.18). For all t, one has $\lambda(t) \geq \delta\beta^2 > 0$, $\forall t \in [0,1]$, and Gronwall's lemma immediately implies

Lemma 2.8
Assume that for some δ,β,h,q, and one has

$$\lim_{N\to\infty} \Delta_N(\beta,h,q,\delta) = 0. \tag{2.24}$$

Then

$$\lim_{N\to\infty}\int_0^1 \nu_s^{(2)}\left(R_N(\sigma,\sigma')-q\right)^2 ds = 0,$$

and consequently

$$f(\beta,h) = \frac{\beta^2(1-q)^2}{4} + \frac{1}{\sqrt{2\pi}}\int \log\cosh\left(h+\beta\sqrt{q}z\right)\mathrm{e}^{-z^2/2}dz.$$

In particular, if (2.24) is true with $q = q(\beta,h)$, then $f(\beta,h) = \mathrm{RS}(\beta,h)$. (Actually, (2.24) is never true unless $q = q(\beta,h)$).

We therefore finish the proof of Theorem 2.7 by proving

Lemma 2.9
If $\lambda_0 = (1+\delta)\beta^2 < 1$.

$$\lim_{N\to\infty}\Delta_N(\beta,h,q(\beta,h),\delta) = 0.$$

Proof. The proof we give is somewhat suboptimal, as it requires $\beta < 1$ also for large h. With *some* additional work one could do better, but it seems to be impossible to prove the result in the full AT-region in the way we proceed.

$$\Delta_N = \frac{1}{2N}\mathbb{E}\log E_{\mathcal{G}_0^{\otimes 2}}\exp\left[\frac{\lambda(0)N}{2}\left(R_N(\sigma,\sigma')-q\right)^2\right],$$

Let's first check what the rôle of (2.13) is: For fixed (g_i), under $\mathcal{G}_0^{\otimes 2} := \mathcal{G}_{t=0}^{\otimes 2}$, the σ_i, σ_i' are independent random variables with distribution

$$\mathcal{G}_0(\sigma_i = 1) = \frac{\mathrm{e}^{\beta\sqrt{q}g_i+h}}{2\cosh\left(\beta\sqrt{q}g_i+h\right)},$$

and therefore with expectation

$$\mathcal{G}_0\left(\sigma_i\right)=\tanh\left(\beta\sqrt{q}g_i+h\right).$$

Under $\mathcal{G}_0^{\otimes 2}$, the $\tau_i := \sigma_i\sigma_i'$ are independent ± 1 random variables with expectation $\tanh^2\left(\beta\sqrt{q}g_i+h\right)$, and therefore

$$\mathcal{E}_0^{\otimes 2}\left(\frac{1}{N}\sum\nolimits_i\tau_i\right)=\frac{1}{N}\sum_i\tanh^2\left(\beta\sqrt{q}g_i+h\right)\simeq\int\tanh^2\left(\beta\sqrt{q}x+h\right)\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-x^2/2}dx$$

with large $\mathbb{P}$-probability, which is exactly q for $q=q\left(\beta,h\right)$. By the LLN, we therefore have that $\frac{1}{N}\sum_i\tau_i-q\simeq 0$ with large $\mathbb{P}\otimes\mathcal{G}_0^{\otimes 2}$-probability. The situation is therefore very similar to the one already encountered in the proof of Theorem 2.4, where the $\sigma_i\sigma_i'$ where symmetric and therefore just coin tossing. Here there is the complication that the tilt is random.

We write Π for the probability $\mathcal{G}_0^{\otimes 2}$, and set

$$m_i:=\tanh^2\left(\beta\sqrt{q}g_i+h\right)=E_\Pi\left(\tau_i\right),$$

$$p_i:=\Pi\left(\tau_i=1\right)=\frac{1+m_i}{2}\geq 1/2.$$

Keep in mind that p_i,m_i depend on the random variables g_i.We use the same trick to linearize the square in an exponent by introducing an auxiliary Gaussian variable, say Z :

$$\begin{aligned}
&E_\Pi\exp\left[\frac{\lambda_0}{2}N\left(R_N\left(\sigma,\sigma'\right)-q\right)^2\right]\\
=\;&E_\Pi\exp\left[\frac{\lambda_0}{2}N\left(\frac{1}{N}\sum\nolimits_{i=1}^N\tau_i-q\right)^2\right]\\
=\;&E_\Pi E_Z\exp\left[\sqrt{\lambda_0 N}Z\left(\frac{1}{N}\sum\nolimits_{i=1}^N\tau_i-q\right)\right] \qquad (2.25)\\
=\;&E_Z\left\{\exp\left[-q\sqrt{\lambda_0 N}Z\right]E_\Pi\exp\left[Z\sqrt{\frac{\lambda_0}{N}}\sum\nolimits_{i=1}^N\tau_i\right]\right\}\\
=\;&E_Z\left\{\exp\left[-q\sqrt{\lambda_0 N}Z\right]\prod\nolimits_{i=1}^N\left[p_i\mathrm{e}^{\sqrt{\lambda_0/N}Z}+\left(1-p_i\right)\mathrm{e}^{-\sqrt{\lambda_0/N}Z}\right]\right\}\\
=\;&E_Z\left\{\exp\left[-q\sqrt{\lambda_0 N}Z\right]\exp\left[\sum\nolimits_{i=1}^N\log\left(p_i\mathrm{e}^{\sqrt{\lambda_0/N}Z}+\left(1-p_i\right)\mathrm{e}^{-\sqrt{\lambda_0/N}Z}\right)\right]\right\}.
\end{aligned}$$

Remark now that for any $p\in(0,1)$ and $\alpha\in\mathbb{R}$ one has

$$\log\left(p\mathrm{e}^\alpha+\left(1-p\right)\mathrm{e}^{-\alpha}\right)\leq\left(2p-1\right)\alpha+\frac{\alpha^2}{2}. \qquad (2.26)$$

To see this, we compute the first two derivatives

$$\begin{aligned}\frac{d}{d\alpha}\log\left(p\mathrm{e}^{\alpha}+(1-p)\,\mathrm{e}^{-\alpha}\right) &= \frac{p\mathrm{e}^{\alpha}-(1-p)\,\mathrm{e}^{-\alpha}}{p\mathrm{e}^{\alpha}+(1-p)\,\mathrm{e}^{-\alpha}}\\ \frac{d^2}{d\alpha^2}\log\left(p\mathrm{e}^{\alpha}+(1-p)\,\mathrm{e}^{-\alpha}\right) &= 1-\left(\frac{p\mathrm{e}^{\alpha}-(1-p)\,\mathrm{e}^{-\alpha}}{p\mathrm{e}^{\alpha}+(1-p)\,\mathrm{e}^{-\alpha}}\right)^2\leq 1.\end{aligned}$$

As the derivative at 0 is $2p-1$, (2.26) follows. Implementing in (2.25), we get

$$\begin{aligned}&E_{\Pi}\exp\left[\frac{\lambda_0}{2}N\left(R_N\left(\sigma,\sigma'\right)-q\right)^2\right]\\ \leq\ & E_Z\left\{\exp\left[-q\sqrt{\lambda_0 N}Z\right]\exp\left[Z\sum\nolimits_{i=1}^N m_i\sqrt{\frac{\lambda_0}{N}}+\frac{\lambda_0}{2}Z^2\right]\right\}\\ =\ & E_Z\left\{\exp\left[Z\sqrt{\lambda_0 N}\left(\frac{1}{N}\sum\nolimits_{i=1}^N m_i-q\right)+\frac{\lambda_0}{2}Z^2\right]\right\}\\ =\ & \frac{1}{\sqrt{2\pi}}\int\exp\left[x\sqrt{\lambda_0 N}\left(\frac{1}{N}\sum\nolimits_{i=1}^N m_i-q\right)+\frac{\lambda_0-1}{2}x^2\right]dx.\end{aligned}$$

The integral is convergent for $\lambda_0<1$, and gives there

$$\frac{1}{\sqrt{1-\lambda_0}}\exp\left[\frac{\lambda_0 N}{2\left(1-\lambda_0\right)}\left(\frac{1}{N}\sum\nolimits_{i=1}^N m_i-q\right)^2\right].$$

Summarizing, we get

$$\begin{aligned}\Delta_N\left(\lambda_0,q\right) &= \frac{1}{2N}\mathbb{E}\log\mathcal{E}_{t=0}^{\otimes 2}\exp\left[\frac{\lambda_0}{2}N\left(R_N\left(\sigma,\sigma'\right)-q\right)^2\right]\\ &\leq \frac{1}{2N}\mathbb{E}\log\frac{1}{\sqrt{1-\lambda_0}}\exp\left[\frac{\lambda_0 N}{2\left(1-\lambda_0\right)}\left(\frac{1}{N}\sum\nolimits_{i=1}^N m_i-q\right)^2\right]\\ &= \frac{1}{2N}\log\frac{1}{\sqrt{1-\lambda_0}}+\frac{\lambda_0}{4\left(1-\lambda_0\right)}\mathbb{E}\left(\frac{1}{N}\sum\nolimits_{i=1}^N m_i-q\right)^2\end{aligned}$$

which converges to 0, provided $\lambda_0<1$. ■

It is evident that the lemma finishes the proof of Theorem 2.7: If $\beta<1$, we can choose $\delta>0$ such that $(1+\delta)\,\beta^2<1$. Then we can apply Lemma 2.9, and then Lemma 2.8.

3 Ruelle's probability cascades

3.1 Derrida's REM and its free energy

The main difficulty of the SK-model is coming from the fact that the Gaussian random variables (2.1) are correlated. Derrida had the brilliant idea to ask if something interesting is happening if one just considers i.i.d. random variables. However, one wants to keep the variance of the right order. The SK-Hamiltonian has a variance of order N. We assume that the variance is exactly N. Evidently, then also the σ need not to carry an internal structure. We therefore assume that we have just 2^N independent Gaussian random variables, call them $X_\sigma^{(N)}$, $1 \leq \sigma \leq 2^N$, defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, which are centered and have variance N. Of course, one may still assume that $\sigma \in \Sigma_N$, but this will be of no relevance here. We then define the "Gibbs measure" on the σ by defining for any $\omega \in \Omega$, and any $\beta > 0$

$$\mathcal{G}_{\omega,\beta,N}(\sigma) = \frac{\exp\left[\beta X_\sigma^{(N)}(\omega)\right]}{Z_{\omega,\beta,N}}, \tag{3.1}$$

where $Z_{\omega,\beta,N} = \sum_\sigma \exp\left[\beta X_\sigma^{(N)}(\omega)\right]$. The free energy is as usual defined by

$$f(\beta) = \lim_{N\to\infty} \frac{1}{N} \log Z_{\omega,\beta,N}.$$

In principle, this could still depend on ω, but we will see in a moment, that limit exists almost $\mathbb{P}$-almost surely, and does not depend on ω. In fact, we have the following result:

Theorem 3.1
$f(\beta)$ exists almost surely and is given by

$$f(\beta) = \begin{cases} \frac{\beta^2}{2} + \log 2 & \text{if } \beta \leq \sqrt{2\log 2} \\ \sqrt{2\log 2}\beta & \text{if } \beta \geq \sqrt{2\log 2} \end{cases}.$$

The high temperature value is again the annealed free energy

$$\lim_{N\to\infty} \frac{1}{N} \log \mathbb{E} Z_{\beta,N}.$$

Curiously, it is *not* true that $\mathbb{E}Z_\beta^2 \leq C\left(\mathbb{E}Z_\beta\right)^2$ up to the correct critical value:

$$\begin{aligned}\mathbb{E}Z_\beta^2 &= \sum_{\sigma,\sigma'} \mathbb{E}\exp\left[\beta\left(X_\sigma^{(N)} + X_{\sigma'}^{(N)}\right)\right] \\ &= \sum_\sigma \exp\left[2\beta^2 N\right] + \sum_{\sigma\neq\sigma''} \exp\left[\beta^2 N\right] \\ &= \exp\left[2\beta^2 N + N\log 2\right] + 2^N\left(2^N - 1\right)\exp\left[N\beta^2\right].\end{aligned}$$

The first summand dominates the second as soon as $\beta > \sqrt{\log 2}$, and in fact, $\frac{\mathbb{E}Z_\beta^2}{\left(\mathbb{E}Z_\beta\right)^2}$ is exponentially growing in this case. One therefore has to argue slightly more subtle than in the SK case. The free energy of the REM is not really of relevance for spin glass theory. It is the Gibbs distribution which is much more interesting.

Proof of Theorem 3.1. The trick is to apply the "second moment method" not directly to Z but to

$$A_N(s) \stackrel{\text{def}}{=} \#\{\sigma : X_\sigma^{(N)} \geq sN\}. \tag{3.2}$$

Let Φ as usual be the standard normal distribution function. Then

$$\mathbb{E}A_N(s) = 2^N \left(1 - \Phi\left(s\sqrt{N}\right)\right) \asymp 2^N \mathrm{e}^{-s^2 N/2}.$$

Here we use the following notation: Given two sequences $\{a_N\}, \{b_N\}$ of positive real numbers, which may depend on other parameters (like s above), then we write $a_N \asymp b_N$, provided for any $\varepsilon > 0$ there exists N_0 (which may depend on the auxiliary parameters), such that

$$\mathrm{e}^{-\varepsilon N} a_N \leq b_N \leq \mathrm{e}^{\varepsilon N} a_N$$

for $N \geq N_0$. N_0 may depend on the parameters (of course it depends on ε anyway). If N_0 can be chosen independently of some parameters involved, like s, then we say that the relation $a_N \asymp b_N$ holds uniformly in these parameters.

We also use the same notation for sequences of positive random variables, meaning then that the relation holds almost surely.

Remark that for $s > \sqrt{2\log 2}$, $\mathbb{E}A_N(s)$ converges to 0, exponentially in N. From the Markov inequality, one gets $\mathbb{P}\left(A_N(s) \neq 0\right)$ converges to 0, exponentially fast, and then by Borel-Cantelli argument, we get that $A_N(s) = 0$ for large enough N, a.s. As this holds true for all $s > \sqrt{2\log 2}$, we get

$$\mathbb{P}\left(\limsup_{N\to\infty} \frac{1}{N} \sup_\sigma X_\sigma^{(N)} \leq \sqrt{2\log 2}\right) = 1 \tag{3.3}$$

For the second moment, we get

$$\mathbb{E}A_N(s)^2 = 2^N \left(1 - \Phi\left(s\sqrt{N}\right)\right) + 2^N(2^N - 1)\left(1 - \Phi\left(s\sqrt{N}\right)\right)^2,$$

we see that for $0 \leq s < \sqrt{2\log 2}$, this is $\left[\mathbb{E}A_N(s)\right]^2$, up to a factor, which is exponentially close to 1. From that we get

$$A_N(s) \asymp \mathbb{E}A_N(s) = 2^N \left(1 - \Phi\left(s\sqrt{N}\right)\right) \asymp \exp\left[N\left(\log 2 - \frac{s^2}{2}\right)\right].$$

As $A_N(s)$ is decreasing in s, this is easily seen to hold uniformly on $0 \leq s \leq s_0$ for an arbitrary $s_0 < \sqrt{2\log 2}$. Precisely: Given an $\varepsilon > 0$, we have that almost sure, one has

$$\exp\left[N\left(\log 2 - \frac{s^2}{2} - \varepsilon\right)\right] \leq A_N(s) \leq \exp\left[N\left(\log 2 - \frac{s^2}{2} + \varepsilon\right)\right] \tag{3.4}$$

for $s \in [0, s_0]$, if N is large enough. For $s \leq 0$, clearly $A_N(s) \geq A_N(0) \asymp 2^N$. We get for arbitrary $s_0 < \sqrt{2\log 2} < s_1$

$$\begin{aligned}\sum_\sigma \mathrm{e}^{\beta X_\sigma^{(N)}} &= N\beta \int_{-\infty}^{\infty} A_N(s)\mathrm{e}^{N\beta s} ds \\ &= N\beta \left\{\int_{-\infty}^{0} + \int_{0}^{s_0} + \int_{s_0}^{s_1} + \int_{s_1}^{\infty}\right\} A_N(s)\mathrm{e}^{N\beta s} ds.\end{aligned}$$

The fourth part is 0, the third, we estimate from above by $A_N(s_0)\,\mathrm{e}^{\beta N s_1}$, and by 0 from below, the second is estimated by (3.4), and the first is estimated from above by 2^N, and from below by 0. Getting everything together the reader will have no difficulty to check that

$$\lim_{N\to\infty}\frac{1}{N}\log\sum_\sigma \mathrm{e}^{\beta X_\sigma^{(N)}} = \sup_{s\le\sqrt{2\log 2}}\left\{-\frac{s^2}{2}+\beta s+\log 2\right\} = \left\{\begin{array}{ll}\frac{\beta^2}{2}+\log 2 & \text{if } \beta\le\sqrt{2\log 2}\\ \sqrt{2\log 2}\beta & \text{if } \beta\ge\sqrt{2\log 2}\end{array}\right. .$$

■

We next want to describe the large N behavior of the Gibbs measure $\mathcal{G}_{\omega,\beta,N}$. We have to distinguish between the high temperature case $\beta<\sqrt{2\log 2}$, and the low temperature case $\beta>\sqrt{2\log 2}$. We abstain from discussing the critical case $\beta=\sqrt{2\log 2}$. The fundamental difference is that in the high temperature case, the Gibbs measure is concentrated on a growing number of energy levels, which become dense and denser packed as $N\to\infty$. In contrast, in the low temperature regime, the Gibbs distribution is essentially concentrated on the top energy levels. We now make this precise.

Exercise 3.2
Assume $\beta<\sqrt{2\log 2}$

a) For any $\varepsilon>0$, *there exist* $K,\delta>0$ *such that*

$$\mathbb{P}\left(\left\{\omega:\mathcal{G}_{\omega,\beta,N}\left(\left\{\sigma:X_\sigma\in\left[\beta N-K\sqrt{N},\beta N+K\sqrt{N}\right]\right\}\right)\ge 1-\varepsilon\right\}\right)\ge 1-\mathrm{e}^{-\delta N},$$

i.e. up to a negligible $\mathbb{P}$*-probability,* $\mathcal{G}$ *is concentrated* σ*'s for which the energy levels are in a window of size of order* $\sqrt{N}$ *around* βN*. (The fact that exactly* β *is the value where the energy levels concentrate under the Gibbs measure is an "accident".)*

b) $\max_\sigma P_{\omega,\beta,N}(\sigma)$ *is exponentially decaying,* $\mathbb{P}$*-a.s.*

The low temperature regime $\beta>\sqrt{2\log 2}$ is more interesting, as there, the energies get a macroscopic but random weight. For any sequence a_N of real numbers, $\sum_\sigma\delta_{X_a-a_N}$ defines a point process on $\mathbb{R}$. We will sometimes just call such an object "the point process $\{X_\sigma-a_N\}_\sigma$".

Proposition 3.3
If $a_N=\sqrt{2\log 2}N-\frac{1}{2\sqrt{2\log 2}}\log N+\frac{1}{2\sqrt{2\log 2}}\log(2\pi)$, *then the above point process converges weakly to a* PPP $\left(\sqrt{2\log 2}\exp\left[-\sqrt{2\log 2}t\right]dt\right)$.

Proof. We denote by Q_N the law of $\sum_\sigma\delta_{X_a-a_N}$. If $\phi\in C_o^+(\mathbb{R})$, the

$$\begin{aligned}L_{Q_N}(\phi) &= \mathbb{E}\exp\left[-\sum_\sigma\phi\left(X_\sigma^{(N)}-a_N\right)\right]\\ &= \left\{\frac{1}{\sqrt{2\pi N}}\int\exp\left[-\phi(x-a_N)-\frac{x^2}{2N}\right]dx\right\}^{2^N}\\ &= \left\{1-\frac{1}{\sqrt{2\pi N}}\int\left(1-\mathrm{e}^{-\phi(x)}\right)\exp\left[-\frac{(x+a_N)^2}{2N}\right]dx\right\}^{2^N}.\end{aligned}$$

We abbreviate

$$\delta\left(x,N\right) := \frac{1}{\sqrt{2\pi N}}\left(1-\mathrm{e}^{-\phi(x)}\right)\exp\left[-\frac{\left(x+a_N\right)^2}{2N}\right],$$

and so

$$L_{Q_N}\left(\phi\right) = \exp\left[2^N\log\left(1-\int\delta\left(x,N\right)dx\right)\right].$$

As ϕ has compact support, there exist $K>0$ such that $\phi=0$ outside $\left[-K,K\right]$, and therefore $\delta\left(x,N\right)=0$, too, outside this interval. On the other hand

$$\exp\left[-\frac{\left(x+a_N\right)^2}{2N}\right] = \sqrt{4\pi\log 2}\mathrm{e}^{-x\sqrt{2\log 2}}\exp\left[-N\log 2\right]\sqrt{N}(1+o(1)),$$

uniformly in $x\in\left[-K,K\right]$, and therefore

$$\delta\left(x,N\right) = 2^{-N}\left(1-\mathrm{e}^{-\phi(x)}\right)\sqrt{2\log 2}\mathrm{e}^{-x\sqrt{2\log 2}}(1+o(1)),$$

uniformly in $x\in[-K,K]$. Expanding $\log\left(1-\varepsilon\right)=-\varepsilon-O\left(\varepsilon^2\right)$ for ε small, it follows from the fact that $\delta\left(x,N\right)=0$ outside $\left[-K,K\right]$:

$$\begin{aligned}&\exp\left[2^N\log\left(1-\int\delta\left(x,N\right)dx\right)\right]\\ =\ &\exp\left[-\int\left(1-\mathrm{e}^{-\phi(x)}\right)\sqrt{2\log 2}\mathrm{e}^{-x\sqrt{2\log 2}}dx(1+o(1))+O\left(2^{-N}\right)\right],\end{aligned}$$

i.e.

$$\lim_{N\to\infty}L_{Q_N}\left(\phi\right) = \exp\left[-\sqrt{2\log 2}\int\left(1-\mathrm{e}^{-\phi(x)}\right)\exp\left[-\sqrt{2\log 2}x\right]dx\right].$$

■

3.2 The Poisson-Dirichlet point process

We now discuss the limiting Gibbs distribution of the Random Energy Model for $\beta>\sqrt{2\log 2}$. First remark that applying Lemma 1.9 to the function $\mathbb{R}\ni y\to\exp\left[\beta y\right]\in\mathbb{R}^+:=(0,\infty)$, we obtain

Corollary 3.4
The point process

$$\left\{\exp\beta\left(X_\sigma^{(N)}-a_N\right)\right\}_\sigma$$

converges weakly to the Poisson point process with intensity measure $mt^{-m-1}dt$, where

$$m=m(\beta)=\frac{\sqrt{2\log 2}}{\beta}\in(0,1).$$

A kind of a proof. Let $[t, t+h]$ be an infinitesimal interval in $\mathbb{R}^+$, then in the $N \to \infty$ limit

$$\begin{aligned}
& P\left(\exists \sigma : \exp \beta\left(X_\sigma^{(N)} - a_N\right) \in [t, t+h]\right) \\
= \; & P\left(\exists \sigma : X_\sigma^{(N)} - a_N \in \left[\frac{\log t}{\beta}, \frac{\log(t+h)}{\beta}\right]\right) \\
= \; & P\left(\exists \sigma : X_\sigma^{(N)} - a_N \in \left[\frac{\log t}{\beta}, \frac{\log t}{\beta} + \frac{h}{t\beta}\right]\right) \\
= \; & \frac{h}{t\beta}\sqrt{2\log 2}\exp\left[-\sqrt{2\log 2}\beta^{-1}\log t\right] = hmt^{-m-1}.
\end{aligned}$$

■

The Poisson point processes with this intensity play an absolutely crucial role in the Parisi theory of spin glasses.

Lemma 3.5
Let Ξ be a PPP $\left(mt^{-m-1}dt\right)$ on $\mathbb{R}^+$ with the parameter $m \in (0,1)$. Then $\Xi(\mathbb{R}^+) = \infty$, almost surely, but $\int_{\mathbb{R}^+} t\Xi(dt) < \infty$ almost surely.

Proof. For any $n > 0$, one has

$$\alpha_n := m\int_{1/n}^{\infty} t^{-m-1}dt < \infty,$$

but the expression diverges if $n \to \infty$. Therefore $U_n = \Xi([1/n, \infty))$ is Poisson with parameter α_n, and $U_n \uparrow \Xi(\mathbb{R}^+)$. As $\lim_{n\to\infty} \alpha_n = \infty$, it follows that $\Xi(\mathbb{R}^+) = \infty$ almost surely.

For the second statement, we would like to apply Lemma 1.4, but this is not directly possible, since

$$E\int_{\mathbb{R}^+} t\Xi(dt) = \int_{\mathbb{R}^+} tmt^{-m-1}dt = \infty.$$

However, we have

$$E\int_{(0,1)} t\Xi(dt) = \int_{(0,1)} tmt^{-m-1}dt < \infty,$$

as we assume $m < 1$, and therefore

$$\int_{(0,1)} t\Xi(dt) < \infty, \text{ a.s.}$$

On the other hand, because $\int_1^\infty t^{-m-1}dt < \infty$, we have that $\Xi([1,\infty)) < \infty$, and as Ξ is a point measure, we have that $\int_{[1,\infty)} t\Xi(dt) < \infty$, almost surely. (If $\Xi = \sum \delta_{\eta_k}$, then $\int_{[1,\infty)} t\Xi(dt)$ is simply the sum over those η_k which are ≥ 1). Together, we conclude that $\int_{\mathbb{R}^+} t\Xi(dt) < \infty$ almost surely. ■

We can represent Ξ by a sequence $\{\eta_k\}$ of random variables, taking values in $\mathbb{R}^+$. As $\Xi(\mathbb{R}^+) = \infty$, this is an infinite sequence. There exists however a largest element, as follows from $\Xi([1,\infty)) < \infty$, and therefore, we can choose the sequence $\{\eta_k\}$ ordered downwards

$$\eta_0 > \eta_1 > \eta_2 > \ldots > 0.$$

We will always do that without usually specially mentioning it. As

$$\int_{\mathbb{R}^+} t\Xi(dt) = \sum \eta_i < \infty, \text{ a.s.}$$

we can transform the point process by normalizing the η_l :

$$\overline{\eta}_i \stackrel{\text{def}}{=} \frac{\eta_i}{\sum_j \eta_j}.$$

Evidently, this defines a point process $\{\overline{\eta}_i\}$ which lives on positive point configurations which sum up to 1. Such a point process cannot be Poissonian. In case $\{\eta_i\}$ is a point process on $\mathbb{R}^+$ with $\sum_i \eta_i$, we write $\{\overline{\eta}_i\} = \mathcal{N}(\{\eta_i\})$. (The reader will have no difficulty to check that this operation is measurable with respect to the Borel-field generated by the vague topology.)

Starting with a PPP $\left(mt^{-m-1}dt\right)$, the point process obtained by this normalization procedure is called the Poisson-Dirichlet point process and is denoted by $\mathrm{PD}(m)$.

The following result is plausible, but a bit cumbersome to prove.

Theorem 3.6
Let $\mathcal{G}_{\beta,N}(\sigma)$ the energies of Derrida's REM with $\beta > \sqrt{2\log 2}$, and let $m \stackrel{\text{def}}{=} \sqrt{2\log 2}/\beta$. Then $\{\mathcal{G}_{\beta,N}(\sigma)\}_\sigma$ converges weakly as $N \to \infty$ to $\mathrm{PD}(m)$ provided $m < 1$.

Remark 3.7
As a preparation to the next section, we check how the parameter m is changed when we change the number of energies, and their variance. So we assume we have $2^{\alpha N}$ variables of variance $\sigma^2 N$. First remark that if

$$\{X_\sigma\}_\sigma \to \mathrm{PPP}(f(t)\,dt),$$

then for $\varepsilon > 0$

$$\{\varepsilon X_\sigma\}_\sigma \to \mathrm{PPP}\left(\frac{1}{\varepsilon} f\left(\frac{t}{\varepsilon}\right) dt\right).$$

Our random variables $X_\sigma,\ 1 \le \sigma \le 2^{\alpha N}$, of variance $\gamma^2 N$ can be written as $\frac{\gamma}{\sqrt{\alpha}}\bar{X}_\sigma$, where the $\bar{X}_\sigma$ have variance $\overline{N} \stackrel{\text{def}}{=} \alpha N$. Therefore, the point process

$$\{X_\sigma\} = \left\{\gamma \bar{X}_\sigma/\sqrt{\alpha}\right\} \to \mathrm{PPP}\left(\frac{\sqrt{\alpha}}{\gamma}\sqrt{2\log 2}\,\mathrm{e}^{-\left(\sqrt{2\alpha\log 2}/\gamma\right)t} dt\right).$$

The point process of the Gibbs measure then converges to $\mathrm{PD}(m)$ with

$$m \stackrel{\text{def}}{=} \frac{\sqrt{2\log 2}\sqrt{\alpha}}{\beta\gamma},$$

provided, of course, that $m < 1$.

The Poisson point processes PPP $\left(a\mathrm{e}^{-at}dt\right)$ on $\mathbb{R}$, and PPP $\left(at^{-a-1}dt\right)$ on $\mathbb{R}^+$ have a number of remarkable properties. An important one is an invariance property under special transformations. We formulate it for the PPP $\left(a\mathrm{e}^{-at}\right)$. This point process has a largest point, and therefore, we can represent it as $\{\xi_i\}$, where $\xi_0 > \xi_1 > \cdots$ are real-valued random variables.

Proposition 3.8

a) *Let* $\{\xi_i\}$ *be a* PPP $\left(a\mathrm{e}^{-at}dt\right)$, *and* $X_0, X_1, \ldots$ *be i.i.d. real random variables satisfying* $M(a) = E\mathrm{e}^{aX_i} < \infty$, *being independent of the point process, too. Then* $\left\{\xi_i + X_i - \frac{1}{a}\log M(a)\right\}$*is also a* PPP $\left(a\mathrm{e}^{-at}dt\right)$

b) *Let* $\left\{\xi_i^k\right\}_i$, $k \in \mathbb{N}$ *be an i.i.d. sequence of* PPP $\left(a\mathrm{e}^{-at}dt\right)$, *and let* (x_k) *be a sequence of real numbers satisfying* $m(a) \stackrel{\text{def}}{=} \sum_k \mathrm{e}^{ax_k} < \infty$. *Then the point process*

$$\left\{\xi_i^k + x_k - (\log m(a))/a\right\}_{i,k}$$

is also a PPP $\left(a\mathrm{e}^{-at}dt\right)$.

Proof. a) Let $\phi \in C_o^+(\mathbb{R})$. Then

$$\begin{aligned} & E\left(\exp\left[-\sum\nolimits_i \phi\left(\xi_i + X_i - \frac{1}{a}\log M(a)\right)\right]\right) \\ = \; & E_\xi \prod\nolimits_i E_X \exp\left[-\phi\left(\xi_i + X_i - \frac{1}{a}\log M(a)\right)\right] \\ = \; & E_\xi \prod\nolimits_i \exp\left[-\psi(\xi_i)\right], \end{aligned}$$

E_ξ being the expectation with respect to the point process, and E_X with respect to the X_i, and where

$$\exp\left[-\psi(x)\right] = E_X \exp\left[-\phi\left(x + X - \frac{1}{a}\log M(a)\right)\right].$$

There is a slight problem as ψ has only compact support provided the X_i are bounded random variables. Assuming this for the moment, we get

$$\begin{aligned} & E\left(\exp\left[-\sum\nolimits_i \phi(\xi_i + X_i)\right]\right) \\ = \; & \exp\left[-a\int\left(1 - \mathrm{e}^{\psi(x)}\right)\mathrm{e}^{-ax}dx\right] \\ = \; & \exp\left[-aE_X\int\left(1 - \exp\left[-\phi\left(x + X - \frac{1}{a}\log M(a)\right)\right]\right)\mathrm{e}^{-ax}dx\right] \\ = \; & \exp\left[-\frac{a}{M(a)}E_X\int\left(1 - \exp\left[-\phi(x)\right]\right)\mathrm{e}^{-ax}\mathrm{e}^{aX}dx\right] \\ = \; & \exp\left[-\frac{a}{M(a)}\int\left(1 - \exp\left[-\phi(x)\right]\right)\mathrm{e}^{-ax}dxE_X\mathrm{e}^{aX}\right] \\ = \; & \exp\left[-a\int\left(1 - \exp\left[-\phi(x)\right]\right)\mathrm{e}^{-ax}dx\right], \end{aligned}$$

which proves the desired result. For general X_i (not necessarily bounded), we use a truncation $X_i^{(K)} := X_i 1_{\{|X_i| \leq K\}}$. Then with $M_K(a) := E\mathrm{e}^{aX_i^{(K)}}$

$$\lim_{K\to\infty} M_K(a) = M(a)$$

is evident, but we have to prove that

$$\left\{\xi_i + X_i^{(K)} - \frac{1}{a}\log M_K(a)\right\} \to \left\{\xi_i + X_i - \frac{1}{a}\log M(a)\right\}$$

weakly. This is left to the reader to check.

b) With ϕ as above, we have

$$\begin{aligned}
& E\left(\exp\left[-\sum\nolimits_{i,k}\phi\left(\xi_i^k + x_k - \frac{\log m(a)}{a}\right)\right]\right) \\
= & \prod\nolimits_k E_k \exp\left[-\sum\nolimits_{i,k}\phi\left(\xi_i^k + x_k - \frac{\log m(a)}{a}\right)\right] \\
= & \prod\nolimits_k \exp\left[-a\int\left(1 - \mathrm{e}^{\phi\left(x + x_k - \frac{\log m(a)}{a}\right)}\right)\mathrm{e}^{-ax}dx\right] \\
= & \prod\nolimits_k \exp\left[-\frac{1}{m(a)}\mathrm{e}^{ax_k}a\int\left(1 - \mathrm{e}^{\phi(x)}\right)\mathrm{e}^{-ax}dx\right] \\
= & \exp\left[-\sum\nolimits_k \frac{1}{m(a)}\mathrm{e}^{ax_k}a\int\left(1 - \mathrm{e}^{\phi(x)}\right)\mathrm{e}^{-ax}dx\right] \\
= & \exp\left[-a\int\left(1 - \mathrm{e}^{\phi(x)}\right)\mathrm{e}^{-ax}dx\right],
\end{aligned}$$

as claimed. ■

We can immediately translate this invariance property in one for PPP $\left(at^{-a-1}dt\right)$ by using the fact that if $\{\xi_i\}$ is a PPP $\left(a\mathrm{e}^{-at}dt\right)$ then $\{\eta_i\}$ with $\eta_i \stackrel{\text{def}}{=} \mathrm{e}^{\xi_i}$ is a PPP $\left(at^{-a-1}dt\right)$.

Corollary 3.9

a) *Let $\{\eta_i\}$ be a PPP $\left(at^{-a-1}\right)$ on $\mathbb{R}^+$, and let $\{Y_i\}$ be a sequence of i.i.d. positive random variables with $C(a) := E(Y_i^a) < \infty$, also independent of the point process. Then*

$$\left\{Y_i\eta_i / C(a)^{1/a}\right\}$$

is a PPP $\left(at^{-a-1}dt\right)$.

b) *Let $\Xi_k = \left\{\eta_i^k\right\}_i$, $k \in \mathbb{N}$ be an i.i.d. sequence of* PPP $\left(at^{-a-1}dt\right)$*'s, and let $\{y_k\}$ be a sequence of positive real numbers satisfying $C(a) \stackrel{\text{def}}{=} \sum_k y_k^a < \infty$. Then the point process $\left\{y_k\eta_i^k / C(a)^{1/a}\right\}_{i,k}$ is also a* PPP $\left(at^{-a-1}dt\right)$.

Proof. We present $\eta_i = \mathrm{e}^{\xi_i}$ as above, and $Y_i = \mathrm{e}^{X_i}$. Then $C(a) = E\left(\mathrm{e}^{aX_i}\right) = M(a)$ from the last proposition. Then

$$Y_i \eta_i / C(a)^{1/a} = \exp\left[\xi_i + X_i - \frac{1}{a}\log M(a)\right],$$

and we can apply the last proposition. b) is similar. ∎

The corollary will have a simple consequence which will be of crucial importance:

Lemma 3.10
Let $0 < a < a' < 1$, $\mathbf{y} = \{y_i\}$ be a sequence with $C(a, \mathbf{y}) \overset{\text{def}}{=} \sum_k y_k^a < \infty$, and $\Xi_k = \left\{\eta_i^k\right\}_i$, $k \in \mathbb{N}$ be an i.i.d. sequence of PPP $\left(at^{-a-1}dt\right)$*'s. Then*

$$\sum_{i,k}\left(y_k \eta_i^k\right)^{a'} < \infty,$$

almost surely, and

$$\mathcal{N}\left(\left\{\left(y_k \eta_i^k\right)^{a'}\right\}_{i,k}\right) =^{\mathcal{L}} \mathrm{PD}\left(\frac{a}{a'}\right).$$

In particular, the law of the right-hand side does not depend on the sequence $\mathbf{y}$.

Proof. By b) of the above corollary, we know that $\left\{y_k \eta_i^k / C(a, \mathbf{y})^{1/a}\right\}_{i,k}$ is a PPP $\left(at^{-a-1}dt\right)$. As $\int_0^1 \left(at^{-1-a}\right)^{a'} dt < \infty$, it follows that

$$\sum_{i,ki}\left(y_k \eta_i^k / C(a, \mathbf{y})^{1/a}\right)^{a'} < \infty,$$

almost surely, i.e.

$$\sum_{i,ki}\left(y_k \eta_i^k\right)^{a'} < \infty, \text{ a.s.}$$

The point process

$$\left\{\left(y_k \eta_i^k / C(a, \mathbf{y})^{1/a}\right)^{a'}\right\}_{i,k}$$

is a PPP $\left(\frac{a}{a'}t^{-a/a'-1}dt\right)$. As the constant $1/C(a, \mathbf{y})^{a'/a}$ cancels out at the normalization procedure, the claim follows. ∎

It will be important to know all the moments of a PD(a). To prepare that we consider the following situation: Let f be a Lebesgue integrable non-negative function on $\mathbb{R}^+$. We write $\alpha := \int_0^\infty f(t)\, dt$. We consider a PPP$(f(t)\, dt)$ on $\mathbb{R}^+$. Then, as the intensity measure of the point process is finite, we can represent it as (see 1.8 in the introduction) $\{\eta_i\}_{1 \leq i \leq S}$, where S is Poisson with parameter α, and $\{\eta_i\}$ is an i.i.d.

sequence with distribution $\overline{f}(t) := f(t)/\alpha$. On the event $\{S \geq 1\}$ we can normalize the points $\eta_1, \ldots, \eta_S$ by putting

$$\overline{\eta}_i := \frac{\eta_i}{Z},\ i \leq S,\ \text{where } Z := \sum\nolimits_{j=1}^{S} \eta_j.$$

For $r > 0$, we want to compute

$$\begin{aligned} E\left(\sum\nolimits_{i=1}^{S} \overline{\eta}_i^r;\ S \geq 1\right) &= E\left(\sum\nolimits_{i=1}^{S} \frac{\eta_i^r}{\left(\sum_{j=1}^{S} \eta_j\right)^r};\ S \geq 1\right) \\ &= \sum_{k=1}^{\infty} P(S=k)\, E\left(\sum\nolimits_{i=1}^{k} \frac{\eta_i^r}{\left(\sum_{j=1}^{k} \eta_j\right)^r}\right) \\ &= \sum_{k=1}^{\infty} P(S=k)\, kE\left(\frac{\eta_1^r}{\left(\eta_1 + \sum_{j=2}^{k} \eta_j\right)^r}\right). \end{aligned}$$

Remark now that $\sum_{j=2}^{k} \eta_j$ is independent of η_1, and furthermore

$$kP(S=k) = \alpha P(S=k-1),$$

(as S is Poisson). Therefore, defining $\sum_{i=1}^{S} \overline{\eta}_i^r = 0$ if $S = 0$, we have

$$E\left(\sum\nolimits_{i=1}^{S} \overline{\eta}_i^r\right) = \int_0^\infty dt f(t)\, t^r E\left(\frac{1}{(t+Z)^r}\right).$$

We can now extend that to density functions f on $\mathbb{R}^+$, satisfying $\int_0^\infty f(t)\,dt = \infty$, $\int_0^\infty tf(t)\,dt < \infty$, $\int_1^\infty f(t)\,dt < \infty$. Then a PPP $(f(t)\,dt)$ has infinitely many point, but only finitely many above a fixed value $\varepsilon > 0$, i.e. the points accumulate at 0. Furthermore $Z := \sum \eta_i$ is finite almost surely. We can approximate such a Poisson point process by point processes PPP $(f_n(t)\,dt)$ where $f_n(t) = f(t)\,1_{t \geq 1/n}$. By a limiting procedure, we then get

Lemma 3.11

$$E\left(\sum\nolimits_i \overline{\eta}_i^r\right) = \int_0^\infty dt f(t)\, t^r E\left(\frac{1}{(t+Z)^r}\right)$$

understanding that if one side is infinite, then the other is, too.

Exercise 3.12
Assume again that $f : \mathbb{R}^+ \to [0,\infty)$ *satisfies* $\int_0^\infty f(t)\,dt = \infty$, $\int_0^\infty tf(t)\,dt < \infty$, *and let* $\{\eta_i\}$ *be a* PPP $(f(t)\,dt)$ *Let furthermore* $r_1, \ldots, r_k > 0$, *and* $N := \sum_{i=1}^{k} r_i$. *Set*

$$X_{\mathbf{r}}^{(k)} := \sum\nolimits_{i_1,\ldots,i_r}^{*} \overline{\eta}_{i_1}^{r_1} \overline{\eta}_{i_2}^{r_2} \cdot \cdots \cdot \overline{\eta}_{i_k}^{r_k}, \tag{3.5}$$

where $\sum_{i_1,\ldots,i_r}^{*}$ *means that we take the sum over* k*-tuples which are all distinct. Then*

$$E\left(X_{\mathbf{r}}^{(k)}\right) = \int_0^\infty dt_1 \cdots \int_0^\infty dt_k \prod\nolimits_{j=1}^{k} \left(t_j^{r_j} f\left(t_j\right)\right) E\left(\frac{1}{\left(\sum_{j=1}^{k} t_j + Z\right)^N}\right), \qquad (3.6)$$

where again

$$Z := \sum\nolimits_j \eta_j.$$

(One should remark that it is not necessary to assume $r_j \in \mathbb{N}$*).*

We apply this to prove the following

Proposition 3.13
Let $\{\overline{\eta}_i\}$*be a* PD (m) *with* $m < 1$*. Let furthermore* $r_1, \ldots, r_k$ *be as in Exercise 3.12, but assuming that they are natural numbers, and put* $N := \sum_{i=1}^{k} r_i$*. Then with* $X_{\mathbf{r}}^{(k)}$ *as in (3.5), one has*

$$EX_{\mathbf{r}}^{(k)} = \frac{(k-1)!}{(N-1)!} m^{k-1} \prod_{i=1}^{k} g\left(r_i, m\right),$$

where

$$g(r,m) := \begin{cases} 1 & \text{if } r = 1 \\ (r-1-m)(r-2-m)\cdot\cdots\cdot(1-m) & \text{if } r \geq 2 \end{cases} \qquad (3.7)$$

Proof. We use (3.6) with $f(t) = mt^{-m-1}$.
Let's first look at the special case $k = 1$, $r_1 = r \geq 2$. We have

$$E\left(\sum\nolimits_i \overline{\eta}_i^r\right) = m \int_0^\infty dt\ t^{-m+r-1} E\left(\frac{1}{(t+Z)^r}\right),$$

and partial integration gives

$$\begin{aligned} m \int_0^\infty dt\ t^{-m+r-1} E\left(\frac{1}{(t+Z)^r}\right) &= m \frac{r-1-m}{r-1} \int_0^\infty dt\ t^{-m+r-2} E\left(\frac{1}{(t+Z)^{r-1}}\right) \\ &= \frac{r-1-m}{r-1} E\left(\sum\nolimits_i \overline{\eta}_i^{r-1}\right). \end{aligned}$$

Iterating, we get

$$E\left(\sum\nolimits_i \overline{\eta}_i^r\right) = \frac{g(r,m)}{(r-1)!} E\left(\sum\nolimits_i \overline{\eta}_i\right) = \frac{g(r,m)}{(r-1)!}.$$

In the same way, one proves by partial integration that for $k \geq 2$,

$$EX_{\mathbf{r}}^{(k)} = \frac{(k-1)!}{(N-1)!} \prod_{i=1}^{k} g\left(r_i\right) EX_{(1,\ldots,1)}^{(k)}.$$

It therefore remains to prove that $EX^{(k)}_{(1,\ldots,1)} = m^{k-1}$. This is evident if $k = 1$. If we have proved it up to $k-1$, then we have also proved the general formula up to $k-1$. For k, we write

$$\begin{aligned}
X^{(k)}_{(1,\ldots,1)} &= \sum\nolimits^*_{i_1,\ldots,i_r} \overline{\eta}_{i_1}\overline{\eta}_{i_2}\cdot\cdots\cdot\overline{\eta}_{i_k} \\
&= \sum\nolimits^*_{i_1,\ldots,i_{k-1}} \overline{\eta}_{i_1}\cdot\cdots\cdot\overline{\eta}_{i_{k-1}} \sum_{j\notin\{i_1,\ldots,i_{k-1}\}} \overline{\eta}_j \\
&= \sum\nolimits^*_{i_1,\ldots,i_{k-1}} \overline{\eta}_{i_1}\cdot\cdots\cdot\overline{\eta}_{i_{k-1}} \left(1-\overline{\eta}_{i_1}-\ldots-\overline{\eta}_{i_{k-1}}\right) \\
&= X^{(k-1)}_{(1,\ldots,1)} - (k-1)\, X^{(k-1)}_{(2,1,\ldots,1)},
\end{aligned}$$

and computing the expectation, we can apply the induction hypothesis, giving

$$\begin{aligned}
EX^{(k)}_{(1,\ldots,1)} &= m^{k-2} - (k-1)\frac{(k-2)!}{(k-1)!} m^{k-2} g(2,m) \\
&= m^{k-2} - m^{k-2}(1-m) = m^{k-1}.
\end{aligned}$$

■

For the special case $k = 1$, $r_1 = 2$, we get

$$E\left(\sum\nolimits_i \overline{\eta}_i^2\right) = 1-m.$$

We can apply this to compute the expected overlaps in the REM in the $N \to \infty$ limit: It is natural to define the "overlap" between σ, σ' to be 1 if $\sigma = \sigma'$, and 0 if $\sigma \neq \sigma'$. Then under the Gibbs measure, we have the expected overlap to be

$$\mathcal{G}^{\otimes 2}_{\omega,\beta,N}\left(\left\{\left(\sigma,\sigma'\right) : \sigma = \sigma'\right\}\right) = \sum_\sigma \left[\mathcal{G}_{\omega,\beta,N}(\sigma)\right]^2,$$

and if we take the expectation over $\mathbb{P}$, we get

$$\mathbb{E}\mathcal{G}^{\otimes 2}_{\omega,\beta,N}\left(\left\{\left(\sigma,\sigma'\right) : \sigma = \sigma'\right\}\right) = \mathbb{E}\sum_\sigma \left[\mathcal{G}_{\omega,\beta,N}(\sigma)\right]^2.$$

Passing to the limit, we get

Proposition 3.14

a) If $\beta < \sqrt{2\log 2}$, then

$$\lim_{N\to\infty} \mathbb{E}\mathcal{G}^{\otimes 2}_{\omega,\beta,N}\left(\left\{\left(\sigma,\sigma'\right) : \sigma = \sigma'\right\}\right) = 0.$$

b) If $\beta > \sqrt{2\log 2}$, then

$$\begin{aligned}
\lim_{N\to\infty} \mathbb{E}\mathcal{G}^{\otimes 2}_{\omega,\beta,N}\left(\left\{\left(\sigma,\sigma'\right) : \sigma = \sigma'\right\}\right) &= E\left(\sum\nolimits_i \overline{\eta}_i^2\right) \\
&= 1 - \frac{\sqrt{2\log 2}}{\beta},
\end{aligned}$$

where $\{\bar{\eta}_i\}$ is a PD$\left(\frac{\sqrt{2\log 2}}{\beta}\right)$.

3.3 The GREM and Ruelle's cascades

Derrida evidently felt that the REM is too simple to shed any light on "real" spin glasses. He therefore invented a modification where the energies are correlated, like in the SK-model, but in a very special hierarchical way.

I will not prove anything of substance about Derrida's GREM, but I will quickly go to the limiting object, the Ruelle cascades.[9] I will however give a description of the GREM and some of its properties.

Consider a tree with a root and K levels. On each level, a bond branches into $2^{N/K}$ "children" branches. The leaves σ can then be written as

$$\sigma = (i_1, i_2, \ldots, i_K), \ 1 \leq i_j \leq 2^{N/K}.$$

The bonds of the graph can be identified with the prefixes $(i_1, \ldots, i_j)$ of σ, $\ j \leq K$. To pass from the root to the leaf σ on passes through the bonds

$$i_1, (i_1, i_2), (i_1, i_2, i_3), \ldots, (i_1, i_2, \ldots, i_K).$$

The energies of the GREM are given by summing independent bond energies along the path from the root to the leaves.

$$X_\sigma \stackrel{\text{def}}{=} X^{(1)}_{i_1} + X^{(2)}_{i_1,i_2} + \cdots + X^{(K)}_{i_1,\ldots,i_K}.$$

All the $X^{(j)}$-variables are assumed to be independent and centered Gaussians. On level $j \leq K$, all variables have the same variances

$$\operatorname{var}\left(X^{(j)}_{i_1,\ldots,i_j}\right) = \alpha_j^2 N.$$

Usually, one assumes that $\sigma_1^2 > \sigma_2^2 > \cdots > \sigma_K^2$, but it is not really necessary. (If it is not satisfied, then some of the levels simply disappear in the limit). We also assume

$$\sum_{i=1}^{K} \alpha_i^2 = 1$$

which is just a normalization of no importance.

The covariances are trivially computed

$$\mathbb{E}\left(X_{\boldsymbol{\sigma}} X_{\boldsymbol{\sigma}'}\right) = N \sum_{i=1}^{q(\boldsymbol{\sigma},\boldsymbol{\sigma}')} \alpha_i^2,$$

where

$$q\left((i_1, \ldots, i_K), (i'_1, \ldots, i'_K)\right) = \max\left(m : (i_1, \ldots, i_m) = \left(i'_1, \ldots, i'_m\right)\right).$$

[9] Ruelle did not prove that Derrida's GREM converges to the object he introduced, although he seemed to have taken it as a kind of "evident", not worth to bother.

Particularly, the variance of the variables is N as in the REM case.

The partition function and the Gibbs measure are defined in the usual way:

$$Z_{N,\beta,\omega} \stackrel{\text{def}}{=} \sum_{\sigma} \exp\left[\beta X_\sigma\right], \ \mathcal{G}_{N,\beta,\omega}\left(\sigma\right) \stackrel{\text{def}}{=} \frac{1}{Z_{N,\beta,\sigma}} \exp\left[\beta X_\sigma\right].$$

The free energy

$$f\left(\beta\right) \stackrel{\text{def}}{=} \lim_{N\to\infty} \frac{1}{N} \log Z_N$$

can be computed explicitly. It is piecewise quadratic with K pieces of different second derivative (provided $\sigma_1^2 > \sigma_2^2 > \cdots > \sigma_K^2$). The model has K critical values:

$$\beta_1^{\text{cr}} = \frac{\sqrt{2\log 2}}{\sqrt{K}\sigma_1} < \beta_2^{\text{cr}} = \frac{\sqrt{2\log 2}}{\sqrt{K}\sigma_2} < \cdots < \beta_K^{\text{cr}} = \frac{\sqrt{2\log 2}}{\sqrt{K}\sigma_K}.$$

For $\beta < \beta_1^{\text{cr}}$, one has that the free energy equals the annealed free energy. $f\left(\beta\right) = \beta^2/2 + \log 2$. For $\beta > \beta_K^{\text{cr}}$, the free energy is linear in β. The second derivative of the free energy in β jumps at all critical values, but the first derivative stays continuous.

For $\beta < \beta_K^{\text{cr}}$ there is no "macroscopic" Gibbs weight, i.e. the Gibbs weights are all exponentially small. However, for $\beta > \beta_n^{\text{cr}}$, the marginals

$$\sum_{i_{n+1},\ldots,i_K} \mathcal{G}\left(\left(i_1,\ldots,i_K\right)\right)$$

are macroscopic, and in the limit $N \to \infty$ given by a Poisson-Dirichlet point process. For $\beta_n^{\text{cr}} < \beta < \beta_{n+1}^{\text{cr}}$, the Gibbs distribution "freezes" at level n, but not at level $n+1$, meaning that the Gibbs measure concentrates at configurations $\left(i_1,\ldots,i_K\right)$ for which $X_{i_1}^1,\ldots,X_{i_1,\ldots,i_n}^n$ are near their maximal possible value, but not $X_{i_1,\ldots,i_{n+1}}^{n+1}$. I don't give any details about that, but now present Ruelle's limiting object.

Ruelle argued that the limiting Gibbs measure (at least for β large) should have the following cascade structure. One chooses K parameters $0 < m_1 < \cdots < m_K < 1$. Then, on a first level one chooses a PPP $\left(m_1 t^{-m_1-1}dt\right)$, $\Xi^1 = \left\{\eta_i^1\right\}_i$, where (for convenience) the points $\eta_0 > \eta_1 > \cdots > 0$ are ordered downwards. On the next level, one chooses for *any* $i \in \mathbb{N}$ a PPP $\left(m_2 t^{-m_2-1}dt\right)$ Ξ_i^2 whose countably many points are denoted by $\left\{\eta_{ij}^2\right\}_{j\in\mathbb{N}}$, and we furthermore assume that these point processes are all independent, and also independent of Ξ^1. In this way, one proceeds: On the third level, one chooses independent point processes $\Xi_{i_1 i_2}^3 \stackrel{\text{def}}{=} \left\{\eta_{i_1 i_2 j}^3\right\}_j$ for any $i_1, i_2 \in \mathbb{N}$, and these point processes have density $m_3 t^{-m_3-1}$.

Such a cascade of point processes $\Xi^1, \Xi_{i_1}^2, \Xi_{i_1 i_2}^3, \ldots, \Xi_{i_1 i_2\ldots i_{K-1}}^K$ is called a **Ruelle cascade** to the parameter $\left(m_1,\ldots,m_K\right)$.

We can multiply the points of the all the point processes: For $\mathbf{i} = \left(i_1,\ldots,i_K\right) \in \mathbb{N}^K$, we put

$$\eta_{\mathbf{i}} \stackrel{\text{def}}{=} \eta_{i_1}^1 \eta_{i_1 i_2}^2 \cdots \eta_{i_1,i_2,\ldots,i_K}^K \tag{3.8}$$

This leads to the point process

$$\Xi^{\text{tot}} \stackrel{\text{def}}{=} \{\eta_{\mathbf{i}}\}$$

One should think of the points of this point process as the unnormalized Gibbs weights of a limiting GREM-type spin glass.

Occasionally, it is convenient, to write things additively, by defining

$$\eta^n_{i_1,\ldots,i_n} = \exp\left[\xi^n_{i_1,\ldots,i_n}\right], \ \xi_{\mathbf{i}} = \sum_{m=1}^{K} \xi^n_{i_1,\ldots,i_n}, \ \eta_{\mathbf{i}} = \mathrm{e}^{\xi_{\mathbf{i}}}.$$

Here, for any $n \leq K$, and any $i_1,\ldots,i_{n-1}$, the point process $\left\{\xi^n_{i_1,\ldots,i_n}\right\}_{i_n}$ is a PPP $\left(m_n \mathrm{e}^{-m_n t} dt\right)$.

The first big surprise is

Proposition 3.15
$\mathcal{N}\left(\Xi^{\text{tot}}\right) =^{\mathcal{L}} \mathrm{PD}\left(m_K\right)$.

Proof. Take first $K = 2$. Remark that because of $m_1 < m_2 < 1$ we have $\int_{(0,1)} t^{m_2} m_1 t^{-m_1-1} dt < \infty$, and therefore

$$C \stackrel{\text{def}}{=} \sum_i \left(\eta^1_i\right)^{m_2} < \infty \tag{3.9}$$

almost surely. We can now apply Lemma 3.10. For that we condition on the first level $\left\{\eta^1_i\right\}$, and apply the lemma with $y_i \stackrel{\text{def}}{=} \eta^1_i$, $m = m_2$, $m' = 1$. Then

$$\mathcal{N}\left(\left\{\eta^1_i \eta^2_{ij}\right\}\right) =^{\mathcal{L}} \mathrm{PD}\left(m_2\right).$$

The general K case follows easily by induction. For instance with $K = 3$, one sees from the identity (**??**) that

$$\sum_{i,j} \left(\eta^1_i \eta^2_{ij} / C^{1/m_2}\right)^{m_3} < \infty, \text{ a.s.}$$

i.e.

$$\sum_{i,j} \left(\eta^1_i \eta^2_{ij}\right)^{m_3} < \infty, \text{ a.s.}$$

and we can apply the same argument as above for the next level, conditioning first on the first two levels. ■

It first sight, this proposition seems to tell that the introduction of the cascade structure does not give anything new which is not already present in the case $K = 1$. Nothing could be more wrong than that, as I explain now.

We can order the $\eta_{\mathbf{i}}$ downwards which leads to a random bijection $\phi : \mathbb{N} \to \mathbb{N}^K : \eta_{\phi(k)}$ is the k-th biggest among the $\eta_{\mathbf{i}}$.

Let $0 \leq k \leq K$ and fix it for the moment. We define a (random) equivalence relation on $\mathbb{N}$ by setting $i \sim j$ if and only if $\phi(i)_r = \phi(j)_r$ for $r \leq k$. In other words, i is equivalent to j if and only if the branching between the i-th largest and the j-th largest is at level k or later.

The equivalence relation induces a partition $\mathcal{Z}_k$ of $\mathbb{N}$ into disjoint subsets, the equivalence classes under the equivalence relations. By the very definition, it is clear that $\mathcal{Z}_{k+1}$ is a finer partition than $\mathcal{Z}_k$. If $\mathcal{Z}$ and $\mathcal{Z}'$ are to partitionings of $\mathbb{N}$ we write $\mathcal{Z}' \prec \mathcal{Z}$ if $\mathcal{Z}$ is obtained by possibly dividing the sets of $\mathcal{Z}'$, i.e. if it is the finer partitioning. Using this notation, we evidently have

$$\mathcal{Z}_K = \{\{i\} : i \in \mathbb{N}\} \succ \mathcal{Z}_{K-1} \succ \cdots \succ \mathcal{Z}_1 \succ \mathcal{Z}_0 = \{\mathbb{N}\}. \tag{3.10}$$

The Ruelle cascade therefore leads to a sequence of random partitionings $(\mathcal{Z}_0, \mathcal{Z}_1, \ldots, \mathcal{Z}_K)$. Of course only the $\mathcal{Z}_k$ with $1 \leq k \leq K-1$ are random. A most remarkable property is

Proposition 3.16
$\mathcal{N}(\{\eta_{\mathbf{i}}\})$ and $(\mathcal{Z}_0, \mathcal{Z}_1, \ldots, \mathcal{Z}_K)$ are stochastically independent.

The crucial point is already seen in the case $K = 2$. We step back to the situation of Lemma 3.10 and use the assumptions and notations from there, i.e. we start with a sequence $\mathbf{y} = \{y_k\}$ of positive reals, and independent point processes $\{\eta_i^k\}_i$ which are PPP $\left(at^{-a-1}dt\right)$. Then consider $\mathcal{N}\left(\{y_k \eta_i^k\}_{i,k}\right)$ which we know is a PD(a). As usual, we can order the points of this point process downwards, and the two-stage procedure to produce the points induces a random partitioning $\mathcal{Z}$ of the natural numbers. This partitioning can be created in the following way: We attach to a point of the point-process $\mathcal{N}\left(\{y_k \eta_i^k\}_{i,k}\right)$ the number k if the point stems from the group $y_k \eta_\cdot^k$. In this way, we obtain what is called a *marked* point process.

What we prove is that the marked point process is a point process with independently attached marks. We need some information about marked point processes. Let M be a (locally compact) space. A point process with values in $\mathbb{R}^+$ with marks in M is a point process with values in $\mathbb{R}^+ \times M$ which has the property that almost surely, one has for all $s \in \mathbb{R}^+$ there is at most one point in $\{s\} \times M$. One also requires that the projection of the points to $\mathbb{R}^+$ gives a point process on $\mathbb{R}^+$ (which is not automatic from the requirement that one has a point process on $\mathbb{R}^+ \times M$). On the other hand, there is no requirement that the projection onto M leads to a point process on M.

A very special case is the one where *independent* marks are attached to a point process on $\mathbb{R}^+$. We can construct that in the following way. Any point process Ξ with values in $\mathbb{R}^+$ can be represented as $\Xi = \sum_i \delta_{\xi_i}$, where $\{\xi_i\}$ is a sequence of positive random variables. For instance, if there is a largest point, as is usually the case of interest to us, we can just arrange the points in decreasing order. The ordering of the points is however of no importance for the construction. Given a probability distribution μ on M, we construct an i.i.d. sequence $\{X_i\}$ of random variables, taking values in M, and law μ. Then we take as a point process

$$\sum_i \delta_{(\eta_i, X_i)}.$$

The Laplace functional is easily computed: If $\phi \in C_o^+(\mathbb{R} \times M)$ and $\mathrm{e}^{-\psi(y)} = E\mathrm{e}^{-\phi(y,X)}$, then

$$E \exp\left[-\sum\nolimits_i \phi(\eta_i, X_i)\right] = E \exp\left[-\sum\nolimits_i \psi(\eta_i)\right].$$

In the special case where $\{\eta_i\}$ is a PPP $(f dt)$ then

$$
\begin{aligned}
E\exp\left[-\sum\nolimits_i \phi(\eta_i, X_i)\right] &= \exp\left[-\int\left(1-\mathrm{e}^{-\psi(t)}\right) f(t)\,dt\right] \\
&= \exp\left[-\int\int\left(1-\mathrm{e}^{-\phi(t,x)}\right) f(t)\,dt\mu(dx)\right].
\end{aligned}
$$

Lemma 3.17
In the situation described before $\left\{\left(y_k\eta_i^k/C(a)^{1/a}, k\right)_{i,k}\right\}$ *is a marked point process where the first component is a* PPP $\left(at^{-a-1}\right)$, *and the marks are independently attached with* $\mathbf{y}$*-dependent distribution*

$$p_{a,\mathbf{y}}(k) \stackrel{\text{def}}{=} \frac{y_k^a}{C(a)}.$$

As a consequence, normalizing the point process $\left\{y_k\eta_i^k\right\}_{i,k}$ *and keeping the marks leads to a* PD (a) *with independently attached marks with the above distribution.*

Proof. Let $\phi : \mathbb{R}^+ \times \mathbb{N} \to \mathbb{R}^+$ be continuous with compact support. Then

$$
\begin{aligned}
E\exp\left[-\sum\nolimits_{i,j}\phi\left(y_k\eta_i^k, k\right)\right] &= \prod\nolimits_k E\exp\left[-\sum\nolimits_i \phi\left(y_k\eta_i^k, k\right)\right] \\
&= \prod\nolimits_k \exp\left[-\int\left(1-\mathrm{e}^{-\phi(y_k t,k)}\right) at^{-a-1}dt\right] \\
&= \prod\nolimits_k \exp\left[-\int\left(1-\mathrm{e}^{-\phi(t,k)}\right)\frac{a}{y_k}\left(\frac{t}{y_k}\right)^{-a-1}dt\right] \\
&= \exp\left[-\sum\nolimits_k\int\left(1-\mathrm{e}^{-\phi(t,k)}\right)p_{a,\mathbf{y}}(k)\,C(a,\mathbf{y})\,at^{-a-1}dt\right],
\end{aligned}
$$

which proves that, conditionally on the first level, the point process

$$\left\{\left(y_k\eta_i^k C(a,\mathbf{y})^{1/a}, k\right)\right\}_{k,i}$$

is a marked point process which is a PPP $\left(at^{-a-1}dt\right)$ with independently attached points in $\mathbb{N}$ with law $p_{a,\mathbf{y}}$. ■

We will also need:

Lemma 3.18

a) *If* $\{\eta_i\}_i$ *is a* PPP $\left(at^{-a-1}dt\right)$, *and* $0 < a < b < 1$, *then* $\left\{\eta_i^b\right\}_i$ *is a* PPP $\left(\frac{a}{b}t^{-a/b-1}dt\right)$

b) *Let* $\eta_{i_1}^1, \ldots, \eta_{i_i,\ldots,i_K}^K$ *be a Ruelle cascade with parameters* $m_1 < \cdots < m_K < 1$, *and let* $m_K < m < 1$. *Then*

$$\mathcal{N}\left(\left\{\left(\eta_{i_1}^1 \cdot \ldots \cdot \eta_{i_i,\ldots,i_K}^K\right)^m\right\}_{\mathbf{i}}\right)$$

is a PD (m_K/m) *which is independent of* $\sigma\left(\eta^1, \ldots, \eta^{K-1}\right)$.

Proof. a) Apply Lemma 1.9. With infinitesimal intervals:

$$\begin{aligned} P\left(\exists i : \eta_i^b \in [t, t+h]\right) &= P\left(\exists i : \left[t^{1/b}, (t+h)^{1/b}\right]\right) \\ &= P\left(\exists i : \left[t^{1/b}, t^{1/b} + b^{-1}t^{1/b-1}h\right]\right) \\ &= hb^{-1}t^{1/b-1}\left[a\left(t^{1/b}\right)^{-a-1}\right] = h\frac{a}{b}t^{a/b-1}. \end{aligned}$$

b) We have that

$$C\left(\eta^1, \ldots, \eta^{K-1}; m_K\right) \stackrel{\text{def}}{=} \sum_{i_1, \ldots, i_{K-1}} \left(\eta_{i_1}^1 \cdot \ldots \cdot \eta_{i_i, \ldots, i_{K-1}}^{K-1}\right)^{m_K} < \infty$$

almost surely. (This follows easily by induction on K. We already use it above). We condition on the first $K-1$ levels. Then

$$\left\{\eta_{i_1}^1 \cdot \ldots \cdot \eta_{i_i, \ldots, i_K}^K / C\left(\eta^1, \ldots, \eta^{K-1}; m_K\right)^{1/m_K}\right\}_{\mathbf{i}}$$

is a PPP $\left(m_K t^{-m_K - 1}\right)$, and therefore

$$\left\{\left[\eta_{i_1}^1 \cdot \ldots \cdot \eta_{i_i, \ldots, i_K}^K / C\left(\eta^1, \ldots, \eta^{K-1}; m_K\right)^{1/m_K}\right]^m\right\}_{\mathbf{i}}$$

is a PPP $\left(\frac{m_K}{m} t^{-m_K/m-1}\right)$. So the conditional distribution of $\mathcal{N}\left(\left\{\left(\eta_{i_1}^1 \cdot \ldots \cdot \eta_{i_i, \ldots, i_K}^K\right)^m\right\}_{\mathbf{i}}\right)$ is PD (m_K/m). As this conditional distribution does not depend on $\eta^1, \ldots, \eta^{K-1}$, the statement follows. ■

Proof of Proposition 3.16. As remarked above, we prove it (for the moment) only for $K = 2$. From the lemma above we see that conditionally on the first level $\left\{\eta_{i_1}^1\right\}_{i_1}$, the clustering is through marks, independently attached to the point process $\Xi = \mathcal{N}\left(\left\{\eta_{i_1}^1 \eta_{i_1 i_2}^2\right\}\right)$. The law of the latter does not depend on the realization of η^1, however the distribution of the marks does. Therefore, the clustering is stochastically independent of $\div$, and is obtained through a two-stage procedure: Depending on η^1, one computes a probability law on $\mathbb{N}$ through

$$\left\{p_{m_2, \eta^1}(i)\right\}_{i \in \mathbb{N}}$$

and chosen conditionally independent marks, given η^1, the marks attached to the point process $\div$. The matching is performed by matching points with the same marks. The total distribution of the matching is obtained by choosing η^1 according to its law. ■

Remark 3.19

For later use, it is important to carefully spell out the mechanism of the matching procedure. Conditionally on the first level η^1, both, the point process $\div$ and the matching depend on the point process η^2, and are stochastically independent, conditionally on η^1. The point process Ξ is however independent of η^1, whereas the matching is not. Nevertheless, this implies that Ξ and the matching are also unconditionally independent.

The point process $\left\{p_{m_2, \eta^1}(i)\right\}_{i \in \mathbb{N}}$ itself is obtained through normalizing $\left\{\left(\eta_i^1\right)^{m_2}\right\}$, i.e. it is a PD (m_2/m_1) according to Lemma 3.18.

3.4 The coalescent process

The aim of this section is to give further information about the structure of the distribution of $(\mathcal{Z}_0, \mathcal{Z}_1, \ldots, \mathcal{Z}_K)$. For that, we define a continuous time , time homogeneous, Markov process $\{\Gamma_t\}_{t\geq 0}$ taking values in the set of partitionings of $\mathbb{N}$ (or equivalently, in the set of equivalence relations). We write $\Pi_{\mathbb{N}}$ for the set of equivalence relations on $\mathbb{N}$. The set of equivalence relations is a subset of the set of all relations on $\mathbb{N}$. The latter is evidently a compact set, as it can be presented as $\{0,1\}^{\mathcal{P}(\mathbb{N})}$, where $\mathcal{P}(\mathbb{N})$ denotes the set of (unordered) pairs of $\mathbb{N}$.. It is readily checked that $\Pi_{\mathbb{N}}$ is a closed subset of the set of relations, and therefore, it is compact as well.

If I is a finite subset of $\mathbb{N}$, then we write Π_I for the set of equivalence relations on I. This is a finite set. We write $\pi_{\mathbb{N},I}$ for the natural projection $\Pi_{\mathbb{N}} \to \Pi_I$, and more generally, if $I \subset J \subset \mathbb{N}$, we write $\pi_{J,I}$ for the projection $\Pi_J \to \Pi_I$.

We construct the process $\{\Gamma_t\}$ via its projections $\Gamma_{t,I} \overset{\text{def}}{=} \pi_{\mathbb{N},I}(\Gamma_t)$ for $I \subset\subset \mathbb{N}$. $\{\Gamma_{t,I}\}$ itself is a Markov process which is not automatic from the Markov property of $\{\Gamma_t\}$, of course. Anyway, $\{\Gamma_{t,I}\}$ being a continuous time Markov process on the finite set Π_I, it is perfectly described by its transition matrix $R_{t,I}(\gamma,\gamma')$, $\gamma,\gamma' \in \Pi_I$, which then can be written as

$$R_{t,I} = \exp[tQ_I],$$

with the Q-matrix $Q_I(\gamma,\gamma')$, satisfying

$$\begin{aligned}\sum_{\gamma'} Q(\gamma,\gamma') &= 0,\ \forall\gamma,\\ Q(\gamma,\gamma') &\geq 0,\ \forall\gamma \neq \gamma'.\end{aligned}$$

Here it is: Transitions are possible only to coarser partitionings, i.e. from γ to a $\gamma' \prec \gamma$. Therefore, if γ has just one class, then no transitions are possible, and this is absorbing. This means that $Q_I(\gamma,\gamma') = 0$ if $|\gamma| = 1$. $|\gamma|$ here the number of classes in γ. If $|\gamma| = N \geq 2$, and if γ' is obtained from γ by clumping together exactly $k \geq 2$ classes, then

$$Q_I(\gamma,\gamma') \overset{\text{def}}{=} \frac{1}{(N-1)\binom{N-2}{k-2}}.$$

All other $Q_I(\gamma,\gamma')$ with $\gamma' \neq \gamma$ are 0. So, infinitesimally, only one clumping act is possible, but the number of clumped sets is not restricted. Furthermore, of course,

$$Q_I(\gamma,\gamma) = -\sum_{\gamma':\gamma'\neq\gamma} Q_I(\gamma,\gamma').$$

This defines in the standard way a Markov process $\{\Gamma_{t,I}\}_{t\geq 0}$.

Exercise 3.20
The transitions of the Markov process $\{\Gamma_{t,I}\}$, $I \subset\subset \mathbb{N}$, *are given in the following way. Conditioned on* $\{\Gamma_{t,I} = \gamma\}$, *the process stays in* γ *for an exponential time with expectation* $(|\gamma|-1)^{-1}$. *(Of course, if* $|\gamma| = 1$, *then the process stays there forever). At the*

jump time, $\xi \in \{2, \ldots, |\gamma|\}$ *classes are clumped with*

$$P(\xi = k) = \frac{|\gamma|}{|\gamma| - 1} \frac{1}{k(k-1)}.$$

Conditioned on $\{\xi = k\}$, *the* k *classes to be clumped together in one new class are chosen with equal probability among the* $\binom{N}{k}$ *possibilities.*

With probability one, the process reaches the absorbing one-class state after a finite time.

Somewhat surprisingly, the transition kernel $R_{t,I}$ for I finite, can be computed explicitly:

Proposition 3.21
Assume $\gamma \in \Pi_I$ *has* N *classes, and* γ' *is obtained by clumping* $r_1, r_2, \ldots, r_k \geq 1$ *classes of* γ, *with* $\sum_i r_i = N$. *Then*

$$R_{t,I}(\gamma, \gamma') = \frac{(k-1)!}{(N-1)!} \mathrm{e}^{-(k-1)t} \prod_{i=1}^{k} g\left(r_j, \mathrm{e}^{-t}\right), \tag{3.11}$$

where $g(r, m)$ *is defined in (3.7).*

Proof. We write $q_t(\gamma, \gamma')$ for the right-hand side of (3.11). Evidently, q_0 is the identity matrix. We prove

$$\frac{dq_t}{dt} = Qq_t.$$

From that, the claim follows.

We write $x = \mathrm{e}^{-t}$, $f_x(s) = s^x$, $s > 0$. Then

$$q_t(\gamma, \gamma') = (-1)^{N-k} \frac{(k-1)!}{(N-1)!} \prod_{i=1}^{k} f_x^{(r_i)}(1),$$

where $f^{(m)}$ denotes the m-th derivative w.r.t s.

For $m \geq 1$, one has

$$\begin{aligned} \frac{\partial f_x^{(m)}(1)}{\partial t} &= \left. \frac{\partial^m}{\partial s^m} (-x \log s) f_x(s) \right|_{s=1} \\ &= x \sum_{j=1}^{m} (-1)^j \binom{m}{j} (j-1)! f_x^{(m-j)}(1). \end{aligned}$$

The functions $f_x(s)$ satisfy the identity

$$x f_x^{(r)}(1) = f_x^{(r+1)}(1) + r f_x^{(r)}(1),$$

and implementing that, we get

$$\frac{\partial f_x^{(m)}(1)}{\partial t} = m! \sum_{j=2}^{m} (-1)^{j-1} \frac{f_x^{(m-j+1)}(1)}{j(j-1)(m-j)!} - m f_x^{(m)}(1),$$

where the sum over j is 0 in case $m = 1$.

Let now γ, γ' be as in the statement of the proposition. Then

$$\begin{aligned}
\frac{dq_t(\gamma,\gamma')}{dt} &= (-N+1)\, q_t(\gamma,\gamma') \\
&\quad + (-1)^{N-k} \frac{(k-1)!}{(N-1)!} \\
&\quad \times \sum_{i:m_i\geq 2} \sum_{r=2}^{m_i} (-1)^{r-1} \frac{m_i! f_x^{(m_i-r+1)}(1)}{(m_i-r)! r(r-1)x} \prod_{j:j\neq i} f_x^{(m_j)}(1) \\
&= (-N+1)\, q_t(\gamma,\gamma') \\
&\quad + \sum_{i:m_i\geq 2} \sum_{r=2}^{m_i} \frac{1}{(N-1)\binom{N-2}{r-2}} \binom{m_i}{r} (-1)^{N-k-r+1} \\
&\quad \times \frac{(k-1)!}{(N-r)!} \frac{1}{x} f_x^{(m_i-r+1)}(1) \prod_{j:j\neq i} f_x^{(m_j)}(1) \\
&= \sum_{\gamma'':\gamma'\prec\gamma''\prec\gamma} Q(\gamma,\gamma'')\, q_t(\gamma'',\gamma').
\end{aligned}$$

This proves the claim. ■

We next claim that the Markov processes on Π_I, $I \subset\subset \mathbb{N}$, are compatible, meaning that if $I \subset J$, then the Markov process constructed with values in Π_J, projected onto Π_I is the Markov process with this state space. This is proved by checking that the Q-matrices have the appropriate compatibility property, namely

Lemma 3.22
Let $\gamma, \gamma' \in \Pi_I$, $\gamma' \prec \gamma$, and $\tilde{\gamma}$ be any element in Π_J with $\pi_{J,I}(\tilde{\gamma}) = \gamma$. Then

$$Q_I(\gamma,\gamma') = \sum_{\substack{\tilde{\gamma}'\in\Pi_J:\tilde{\gamma}'\prec\tilde{\gamma}, \\ \pi_{J,I}(\tilde{\gamma}')=\gamma'}} Q_J(\tilde{\gamma},\tilde{\gamma}').$$

Proof. We only have to check the formula when γ' is obtained from γ by clumping k classes, γ having $N \geq k$ classes, $k \geq 2$. The chosen $\tilde{\gamma}$ may have N classes, too, or more. Say, it has $\tilde{N} \geq N$ classes. Now, in order to get by a simple clumping a partitioning $\tilde{\gamma}'$ which when restricted to I equals γ', one has several possibilities, but certainly, the extensions of the classes clumped in γ have to be clumped. Of the $\tilde{N} - N$ new classes in $\tilde{\gamma}$, the clumping of them has no influence on the trace on I. Therefore, if we decide to clump $l \leq \tilde{N} - N$ of them, there are simply $\binom{\tilde{N}-N}{l}$ to select this group which should be clumped, and in this case, we have

$$Q_J\left(\tilde{\gamma}, \tilde{\gamma}'\right) = \frac{1}{\left(\tilde{N}-1\right)\binom{\tilde{N}-2}{k+l-2}}.$$

Therefore, all we have to check is

$$\frac{1}{(N-1)\binom{N-2}{k-2}} = \sum_{l=0}^{\tilde{N}-N} \binom{\tilde{N}-N}{l} \frac{1}{\left(\tilde{N}-1\right)\binom{\tilde{N}-2}{k+l-2}}$$

which is elementary. ■

As a consequence, one obtains the compatibility of the semigroups $R_{t,I}$, namely that if $I \subset J$, and $\tilde{\gamma} \in \Pi_J$, then

$$R_{t,I}\left(\pi_{J,I}\left(\tilde{\gamma}\right), \cdot\right) = R_{t,J}\left(\tilde{\gamma}, \pi_{J,I}^{-1}\left(\cdot\right)\right). \tag{3.12}$$

and then, by soft arguments, one can extend the semigroup to a Feller semigroup $\{R_t\}$ on $\Pi_{\mathbb{N}}$, satisfying

$$R_{t,I}\left(\pi_{\mathbb{N},I}\left(\tilde{\gamma}\right), \cdot\right) = R_t\left(\tilde{\gamma}, \pi_{\mathbb{N},I}^{-1}\left(\cdot\right)\right) \tag{3.13}$$

for any $I \subset\subset \mathbb{N}$, $\tilde{\gamma} \in \Pi_{\mathbb{N}}$. This leads to a Feller process $\{\Gamma_t\}_{t\geq 0}$ taking values in $\Pi_{\mathbb{N}}$ which we start with the trivial partitioning of $\mathbb{N}$ into single points. This process is characterized by the property that its projections to $\Pi_I, I \subset\subset \mathbb{N}$, are Markov with the semigroup $R_{t,I}$.

Exercise 3.23
Prove that for any $t > 0$, Γ_t has infinitely many countably infinite classes, and no finite classes, almost surely. In particular, Γ_t is non-trivial for any $t > 0$.

Our next task is to relate the above semigroup to the clusterings coming from the Ruelle cascades.

To do that, we describe the semigroup in a different way.

Assume that γ is a partitioning of $\mathbb{N}$, $\gamma = \{C_1, C_2, \ldots\}$, and let $t > 0$. We first choose a PD $\left(\mathrm{e}^{-t}\right)$, leading to a random probability distribution $\bar{\eta} = \{\bar{\eta}_i\}_{i\in\mathbb{N}}$. Conditioned on this realization of the Poisson-Dirichlet process, we choose for every C_k independent random numbers Y_k where

$$P\left(Y_k = j \middle| \bar{\eta}\right) = \bar{\eta}_j.$$

Then we cluster the sets with the same number. This constructs a random partitioning $\gamma' \prec \gamma$. The corresponding kernel is denoted by S_t, i.e. $S_t(\gamma, \cdot)$ is the distribution of the above constructed random γ'.

Lemma 3.24

$$S_t = R_t, \ \forall t \geq 0.$$

Proof. For any finite $I \subset\subset \mathbb{N}$, we can define kernels $S_{t,I}$ in an evident way by restricting the above random matching mechanism to finitely many classes. By the very construction, one has (3.13) satisfied for the kernels S_t, $S_{t,I}$. It therefore suffices to prove $S_{t,I} = R_{t,I}$ for all finite I.

Let $\gamma \in \Pi_I$ have N classes, and γ' be obtained by clumping $r_1, \ldots, r_k$ classes together, $\sum r_i = N$. Then, conditioned on $\bar{\eta}$, the probability that under $S_{t,I}$ one has this clumping is

$$\sum\nolimits^*_{i_1,\ldots,i_k} \bar{\eta}_{i_1}^{r_1} \bar{\eta}_{i_2}^{r_2} \cdots \bar{\eta}_{i_k}^{r_k},$$

and so

$$S_{t,I}(\gamma,\gamma') = E \sum\nolimits^*_{i_1,\ldots,i_k} \bar{\eta}_{i_1}^{r_1} \bar{\eta}_{i_2}^{r_2} \cdots \bar{\eta}_{i_k}^{r_k},$$

the expectation with respect to the Poisson-Dirichlet process. This quantity, we have computed in Proposition 3.13:

$$E \sum\nolimits^*_{i_1,\ldots,i_k} \bar{\eta}_{i_1}^{r_1} \bar{\eta}_{i_2}^{r_2} \cdots \bar{\eta}_{i_k}^{r_k} = \frac{(k-1)!}{(N-1)!} \mathrm{e}^{-t(k-1)} \prod_{i=1}^{k} g\left(r_i, \mathrm{e}^{-t}\right),$$

where $g(r,m)$ is from (3.7). This is exactly the expression, we obtained Proposition 3.21 for $S_{t,I}(\gamma,\gamma')$. ■

We are now in the position to identify the law of the Ruelle-clustering in terms of the coalescent process:

We take as before $0 < m_1 < \cdots < m_K < 1$, and define the clustering $\mathcal{Z}_j,\ 0 \leq j \leq K$, by (3.10).

Theorem 3.25
The law of $(\mathcal{Z}_K, \mathcal{Z}_{K-1}, \ldots, \mathcal{Z}_1)$ is the same as that of $\left(\Gamma_0, \Gamma_{t_1}, \ldots, \Gamma_{t_{K-1}}\right)$ with

$$\mathrm{e}^{-t_i} = \frac{m_{K-i}}{m_K}.$$

Proof. We first check the case $K = 2$ where there is only one non-trivial partitioning, namely $\mathcal{Z}_1$, and where we have already done the computation in the proof of Proposition 3.16. There we have proved that $\mathcal{Z}_1$ is obtained by attaching marks to $\mathbb{N}$ coming from a PD $\left(\frac{m_1}{m_2}\right)$, and identifying points with the same marks. (See Remark 3.19). So this is exactly the procedure we have for the kernel S_{t_1} with $\mathrm{e}^{-t_1} = \frac{m_1}{m_2}$.

For the general $K \geq 2$ case, the same argument together with Lemma 3.18 shows that $\mathcal{Z}_{K-1}$ is obtained from the trivial (i.e. non-)clustering $\mathcal{Z}_K$ by applying the kernel S_{t_1} with $\mathrm{e}^{-t_1} = m_{K-1}/m_K$. Now, the way $\mathcal{Z}_{K-2}$ is obtained from $\mathcal{Z}_{K-1}$ is again simply by setting marks to the points of the $(K-1)$-st level, coming from the $(K-2)$-nd level, and matching points (i.e. clusters of the finite point process) which have the same marks. This transition is done via the kernel S_{t_2} where $t_2 = m_{K-2}/m_{K-1}$. There is however one difficulty: One has to check that the new clustering is not influenced (stochastically) by the first clustering.

For that, remember that $\mathcal{Z}_{K-1}$ is obtained through a two-stage procedure: One chooses the marks conditionally independent, according to a probability distribution which is computed from $\eta^1, \ldots, \eta^{K-1}$ trough

$$p^{(K-1)}\left(i_1, \ldots, i_{K-1}\right) = \frac{\left(\eta^1_{i_1}\eta^2_{i_1,i_2}\cdots\eta^{K-1}_{i_1,\ldots,i_{K-1}}\right)^{m_K}}{\text{normalization}}.$$

This is a PD $\left(m_{K-1}/m_K\right)$ which is independent of of $\eta^1, \ldots, \eta^{K-2}$. Now, the clustering from $\mathcal{Z}_{K-1}$ to $\mathcal{Z}_{K-2}$ is obtained again through conditionally independent marks, the distribution of the marks being given by

$$p^{(K-2)}\left(i_1, \ldots, i_{K-2}\right) = \frac{\left(\eta^1_{i_1}\eta^2_{i_1,i_2}\cdots\eta^{K-2}_{i_1,\ldots,i_{K-2}}\right)^{m_K-2}}{\text{normalization}}$$

which is a PD $\left(m_{K-2}/m_{K-1}\right)$, depends on η^{K-2}, of course, but is independent of $\eta^1, \ldots, \eta^{K-3}$. Therefore, the clustering procedure from $\mathcal{Z}_{K-1}$ to $\mathcal{Z}_{K-2}$ is independent of the clustering procedure from $\mathcal{Z}_K$ to $\mathcal{Z}_{K-1}$: One just takes the clusters, puts the marks according to $p^{(K-2)}$ which itself is a PD $\left(m_{K-2}/m_{K-1}\right)$, so that the transition is simply by S_{t_2}. Then again, this clustering is independent of $\eta^1, \ldots, \eta^{K-3}$, and one proceeds in this way. $\blacksquare$

4 Guerra's replica symmetry breaking bound: The Aizenman-Sims-Starr proof

4.1 The Aizenman-Sims-Starr random overlap structures

Definition 4.1
A ***random overlap structure*** $\mathcal{R}$ *(ROSt for short) consists of a finite or countable set* A, *a probability space* $(\Gamma, \mathcal{G}, \mathbb{P})$, *and random variables* $\eta_\alpha \geq 0$, $q_{\alpha,\alpha'}$, $\alpha, \alpha' \in A$, *satisfying the following properties*

1. $\sum_\alpha \eta_\alpha < \infty$
2. $\left(q_{\alpha,\alpha'}\right)$ *is positive definite and satisfies* $q_{\alpha,\alpha} = 1$.

The η_α play the rôle of (unnormalized) Gibbs weights, and the q's are the abstract overlaps.

Example 4.2
As an example take $A = \Sigma_N \stackrel{\text{def}}{=} \{-1,1\}^N$. *For* η_σ, $\sigma \in \Sigma_N$, *we take*

$$\eta_\sigma \stackrel{\text{def}}{=} \exp\left[\frac{\beta}{\sqrt{N}}\sum_{1\leq i<j\leq N} g_{ij}\sigma_i\sigma_j + h\sum_{i=1}^N \sigma_i\right].$$

For $q_{\sigma,\sigma'}$ *we take the standard overlap* $R_N\left(\sigma,\sigma'\right)$, *as introduced before. We write* $\mathcal{R}_N^{\text{SK}}$ *for this overlap structure. The* q *here are nonrandom. On the other hand, we can use a* *(random) reordering of the set A by ordering the η_σ downwards: $\eta_1 > \eta_2 > \ldots > \eta_{2^N}$. After this random reordering, the q become random: $q_{1,2}$ for instance is the overlap of the two indices with the largest η-weight.*

Example 4.3
Another overlap structure is defined by Ruelle's probability cascades introduced in the last section. Fix $0 = m_0 < m_1 < \ldots < m_K = 1$. We take $A = \mathbb{N}^K$, and the η are the (unnormalized) weights $\eta_{\mathbf{i}}$ as in the last section with $m_i,\ 1 \le i \le K$, see (3.8). There is a slight problem because we have to take the last parameter $m_K = 1$, which implies that $\sum_{\mathbf{i}} \eta_{\mathbf{i}} = \infty$. This will not cause any difficulties for what we do below. The overlaps are defined in the following way. Fix a sequence $0 \le q(1) < q(2) < \ldots < q(K) < q(K+1) = 1$, and we set

$$q_{\mathbf{i},\mathbf{i}'} = q\left(\max\left\{k : (i_1, \ldots, i_k) = \left(i'_1, \ldots, i'_k\right)\right\} + 1\right),$$

i.e. we measure the hierarchical distance on the tree, and weight it with the function q. For this random overlap structure, we write $\mathcal{R}_K^{\mathrm{Ruelle}}$.

Given any ROSt, we attach to it families of Gaussian random variables $\left(y_{\alpha,i}\right)_{\alpha\in A,\ i\in\mathbb{N}}$, $\left(\kappa_\alpha\right)_{\alpha\in A}$ by requiring

$$\mathbb{E}\left(\kappa_\alpha \kappa_{\alpha'}\right) = q^2_{\alpha,\alpha'}, \tag{4.1}$$

and the "cavity field" by

$$\mathbb{E}\left(y_{\alpha,j} y_{\alpha',j'}\right) = q_{\alpha,\alpha'} \delta_{j,j'}. \tag{4.2}$$

The κ and the y are independent. In case, the q's itself are random variables, these are just the conditional distributions, given (η, q). It is not difficult to see that such random variables exist. By an extension of the probability space, we can assume that all the random variables are defined on a single probability space.

For later use, we give the construction of the cavity variables for $\mathcal{R}_K^{\mathrm{Ruelle}}$. We simply write

$$y_{\mathbf{i}} = \sqrt{q(1)}\, g^{(0)} + \sum_{k=1}^{K} \sqrt{q(k+1) - q(k)}\, g^{(k)}_{i_1,\ldots,i_k}, \tag{4.3}$$

where the g's are independent standard Gaussians. Furthermore, the $y_{\mathbf{i},j},\ j \in \mathbb{N}$, are independent copies of $y_{\mathbf{i}}$. The $\kappa_{\mathbf{i}}$ are constructed in a similar way.

The above notion of a ROSt needs some explanation. The basic idea comes from what in the physics literature is called the "cavity method". We consider the standard SK-Hamiltonian, but now with $N + M$ spins, where one should think of N being much larger than M. We then try to write the Hamiltonian in terms of the Hamiltonian on N

spin variables acting on the M "newcomers". We write $\tau_i = \sigma_{N+i}$ for the newcomers.

$$
\begin{aligned}
&\frac{\beta}{\sqrt{N+M}} \sum_{1\leq i<j\leq N+M} g_{ij}\sigma_i\sigma_j + h\sum_{i=1}^{N+M}\sigma_i \\
&= \frac{\sqrt{N}}{\sqrt{N+M}}\frac{\beta}{\sqrt{N}} \sum_{1\leq i<j\leq N} g_{ij}\sigma_i\sigma_j + h\sum_{i=1}^{N}\sigma_i + \frac{\beta}{\sqrt{N+M}}\sum_{i=1}^{N}\left(\sum_{j=1}^{M} g_{i,N+j}\sigma_i\right)\tau_j \\
&+ \frac{\beta}{\sqrt{N+M}} \sum_{1\leq i<j\leq M} g_{N+i,N+j}\tau_i\tau_j + h\sum_{j=1}^{M}\tau_j.
\end{aligned}
$$

We neglect parts which are stochastically $o(1)$ for $N\to\infty$, M fixed. In particular, we can neglect the interaction among the newcomers, i.e. we can drop the fourth summand on the right hand side above. Furthermore, we may as well replace $\sqrt{N+M}$ by $\sqrt{N}$ in the third summand. Defining the cavity variables

$$
y_{\sigma,j} \stackrel{\text{def}}{=} \frac{1}{\sqrt{N}}\sum_{j=1}^{M} g_{i,N+j}\sigma_i,
$$

we see that they have exactly the right distribution as required in (4.2), with respect to the random overlap structure $\mathcal{R}_N^{\text{SK}}$ coming from the N system. In the first summand, we have to be more careful: Put $U(\sigma) \stackrel{\text{def}}{=} \sum_{1\leq i<j\leq N} g_{ij}\sigma_i\sigma_j$. Then by (2.2)

$$
\begin{aligned}
\mathbb{E}\left(\frac{U(\sigma)}{\sqrt{N+M}}\frac{U(\sigma')}{\sqrt{N+M}}\right) &= \frac{N}{(N+M)}\mathbb{E}\left(\frac{U(\sigma)}{\sqrt{N}}\frac{U(\sigma')}{\sqrt{N}}\right) = \frac{N}{(N+M)}\left(\frac{N}{2}R_N(\sigma,\sigma')^2 - \frac{1}{2}\right) \\
&\approx \mathbb{E}\left(\frac{U(\sigma)}{\sqrt{N}}\frac{U(\sigma')}{\sqrt{N}}\right) - \frac{M}{2}q_{\sigma,\sigma'}^2
\end{aligned}
$$

taking here the SK-ROSt with $q_{\sigma,\sigma'} = R_N(\sigma,\sigma')$. Therefore,

$$
\left\{\frac{U(\sigma)}{\sqrt{N}}\right\}_{\sigma\in\Sigma_N} \approx^{\mathcal{L}} \left\{\frac{U(\sigma)}{\sqrt{N+M}} + \sqrt{\frac{M}{2}}\kappa_\sigma\right\}_{\sigma\in\Sigma_N},
$$

up to an error which disappears in the $N\to\infty$ limit, M fixed.

If we set

$$
\eta_\sigma \stackrel{\text{def}}{=} \exp\left[\frac{\beta}{\sqrt{N+M}}\sum_{i<j\leq N} g_{ij}\sigma_i\sigma_j + h\sum_{i=1}^{N}\sigma_i\right],
$$

we see that

$$
\begin{aligned}
Z_{N+N} &= \textstyle\sum_{\sigma\in\Sigma_N,\ \tau\in\Sigma_M} \eta_\sigma \exp\left[\sum_{i=1}^{M}(\beta y_{\sigma,i} + h)\tau_i\right], \\
Z_N &\approx \textstyle\sum_{\sigma\in\Sigma_N} \eta_\sigma \exp\left[\beta\sqrt{M/2}\kappa_\sigma\right],
\end{aligned}
$$

and therefore

$$\frac{Z_{N+M}}{Z_N} \approx \frac{\sum_{\sigma\in\Sigma_N,\ \tau\in\Sigma_M} \eta_\sigma \exp\left[\sum_{i=1}^{M}\left(\beta y_{\sigma,i}+h\right)\tau_i\right]}{\sum_{\sigma\in\Sigma_N}\eta_\sigma \exp\left[\beta\sqrt{M/2}\kappa_\sigma\right]}. \tag{4.4}$$

Here we have used the ROSt from the N-spin SK model (with Gibbs weights coming from a slightly changed temperature parameter). Aizenman, Sims and Starr had the idea to consider the above object when the N system is replaced by an *arbitrary* ROSt $\mathcal{R}$, and they consider the "relative finite M free energy" in the following way

$$G_M\left(\beta,h,\mathcal{R}\right) \stackrel{\text{def}}{=} \frac{1}{M}\mathbb{E}\left(\log\frac{\sum_{\alpha,\tau\in\Sigma_M}\eta_\alpha\exp\left[\sum_{i=1}^{N}\left(\beta y_{\alpha,i}+h\right)\tau_i\right]}{\sum_\alpha \eta_\alpha \exp\left[\beta\sqrt{M/2}\kappa_\alpha\right]}\right), \tag{4.5}$$

where the $\mathbb{E}$ expectation is taken with respect both to the law of the random overlap structure and the cavity variables $y_{\alpha,i}$ and κ_α. The idea is that when taking $\mathcal{R}=\mathcal{R}_N^{\mathrm{SK}}$, one has, up to a negligible correction,

$$G_M\left(\beta,h,\mathcal{R}_N^{\mathrm{SK}}\right) = \frac{1}{M}\log Z_{N+M} - \frac{1}{M}\log Z_N,$$

which should satisfy

$$\lim_{M\to\infty}\lim_{N\to\infty} G_M\left(\beta,h,\mathcal{R}_N^{\mathrm{SK}}\right) = f\left(\beta,h\right).$$

This is in fact correct, as we will see below, but is not very interesting, as one does not see why $G_M\left(\beta,h,\mathcal{R}_N^{\mathrm{SK}}\right)$ should be easier to compute than $f_N = N^{-1}\mathbb{E}\log Z_N$. The real issue is the following variant of Guerra's theorem:

Theorem 4.4 (Guerra, Aizenman-Sims-Starr)
For any M, and any random overlap structure $\mathcal{R}$ one has

$$f_M\left(\beta,h\right) \stackrel{\text{def}}{=} \frac{1}{M}\mathbb{E}\log Z_M \le G_M\left(\beta,h,\mathcal{R}\right), \tag{4.6}$$

Z_M here being the SK-partition function.

Proof. To a large extent it is a rerun of the computation done in Section 2.3.2. One uses the following interpolation:

$$-H_M\left(\tau,\alpha,t\right) \stackrel{\text{def}}{=} \frac{\sqrt{1-t}}{\sqrt{M}}\sum_{1\le i<j\le M} g_{ij}\tau_i\tau_j + \sqrt{\frac{M\left(1-t\right)}{2}}\kappa_\alpha + \sqrt{t}\sum_{i=1}^{M} y_{\alpha,i}\tau_i$$

and defines

$$\begin{aligned}\hat{G}_M\left(\beta,h,t,\mathcal{R}\right) &\stackrel{\text{def}}{=} \frac{1}{M}\mathbb{E}\left(\log\sum_{\alpha\in A,\ \tau\in\Sigma_M}\eta_\alpha\exp\left[-\beta H_M\left(\tau,\alpha,t\right)+h\sum_i \tau_i\right]\right) \\ &\quad -\frac{1}{M}\mathbb{E}\left(\log\sum_{\alpha\in A}\eta_\alpha\exp\left[\beta\sqrt{M/2}\kappa_\alpha\right]\right).\end{aligned} \tag{4.7}$$

where $\mathbb{E}$ is taken with respect to the overlap structure, i.e. the y's and the κ's, *and* the g's (which are supposed to be independent). For $t = 0$, the κ-part cancels, and one just gets $f_M(\beta, h)$. For $t = 1$, one gets $G_M(\beta, h, \mathcal{R})$.

We write $\mathcal{G}_t$ for the Gibbs distribution on $A \times \Sigma_M$ given by the unnormalized weights

$$u(\alpha, \tau) \overset{\text{def}}{=} \eta_\alpha \exp\left[-\beta H_M(\tau, \alpha, t) + h \sum\nolimits_i \tau_i\right],$$

i.e. $\mathcal{G}_t(\alpha, \tau) \overset{\text{def}}{=} u(\alpha, \tau)/Z$, with $Z \overset{\text{def}}{=} \sum_{\alpha,\tau} u(\alpha, \tau)$. Then we write $\nu^{(k)}$ for the expectation under $\mathbb{P} \otimes \mathcal{G}_t^{\otimes k}$, where $\mathbb{P}$ is the probability law, governing the cavity variables $y_{\alpha,i}$, the κ's, the g's, and the q's, if they are random. Remark however, that the g's and the rest are independent, and conditionally on the q's, the y's and the κ's are independent.

We compute the t-derivative of $\hat{G}_M$. Remark that the denominator on the right hand side does not depend on t, so it does not appear. We therefore get

$$\begin{aligned}
\frac{d\hat{G}}{dt} &= \frac{\beta}{M}\nu_t^{(1)}\left(\frac{d(-H_M)}{dt}\right) \\
&= \frac{\beta}{M}\mathbb{E}\left[\sum\nolimits_{\alpha,\tau} \frac{d(-H_M(\alpha,\tau))}{dt}\frac{u(\alpha,\tau)}{Z}\right]
\end{aligned}$$

$$\begin{aligned}
-\frac{dH_M(\tau,\alpha,t)}{dt} &= -\frac{1}{2\sqrt{M}\sqrt{1-t}}\sum\nolimits_{1\le i<j\le M} g_{ij}\tau_i\tau_j - \sqrt{\frac{M}{2}}\frac{1}{2\sqrt{1-t}}\kappa_\alpha \\
&\quad + \frac{1}{2\sqrt{t}}\sum_{i=1}^{M} y_{\alpha,i}\tau_i,
\end{aligned}$$

so we get

$$\frac{d\hat{G}}{dt} = -S_1 - S_2 + S_3, \text{ say.}$$

We use Gaussian partial integration

$$\begin{aligned}
\mathbb{E}\left(g_{ij}\frac{u(\alpha,\tau)}{Z}\right) &= \mathbb{E}\left(Z^{-1}\frac{\partial u(\alpha,\tau)}{\partial g_{ij}}\right) - \mathbb{E}\left(Z^{-2}u(\alpha,\tau)\frac{\partial Z}{\partial g_{ij}}\right) \\
&= \mathbb{E}\left(\frac{u(\alpha,\tau)}{Z}\frac{\beta\sqrt{1-t}}{\sqrt{M}}\tau_i\tau_j\right) \\
&\quad -\mathbb{E}\left(\frac{u(\alpha,\tau)}{Z^2}\sum\nolimits_{\alpha',\tau'} u(\alpha',\tau')\frac{\beta\sqrt{1-t}}{\sqrt{M}}\tau_i'\tau_j'\right).
\end{aligned}$$

So

$$\begin{aligned}
S_1 &= \frac{\beta^2(M-1)}{4M} - \frac{\beta^2}{2M^2}\sum_{\alpha,\tau}\sum\nolimits_{1\le i<j\le M}\tau_i\tau_j\frac{u(\alpha,\tau)}{Z^2}\sum\nolimits_{\alpha',\tau'} u(\alpha',\tau')\tau_i'\tau_j' \\
&= \frac{\beta^2}{4}\left[1 - \nu_t^{(2)}\left(R_M(\tau,\tau')^2\right)\right].
\end{aligned}$$

Here we stress a bit the notation. $\nu_t^{(2)}$ is a probability measure on $(A \times \Sigma_M)^2$. When we write $\nu_t^{(2)}\left(R_M(\tau,\tau')^2\right)$, we mean in fact that we sum $R_M(\tau,\tau')^2$ over $((\alpha,\tau),(\alpha',\tau'))$, weighted with the probabilities from $\nu_t^{(2)}$.

We do the same type of computation for S_2, but here we have to use Gaussian partial integration for correlated Gaussians, i.e. the Wick Theorem 1.2.

$$
\begin{aligned}
\mathbb{E}\left(\kappa_\alpha \frac{u(\alpha,\tau)}{Z}\right) &= \mathbb{E}\left(Z^{-1} q(\alpha,\alpha)^2 \frac{\partial u(\alpha,\tau)}{\partial \kappa_\alpha}\right) - \mathbb{E}\left(Z^{-2} u(\alpha,\tau) \frac{\partial Z}{\partial \kappa_\alpha}\right) \\
&= \beta \mathbb{E}\left(\frac{u(\alpha,\tau)}{Z}\sqrt{\frac{M(1-t)}{2}}\right) \\
&\quad -\mathbb{E}\left(Z^{-2} u(\alpha,\tau) \sum\nolimits_{\alpha'.\tau'} q(\alpha,\alpha')^2 \frac{u(\alpha',\tau')}{\partial \kappa_{\alpha'}}\right) \\
&= \beta \mathbb{E}\left(\frac{u(\alpha,\tau)}{Z}\sqrt{\frac{M(1-t)}{2}}\right) \\
&\quad -\mathbb{E}\left(Z^{-2} u(\alpha,\tau) \sum\nolimits_{\alpha'.\tau'} u(\alpha',\tau') q(\alpha,\alpha')^2 \beta \sqrt{\frac{M(1-t)}{2}}\right)
\end{aligned}
$$

Therefore

$$
\begin{aligned}
S_2 &= \frac{\beta}{M}\sqrt{\frac{M}{2}} \frac{1}{2\sqrt{1-t}} \sum_{\alpha,\tau} \mathbb{E}\left(\kappa_\alpha \frac{u(\alpha,\tau)}{Z}\right) \\
&= \frac{\beta^2}{4} \sum_{\alpha,\tau} \mathbb{E}\left\{\left(\frac{u(\alpha,\tau)}{Z}\right) - \left(Z^{-2} u(\alpha,\tau) \sum\nolimits_{\alpha'.\tau'} u(\alpha',\tau') q(\alpha,\alpha')^2\right)\right\} \\
&= \frac{\beta^2}{4}\left[1 - \nu_t^{(2)}\left(q(\alpha,\alpha')^2\right)\right].
\end{aligned}
$$

The same type of computation leads to

$$
S_3 = \frac{\beta^2}{2}\left[1 - \nu_t^{(2)}\left(R_M(\tau,\tau') q(\alpha,\alpha')\right)\right],
$$

and therefore

$$
-S_1 - S_2 + S_3 = \frac{\beta^2}{4} \nu_t^{(2)}\left(\left[R_M(\tau,\tau') - q(\alpha,\alpha')\right]^2\right) \geq 0.
$$

So, we get

$$
\frac{d\hat{G}_M(\beta,h,t,\mathcal{R})}{dt} \geq 0, \tag{4.8}
$$

and therefore,

$$
G_M(\beta,h,\mathcal{R}) = \hat{G}_M(\beta,h,1,\mathcal{R}) \geq \hat{G}_M(\beta,h,0,\mathcal{R}) = f_M(\beta,h)
$$

which immediately implies the theorem. ■

The theorem gives upper bounds for $f_M(\beta, h)$ by choosing any random overlap structure. Of course, the "correct" choice would be to pick the ROSt from SK, but then, one cannot do any computation. The really interesting bound comes from taking the Ruelle ROSt which gives the Parisi expression as an upper bound.

It is actually not difficult to see that one gets a full variational formula by optimizing over all ROSt's:

Theorem 4.5

$$f(\beta, h) = \lim_{M\to\infty} \inf_{\mathcal{R}} G_M(\beta, h, \mathcal{R}).$$

The result is not very useful as one cannot perform the infimum over all ROSt's.

Proof. The proof is by a simple application of the Guerra-Toninelli superadditivity result of Section 2.2. Let $\chi_N \overset{\text{def}}{=} N f_N(\beta, h)$. The Guerra-Toninelli result was

$$\chi_{N+M} \geq \chi_N + \chi_M.$$

This easily implies

$$\begin{aligned} f(\beta, h) &= \lim_{N\to\infty} \frac{\chi_N}{N} = \lim_{M\to\infty} \liminf_{N\to\infty} \frac{\chi_{N+M} - \chi_N}{M} \\ &= \lim_{M\to\infty} \liminf_{N\to\infty} \frac{1}{M} \mathbb{E}\left(\log \frac{Z_{N+M}}{Z_N}\right). \end{aligned}$$

The discussion previous to (4.4) implies that for any M

$$\liminf_{N\to\infty} \frac{1}{M} \mathbb{E}\left(\log \frac{Z_{N+M}}{Z_N}\right) = \liminf_{N\to\infty} \hat{G}_M\left(\beta, h, \mathcal{R}_N^{\text{SK}}\right) \geq \inf_{\mathcal{R}} \hat{G}_M(\beta, h, \mathcal{R}).$$

Therefore, the claim follows. ■

4.2 Guerra's replica symmetry breaking bound

We first have to explain the Parisi formula for the SK-model:

Let $K \in \mathbb{N}$ (the number of symmetry breakings), and then we choose parameters

$$0 = m_0 < m_1 < \ldots < m_{K-1} < m_K = 1, \tag{4.9}$$

$$0 = q_0 \leq q_1 < \ldots < q_K < q_{K+1} = 1. \tag{4.10}$$

For $i = 0, \ldots, K$ let g_i be Gaussian with variance $\beta^2(q_{i+1} - q_i)$, and set $Y_{K+1} \overset{\text{def}}{=} \cosh\left(h + \sum_{i=0}^{K} g_i\right)$. Then one defines

$$Y_K \overset{\text{def}}{=} \left[E_K\left(Y_{K+1}^{m_K}\right)\right]^{1/m_K} = E_K(Y_{K+1}). \tag{4.11}$$

where E_K means that one integrates out g_K, so that Y_K still depends on $g_0, \ldots, g_{K-1}$. Then one defines

$$Y_{K-1} \stackrel{\text{def}}{=} \left[E_{K-1}\left(Y_K^{m_{K-1}}\right)\right]^{1/m_{K-1}}$$

and so on, until one gets Y_1. Y_1 is still a random variable as it depends on g_0. Remark however, that in case $q_1 = 0$ which we don't exclude, there is no randomness left. In any case, we set

$$\mathcal{P}_K\left(m, q; \beta, h\right) \stackrel{\text{def}}{=} E \log Y_1 - \frac{\beta^2}{4} \sum_{i=1}^{K} m_i \left(q_{i+1}^2 - q_i^2\right) + \log 2. \tag{4.12}$$

Theorem 4.6 (Parisi, Guerra, Talagrand)
For the SK-model free energy $f\left(\beta, h\right)$ one has

$$f\left(\beta, h\right) = \inf_{K,m,q} \mathcal{P}_K\left(m, q; \beta, h\right) = \lim_{K\to\infty} \inf_{m,q} \mathcal{P}_K\left(m, q; \beta, h\right).$$

I cannot give a proof of this here, and I restrict to prove the spectacular one-sided bound by Guerra which is based on the following computation.

Lemma 4.7
We take $\mathcal{R}_K^{\text{Ruelle}}$ as the ROSt from Example 4.3. Then

$$\hat{G}_M\left(\beta, h, \mathcal{R}\right) = \hat{G}_1\left(\beta, h, \mathcal{R}\right) = \mathcal{P}_K\left(m, q; \beta, h\right). \tag{4.13}$$

Proof. We handle the two parts in (4.5) separately. We take $M = 1$. It will be clear after the computation that for general M the outcome will be the same. We use the representation of the cavity variables (4.3):

$$\begin{aligned}
\frac{1}{2} \sum_{\mathbf{i}, \sigma \in \Sigma_1} \eta_{\mathbf{i}} \exp\left[\left(\beta y_{\mathbf{i}} + h\right)\sigma\right] &= \sum_{\mathbf{i}} \eta_{\mathbf{i}} \cosh\left(\beta y_{\mathbf{i}} + h\right) \\
&= \sum_{(i_1, \ldots, i_K)} \eta_{i_1}^1 \eta_{i_1 i_2}^2 \cdot \cdots \cdot \eta_{i_1 i_2 \ldots i_K}^K \\
&\quad \times \cosh\left(\beta \sqrt{q\left(1\right)} g^{(0)} + \sum_{k=1}^{K} \sqrt{q\left(k+1\right) - q\left(k\right)} g_{i_1, \ldots, i_k}^{(k)} + h\right).
\end{aligned}$$

We condition on $\eta_{i_1}^1, \eta_{i_1 i_2}^2, \ldots, \eta_{i_1 i_2 \ldots i_{K-1}}^{K-1}$ and $g^{(0)}, \ldots, g^{(K-1)}$. Then $\left(\eta_{i_1 i_2 \ldots i_K}^K\right)_{i_K \in \mathbb{N}}$ is a PPP $\left(t \to m_K t^{-m_K - 1}\right)$ whose points are multiplied by the independent random variables $\left(\cosh\left(\beta \sum_{n=0}^K g_{i_1, \ldots, i_n}^{(n)} + h\right)\right)_{i_K}$ .We know that the conditional law (conditioned on anything up to level $K-1$) of

$$\left\{\eta_{i_1 i_2 \ldots i_K}^K \cosh\left(\beta \sum_{n=0}^{K} g_{i_1, \ldots, i_n}^{(n)} + h\right)\right\}_{i_K}$$

is the same as that of

$$\{\bar{\eta}^K_{i_1,\ldots,i_K}\}_{i_K}$$

where

$$\bar{\eta}^K_{i_1,\ldots,i_K} \stackrel{\mathrm{def}}{=} C_K\left(\beta \sum\nolimits_{n=0}^{K-1} g^{(n)}_{i_1,\ldots,i_n}\right)\eta^K_{i_1 i_2\ldots i_K},$$

and

$$C_K(\xi) \stackrel{\mathrm{def}}{=} \left[E_Z \cosh^{m_K}\left(\xi + h + \beta\sqrt{q_{K+1} - q_K}Z\right)\right]^{1/m_K},\ \xi \in \mathbb{R},$$

Z being a standard Gaussian random variable, and E_Z the expectation with respect to Z. C_K is a random variable which still depends on the $g^{(n)}$ up to $n = K-1$. So, we have

$$\{\eta_{\mathbf{i}} \cosh(\beta y_{\mathbf{i}} + h)\}_{\mathbf{i}} =^{\mathcal{L}} \eta^1_{i_1}\eta^2_{i_1 i_2}\cdots\eta^{K-1}_{i_1,\ldots,i_{K-1}} C_K\left(\beta\sum\nolimits_{n=0}^{K-1} g^{(n)}_{i_1,\ldots,i_n}\right)\eta^K_{i_1,\ldots,i_K}.$$

Now, the law of the point process

$$\left\{\eta^{K-1}_{i_1,\ldots,i_{K-1}} C_K\left(\beta\sum\nolimits_{n=0}^{K-1} g^{(n)}_{i_1,\ldots,i_n}\right)\right\}_{i_{K-1}}$$

by the same reasoning, is the same is that of

$$\left\{C_{K-1}\left(\beta\sum\nolimits_{n=0}^{K-2} g^{(n)}_{i_1,\ldots,i_n}\right)\eta^{K-1}_{i_1 i_2\ldots i_{K-1}}\right\}_{i_{K-1}}$$

where

$$C_{K-1}(\xi) = \left[E_Z C_K^{m_{K-1}}\left(\xi + h + \beta\sqrt{q_{K+1} - q_K}Z\right)\right]^{1/m_{K-1}},$$

and therefore,

$$\{\eta_{\mathbf{i}} \cosh(\beta y_{\mathbf{i}} + h)\}_{\mathbf{i}} =^{\mathcal{L}} \eta^1_{i_1}\eta^2_{i_1 i_2}\cdots\eta^{K-2}_{i_1,\ldots,i_{K-2}} C_{K-1}\left(\beta\sum\nolimits_{n=0}^{K-2} g^{(n)}_{i_1,\ldots,i_n}\right)\eta^{K-1}_{i_1,\ldots,i_{K-1}}\eta^K_{i_1,\ldots,i_K}.$$

In this way, one proceeds, and arrives at

$$\{\eta_{\mathbf{i}} \cosh(\beta y_{\mathbf{i}} + h)\}_{\mathbf{i}} =^{\mathcal{L}} \{E \log Y_1 \eta_{\mathbf{i}}\}_{\mathbf{i}},$$

where $E\log Y_1$ is the constant from (4.12).

The second part in (4.5) is simpler because there one has in every step just an integration of a Gaussian in the exponent. We therefore see that multiplying the points $\eta_{\mathbf{i}}$ by $\exp\left[\left(\beta/\sqrt{2}\right)\kappa_{\mathbf{i}}\right]$ simply leads to a multiplication of the point process by $\exp\left[\left(\beta^2/4\right)\sum_{i=0}^K m_i\left(q^2_{i+1} - q^2_i\right)\right]$.

In the definition of G_1 (4.5), we would now like to argue that $\sum_{\mathbf{i}}\eta_{\mathbf{i}}$ cancels out. There is the slight difficulty that this sum diverges almost surely, because of $m_K = 1$, but we can choose m_K slightly less than 1, in which case the sum is finite, and so cancels, and then we can let $m_K \to 1$ in the end.

The upshot of this computation is that

$$\begin{aligned} G_1(\beta, h, \mathcal{R}) &= E \log Y_1 - \frac{\beta^2}{4} \sum_{i=0}^{K} m_i \left(q_{i+1}^2 - q_i^2 \right) + \log 2 \\ &= \mathcal{P}_K(m, q; \beta, h), \end{aligned}$$

the $\log 2$ is coming from dividing by 2 in (**??**). It is fairly evident from this computation that we get the same for arbitrary M. (One is just having M factors of $\cosh(\cdot)$ with independent contents, so in every step of the above argument, the factoring remains). ■

Combining this result with Theorem 4.4, one gets Guerra's result:

Theorem 4.8 (Guerra)

$$f_M(\beta, h) \leq \mathcal{P}_K(m, q; \beta, h)$$

for any K, and any sequence m and q. Therefore

$$f_M(\beta, h) \leq \inf_{K,m,q} \mathcal{P}_K(m, q; \beta, h).$$