Lectures on Schramm–Loewner evolution[1]

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May 9, 2010

These notes are based on a course given to Masters-level students in Cambridge. Their scope is the basic theory of Schramm–Loewner evolution, together with some underlying and related theory for conformal maps and complex Brownian motion. The structure of the notes is influenced by my attempt to make the material accessible to students having a working knowledge of basic martingale theory and Itô calculus.

[1]Draft version for Berlin mini-course, May 2010

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We review the notion of conformal isomorphism of complex domains and the question of existence and uniqueness of conformal isomorphisms between proper simply connected complex domains. Then we illustrate, by a simple special case, Loewner's idea of encoding the evolution of complex domains using a differential equation.

1.1 Conformal isomorphisms

A (*complex*) *domain* is a non-empty connected open subset of the complex plane $\mathbb{C}$. A holomorphic function f on a domain D is a *conformal map on* D if its derivative f' vanishes nowhere. If a conformal map f on D is injective, then it can be shown that its image $D' = f(D)$ is also a domain and its inverse $f^{-1} : D' \to D$ is also a conformal map. We call a bijective conformal map $f : D \to D'$ a *conformal isomorphism*. A domain is *proper* if it is not the whole of $\mathbb{C}$. Write $\mathbb{D}$ for the open disc having centre 0 and radius 1. We note the following fundamental result.

Theorem 1.1 (Riemann mapping theorem). *Let D be a proper simply connected domain. Then there exists a conformal isomorphism $\Phi : D \to \mathbb{D}$.*

1.2 Möbius transformations

For $\theta \in [0, 2\pi)$, the rotation map $z \mapsto e^{i\theta} z$ is a conformal automorphism of $\mathbb{D}$ fixing (the interior point) 0. For $\sigma \in (0, \infty)$ and $b \in \mathbb{R}$, the scaling and translation maps $z \mapsto \sigma z$ and $z \mapsto z + b$ are conformal automorphisms of the upper half-plane $\mathbb{H}$ fixing (the boundary point) ∞. The map $z \mapsto \Psi(z) = i(1+z)/(1-z) : \mathbb{D} \to \mathbb{H}$ is a conformal isomorphism. By taking compositions of these maps we obtain the *Möbius group* of conformal automorphisms of $\mathbb{D}$, each element of which has the form

$$\Phi_{\theta,w}(z) = e^{i\theta} \frac{z - w}{1 - \bar{w} z}, \quad z \in \mathbb{D}$$

for some $\theta \in [0, 2\pi)$ and $w \in \mathbb{D}$. Note that every Möbius transformation extends to a homeomorphism of the closed unit disc $\bar{\mathbb{D}}$.

The Möbius group has three real parameters. We now discuss three different ways in which a Möbius transformation Φ may be specified uniquely by three real constraints. First, the constraints $\Phi(w) = 0$ and $\Phi'(w) > 0$ are satisfied uniquely by $\Phi_{0,w}$. Then, given a boundary point b of $\mathbb{D}$, we can choose θ so that $e^{i\theta}\Phi_{0,w}(b) = 1$ and then the constraints $\Phi(w) = 0$ and $\Phi(b) = 1$ are satisfied uniquely by $\Phi_{\theta,w}$. Finally, given any three distinct boundary points b_1, b_2, b_3 of $\mathbb{D}$, ordered anticlockwise, we can rotate to put b_1 at 1, then apply Ψ to map to $\mathbb{H}$, then scale and translate to send $\Psi(b_2)$ to $-1 = \Psi(i)$ and $\Psi(b_3)$ to $0 = \Psi(-1)$ while fixing $\infty = \Psi(1)$, and finally map back to $\mathbb{D}$ by Ψ^{-1}. The resulting Möbius transformation takes b_1, b_2, b_3 to 1, i, -1 respectively and is the only one to do so.

The following is a basic result of complex analysis. We shall give a proof using Brownian motion in the next section.

Lemma 1.2 (Schwarz lemma). *Let $f : \mathbb{D} \to \mathbb{D}$ be a holomorphic function with $f(0) = 0$. Then $|f(z)| \leqslant |z|$ for all z. Moreover, if $|f(z)| = |z|$ for some $z \neq 0$, then $f(w) = e^{i\theta} w$ for all w, for some $\theta \in \mathbb{R}$.*

Corollary 1.3. *The Möbius transformations are the only conformal automorphisms of $\mathbb{D}$.*

Proof. Let Φ be a conformal automorphism of $\mathbb{D}$. Set $w = \Phi^{-1}(0)$. Then $f = \Phi \circ \Phi_{0,w}^{-1}$ is a conformal automorphism of $\mathbb{D}$ and $f(0) = 0$. Pick $u \in \mathbb{D} \setminus \{0\}$ and set $v = f(u)$. Note that $v \neq 0$. Now, either $|f(u)| = |v| \geqslant |u|$ or $|f^{-1}(v)| = |u| \geqslant |v|$. In any case, by the Schwarz lemma, there exists $\theta \in \mathbb{R}$ such that $f(z) = e^{i\theta} z$ for all z, and so $\Phi = f \circ \Phi_{0,w} = \Phi_{\theta,w}$. □

1.3 Boundary points

Let D be a proper simply connected domain. We shall be interested in the 'boundary of D seen from inside D'. This is not simply the set of limit points of D in $\mathbb{C} \setminus D$, and indeed sometimes may not even be identified with a subset of $\mathbb{C}$. Choose a conformal isomorphism $\Phi : D \to \mathbb{D}$. We say that a sequence $(x_n : n \in \mathbb{N})$ in D is *D-Cauchy* if $(\Phi(x_n) : n \in \mathbb{N})$ is Cauchy in $\mathbb{D}$. Since the image of a Cauchy sequence in $\mathbb{D}$ under any Möbius tranformation is another Cauchy sequence in $\mathbb{D}$, this notion does not depend on the choice of Φ. Call two D-Cauchy sequences $x = (x_n : n \in \mathbb{N})$ and $y = (y_n : n \in \mathbb{N})$ *equivalent* if $(x_1, y_1, x_2, y_2, \dots)$ is also a D-Cauchy sequence. Let $\hat{D}$ denote the set of equivalence classes of D-Cauchy sequences. We can define an injection $\iota : D \to \hat{D}$ by $\iota(z) = [(z, z, z, \dots)]$ and a bijection $\hat{\Phi} : \hat{D} \to \bar{\mathbb{D}}$ by $\hat{\Phi}(x_n : n \in \mathbb{N}) = \lim_n \Phi(x_n)$. We give $\hat{D}$ the topology of $\bar{\mathbb{D}}$. Define the *boundary* $\partial D = \hat{D} \setminus \iota(D)$. Note that $\hat{\Phi} \circ \iota = \Phi$ so $\hat{\Phi}$ maps ∂D onto the unit circle $C = \{z \in \mathbb{C} : |z| = 1\}$. For $b \in \partial D$, we say that a simply connected subdomain $N \subseteq D$ is a *neighbourhood of b in D* if $\{z \in \mathbb{D} : |z - \hat{\Phi}(b)| < \varepsilon\} \subseteq \Phi(N)$ for some $\varepsilon > 0$.

A *Jordan curve* is a continuous one-to-one map $\gamma : C \to \mathbb{C}$. Say D is a *Jordan domain* if $\bar{D} \setminus D$ is the image of a Jordan curve. It can be shown in this case that Φ extends to a homeomorphism of $\bar{D}$ to $\bar{\mathbb{D}}$, so ι extends naturally to a homeomorphism of $\bar{D}$ to $\hat{D}$ and we can identify ∂D with $\bar{D} \setminus D$.

On the other hand, an $\mathbb{H}$-Cauchy sequence is a sequence $(z_n : n \in \mathbb{N})$ in $\mathbb{H}$ which either converges in $\mathbb{C}$ or is such that $|z_n| \to \infty$ as $n \to \infty$. Hence we write $\partial \mathbb{H} = \mathbb{R} \cup \{\infty\}$. In the domain $D = \mathbb{H} \setminus (0, i]$, for $z \in [0, i)$, the D-Cauchy sequences $(z + (1 + i)/n : n \in \mathbb{N})$ and $(z + (-1 + i)/n : n \in \mathbb{N})$ are not equivalent, so their equivalence classes $z+$ and $z-$ are distinct boundary points.

By Corollary 1.3, the conformal isomorphism $\Phi : D \to \mathbb{D}$ is unique up to a Möbius transformation of $\mathbb{D}$. Let $w \in D$ and let $b_1, b_2, b_3 \in \partial D$ be distinct and ordered anti-clockwise. Then Φ may be specified uniquely by imposing any one of the three following additional conditions:

$$\Phi(w) = 0 \text{ and } \Phi'(w) > 0; \quad \Phi(w) = 0 \text{ and } \Phi(b_1) = 1; \quad \Phi(b_1) = 1, \Phi(b_2) = i, \Phi(b_3) = -1.$$

1.4 $SLE(0)$

Consider the (deterministic) process $\gamma = (\gamma_t : t \geqslant 0)$ in the closed upper half-plane $\bar{\mathbb{H}}$ given by

$$\gamma_t = 2i\sqrt{t}.$$

This process belongs to the family of processes $(SLE(\kappa) : \kappa \in [0, \infty))$ to which these notes are devoted, corresponding to the parameter value $\kappa = 0$. Think of γ as progressively eating away the upper half-plane so that what remains at time t is the subdomain $H_t = \mathbb{H} \setminus K_t$, where $K_t = \gamma(0, t] = \{\gamma_s : s \in (0, t]\}$. There is a conformal isomorphism $g_t : H_t \to \mathbb{H}$ given by

$$g_t(z) = \sqrt{z^2 + 4t}$$

which has the following asymptotic behaviour as $|z| \to \infty$

$$g_t(z) = z + \frac{2t}{z} + O(|z|^{-2}).$$

As we shall explain in Proposition 4.2, there is only one conformal isomorphism $H_t \to \mathbb{H}$ such that $g_t(z) - z \to 0$ as $|z| \to \infty$. Thus we can think of the family of maps $(g_t : t \geqslant 0)$ as a canonical encoding of the path γ.

Consider the vector field b on $\bar{\mathbb{H}} \setminus \{0\}$ defined by

$$b(z) = \frac{2}{z} = \frac{2(x - iy)}{x^2 + y^2}.$$

Fix $z \in \bar{\mathbb{H}} \setminus \{0\}$ and define

$$\zeta(z) = \inf\{t \geqslant 0 : \gamma_t = z\} = \begin{cases} y^2/4, & \text{if } z = iy \\ \infty, & \text{otherwise.} \end{cases}$$

Then $\zeta(z) > 0$ and $z \in \bar{K}_t$ if and only if $\zeta(z) \leqslant t$. Set $z_t = g_t(z)$. Then $(z_t : t < \zeta(z))$ satisfies the differential equation

$$\dot{z}_t = b(z_t)$$

and, if $\zeta(z) < \infty$, then $z_t \to 0$ as $t \uparrow \zeta(z)$. Thus $(g_t(z) : z \in \bar{\mathbb{H}} \setminus \{0\}, t < \zeta(z))$ is the (unique) maximal flow of the vector field b in $\bar{\mathbb{H}} \setminus \{0\}$. By *maximal* we mean that $(z_t : t < \zeta(z))$ cannot be extended to a solution of the differential equation on a longer time interval.

1.5 Loewner evolutions

Think of $SLE(0)$ as obtained via the associated flow $(g_t : t \geqslant 0)$ by iterating continuously a map $g_{\delta t}$, which nibbles an infinitesimal piece $(0, 2i\sqrt{\delta t}]$ of $\mathbb{H}$ near 0. Charles Loewner, in the 1920's, showed that one could evolve families of complex domains by more general continuous iterations, where the nibbling point ξ_t moves over time. Although it is not initially clear what this would mean for the path γ, one can certainly consider the flow in $\mathbb{H}$ of the time-dependent vector field

$$b(t,z) = \frac{2}{z - \xi_t}, \quad t \geqslant 0, z \in \mathbb{H}$$

and may hope to use this to describe implicitly a family of domains $(H_t : t \geqslant 0)$, and possibly a path γ.

Oded Schramm, in 1999, realized that for some conjectural conformally invariant scaling limits γ of planar random process, with a certain spatial Markov property, the process $\xi = (\xi_t : t \geqslant 0)$ would have to be a Brownian motion, of some diffusivity κ. The associated processes γ were at that time totally new and have since revolutionized our understanding of conformally invariant planar random processes.

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We review briefly some essentials of stochastic calculus and Brownian motion, including optional stopping, Itô's calculus and the strong Markov property of Brownian motion. We give precise definitions and statements, adapted to our later needs, but do not always state the definitive form of a result, nor do we give proofs.

2.1 Martingales and stopping times

For the purposes of our general discussion, we suppose given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a filtration $(\mathcal{F}_t)_{t \geqslant 0}$. An adapted integrable process $M = (M_t)_{t \geqslant 0}$ is a *martingale* if $\mathbb{E}(M_t|\mathcal{F}_s) = M_s$ almost surely for all $s, t \geqslant 0$ with $s \leqslant t$. We consider here only continuous martingales. A random variable T in $[0, \infty]$ is a *stopping time* if $\{T \leqslant t\} \in \mathcal{F}_t$ for all $t \geqslant 0$. We define

$$\mathcal{F}_T = \{A \in \mathcal{F} : A \cap \{T \leqslant t\} \in \mathcal{F}_t \text{ for all } t \geqslant 0\}.$$

By the optional stopping theorem, if M is a continuous martingale and T is a stopping time, then the stopped process M^T is also a continuous martingale, where $M_t^T = M_{T \wedge t}$. An adapted process M is a *local martingale* if there is a sequence of stopping times $(T_n : n \in \mathbb{N})$ with $T_n \uparrow \infty$ almost surely such that M^{T_n} is a martingale for all n. Then $(T_n : n \in \mathbb{N})$ is called a *reducing sequence* for M. If M is a continuous local martingale starting from 0, then we may obtain a reducing sequence by setting $T_n = \inf\{t \geqslant 0 : |M_t| \geqslant n\}$, which has the useful extra property that the martingales M^{T_n} are uniformly bounded. If M is a continuous local martingale and T is a finite-valued stopping time such that M^T is uniformly bounded, then $\mathbb{E}(M_T) = \mathbb{E}(M_0)$. Even if T can take the value ∞, if M^T is uniformly bounded, then the limit $M_\infty = \lim_{t \to \infty} M_t$ exists almost surely on $\{T = \infty\}$ and the identity $\mathbb{E}(M_T) = \mathbb{E}(M_0)$ remains true. Moreover, if S is another stopping time, with $S \leqslant T$, then $\mathbb{E}(M_T|\mathcal{F}_S) = M_S$ almost surely. These are all aspects of the optional stopping theorem.

To every continuous local martingale M there corresponds a unique continuous non-decreasing adapted process $[M]$ starting from 0, called the *quadratic variation* of M, which is characterized by the property that $(M_t^2 - [M]_t : t \geqslant 0)$ is a local martingale, and is given by

$$[M]_t = \lim_{n \to \infty} \sum_{k=0}^{\lfloor 2^n t \rfloor - 1} (M_{(k+1)2^{-n}} - M_{k2^{-n}})^2.$$

where the limit holds in probability, and uniformly on compact time intervals. From the preceding statement follows a polarized extension: to every pair of continuous local martingales M and N, there corresponds a unique continuous finite-variation adapted process $[M, N]$ starting from 0, called the *covariation* of M and N, which is characterized by the property that $(M_tN_t - [M,N]_t : t \geqslant 0)$ is a local martingale, and is given by

$$[M,N]_t = \lim_{n\to\infty} \sum_{k=0}^{\lfloor 2^n t\rfloor -1} (M_{(k+1)2^{-n}} - M_{k2^{-n}})(N_{(k+1)2^{-n}} - N_{k2^{-n}})$$

the limit holding in the same sense.

2.2 Brownian motion

A continuous adapted $\mathbb{R}^d$-valued process $B = (B_t)_{t\geqslant 0}$ is an $(\mathcal{F}_t)_{t\geqslant 0}$-*Brownian motion* if, for all $s, t \geqslant 0$ with $s \leqslant t$, the increment $B_t - B_s$ is independent of $\mathcal{F}_s$ and is normally distributed with mean 0 and covariance matrix $(t-s)I$. This property suffices to determine the law of B conditional on $\mathcal{F}_0$: for any non-negative measurable function F on the set of continuous paths $W = C([0,\infty), \mathbb{R}^n)$, we have

$$\mathbb{E}(F(B)|\mathcal{F}_0) = f(B_0) \quad \text{almost surely} \tag{1}$$

where

$$f(x) = \int_W F(w)\mu_x(dw), \quad x \in \mathbb{R}^d$$

and where μ_x is *Wiener measure with starting point* x.

Given a Brownian motion B and a stopping time T, we can define a new filtration $(\tilde{\mathcal{F}}_t)_{t\geqslant 0}$ by $\tilde{\mathcal{F}}_t = \mathcal{F}_{T+t}$ and, on the event $\{T < \infty\}$, we can define a new process $\tilde{B}$ by setting $\tilde{B}_t = B_{T+t}$. Then, conditional on $\{T < \infty\}$, $\tilde{B}$ is a $(\tilde{\mathcal{F}}_t)_{t\geqslant 0}$-Brownian motion. This is called the *strong Markov property* of Brownian motion. It is a powerful way to do computations for Brownian motion, taken in conjunction with equation (1) and using the properties of conditional expectation. We usually omit reference to the filtration unless we wish to make a statement involving more than one filtration, such as the strong Markov property.

A complex-valued process $Z = X + iY$ is a *complex Brownian motion* if (X, Y) is a Brownian motion in $\mathbb{R}^2$.

Lévy's characterization is a useful way to identify Brownian motions: for an $\mathbb{R}^d$-valued process $B = (B^1, \ldots, B^d)$, if B^i is a continuous local martingale for all i and if $[B^i, B^j]_t = t\delta_{ij}$ for all i, j and all $t \geqslant 0$, then B is a Brownian motion.

2.3 Itô's integral

Given a continuous finite-variation adapted process A and a continuous adapted process H, we can form the Lebesgue–Stieltjes integral $H \cdot A$. This is a continuous finite-variation adapted process starting from 0 and is given by

$$(H \cdot A)_t = \lim_{n\to\infty} \sum_{k=0}^{\lfloor 2^n t\rfloor -1} H_{k2^{-n}}(A_{(k+1)2^{-n}} - A_{k2^{-n}})$$

where the limit holds in probability, and uniformly on compact time intervals. Given a continuous local martingale M and a continuous adapted process H, we can form the *Itô integral* $H \cdot M$. This is a continuous local martingale starting from 0 and is given by

$$(H \cdot M)_t = \lim_{n \to \infty} \sum_{k=0}^{\lfloor 2^n t \rfloor - 1} H_{k2^{-n}} (M_{(k+1)2^{-n}} - M_{k2^{-n}})$$

the limit holding in the same sense. It is characterized among continuous local martingales starting from 0 by the property that $[H \cdot M] = H^2 \cdot [M]$. We usually write

$$(H \cdot M)_t = \int_0^t H_s dM_s, \quad (H \cdot A)_t = \int_0^t H_s dA_s.$$

A *continuous semimartingale* X is any process having a decomposition $X_t = X_0 + M_t + A_t$ for all $t \geqslant 0$, where X_0 is an $\mathcal{F}_0$-measurable random variable, M is a continuous local martingale starting from 0, and A is a continuous adapted process of finite variation, also starting from 0 The decomposition $X = X_0 + M + A$ is then unique, and we extend the Itô integral to continuous semimartingales by setting

$$\int_0^t H_s dX_s = \int_0^t H_s dA_s + \int_0^t H_s dM_s.$$

An $\mathbb{R}^d$-valued process is called a *continuous semimartingale* if each of its components is a continuous semimartingale (in the same filtration).

We shall make extensive use of Itô's formula: if D is an open set in $\mathbb{R}^d$ and $f : D \to \mathbb{R}$ is a C^2 function, and if $X = (X^1, \ldots, X^d)$ is a continuous semimartingale with values in D, then

$$f(X_t) = f(X_0) + \sum_{i=1}^d \int_0^t \partial_i f(X_s) dX_s^i + \frac{1}{2} \sum_{i,j=1}^d \int_0^t \partial_i \partial_j f(X_s) d[X^i, X^j]_s, \quad t \geqslant 0.$$

This is conveniently written in *differential form*

$$df(X_t) = \partial_i f(X_t) dX_t^i + \frac{1}{2} \partial_i \partial_j f(X_t) dX_t^i dX_t^j$$

where we sum over the repeated indices.

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We discuss the relation of Brownian motion to harmonic functions and then illustrate how this can be used by giving a proof of the Schwarz lemma. Then we show that the image of complex Brownian motion under a holomorphic function is a local martingale, indeed is itself a complex Brownian motion, up to a random change of time-scale. This leads to some useful formulae for harmonic measure.

3.1 Probabilistic solution of the Dirichlet problem

Let u be a C^2 function on $\mathbb{C} = \mathbb{R}^2$ which is harmonic in a bounded domain D. Fix $z \in D$ and let B be a complex Brownian motion starting from z. Consider the stopping time $T = T_D = \inf\{t \geqslant 0 : B_t \notin D\}$, then $T < \infty$ and $B_T \in \bar{D} \setminus D$, almost surely. Set

$$M_t = u(z) + \int_0^t \nabla u(B_s) dB_s$$

then M is a continuous local martingale. By Itô's formula, $u(B_t) = M_t$ for all $t \leqslant T$, so M^T is uniformly bounded and, by optional stopping,

$$u(z) = M_0 = \mathbb{E}(M_T) = \mathbb{E}(u(B_T)).$$

Hence u can be recovered from its restriction to $\bar{D} \setminus D$. (This argument is not special to two dimensions.)

Suppose now that $f = u + iv$ is a holomorphic function defined on a domain D_0 and that D is a bounded domain with $\bar{D} \subseteq D_0$. By the Cauchy–Riemann equations, the real and imaginary parts of f are harmonic in D_0 so, by a simple patching argument, there exist C^2 functions u and v on $\mathbb{R}^2$ which are harmonic on D and such that $f = u + iv$ on D. Thus we obtain the useful formula

$$f(z) = \mathbb{E}_z(f(B_{T_D})).$$

3.2 Proof of the Schwarz lemma

A number of results of complex analysis can be understood well using Brownian motion. Here we give a proof of Lemma 1.2. Let $f : \mathbb{D} \to \mathbb{D}$ be a holomorphic function with $f(0) = 0$. Fix $z \in \mathbb{D}$ and consider a complex Brownian motion B starting from z. Fix $r \in (|z|, 1)$ and $\varepsilon \in (0, 1 - |z|)$ and consider the (almost surely finite) stopping times

$$S = \inf\{t \geqslant 0 : |B_t| = r\}, \quad T = \inf\{t \geqslant 0 : |B_t - z| = \varepsilon\}.$$

Consider the function $g(w) = f(w)/w$. By Taylor's theorem, g is analytic and hence holomorphic in $\mathbb{D}$ so

$$g(z) = \mathbb{E}(g(B_S)) = \mathbb{E}(g(B_T)).$$

Now $|g(B_S)| \leqslant 1/r$, so, letting $r \uparrow 1$, we deduce that $|g(z)| \leqslant 1$ and hence $|f(z)| \leqslant |z|$.

Suppose now that $e^{i\theta}g(z) = 1$. We know that B_T is uniformly distributed on the set $C = \{w : |w - z| = \varepsilon\}$ and $|e^{i\theta}g(w)| \leqslant 1$ for all $w \in \mathbb{D}$. Hence we must have $e^{i\theta}g(w) = 1$ for all $w \in C$. We deduce that the set $A = \{w \in \mathbb{D} : e^{i\theta}g(w) = 1\}$ contains every open ball in $\mathbb{D}$ centred at z. By a finite number of iterations of this argument, it follows that $A = \mathbb{D}$.

3.3 Conformal invariance of Brownian motion

Theorem 3.1. *Let D and D' be complex domains and let $z \in D$ and $z' \in D'$. Let B and B' be complex Brownian motions starting from z and z' respectively. Set*

$$T = \inf\{t \geqslant 0 : B_t \notin D\}, \quad T' = \inf\{t \geqslant 0 : B'_t \notin D'\}.$$

Suppose that there exists a conformal isomorphism $\Phi : D \to D'$ such that $\Phi(z) = z'$. Set $\tilde{T} = \int_0^T |\Phi'(B_t)|^2 dt$ and define for $t < \tilde{T}$

$$\tau(t) = \inf\left\{s \geqslant 0 : \int_0^s |\Phi'(B_r)|^2 dr = t\right\}, \quad \tilde{B}_t = \Phi(B_{\tau(t)}).$$

Then $(\tilde{T}, (\tilde{B}_t)_{t<\tilde{T}})$ and $(T', (B'_t)_{t<T'})$ have the same distribution.

Proof. Assume for now that Φ extends to a conformal map on a neighbourhood of $\bar{D}$ in $\mathbb{C}$. Then we may define a continuous semimartingale Z and a continuous and strictly increasing adapted process A by setting

$$Z_t = \Phi(B_{T\wedge t}) + (B_t - B_{T\wedge t}), \quad A_t = \int_0^{T\wedge t} |\Phi'(B_s)|^2 ds + (t - T \wedge t).$$

and we may extend τ to a continuous function on $[0, \infty)$ by setting

$$\tau(t) = \inf\{s \geqslant 0 : A_s = t\}.$$

Write $\Phi = u + iv$, $B_t = X_t + iY_t$ and $Z_t = M_t + iN_t$. By Itô's formula, for $t \leqslant T$,

$$dM_t = \frac{\partial u}{\partial x}(B_t)dX_t + \frac{\partial u}{\partial y}(B_t)dY_t, \quad dN_t = \frac{\partial v}{\partial x}(B_t)dX_t + \frac{\partial v}{\partial y}(B_t)dY_t$$

and so, using the Cauchy–Riemann equations,

$$dM_t dM_t = |\Phi'(B_t)|^2 dt = dA_t = dN_t dN_t, \quad dM_t dN_t = 0.$$

On the other hand, for $t > T$,

$$dM_t = dX_t, \quad dN_t = dY_t, \quad dM_t dM_t = dt = dA_t = dN_t dN_t, \quad dM_t dN_t = 0.$$

Hence M, N, $M^2 - A$, $N^2 - A$ and MN are all continuous local martingales. Set $\tilde{M}_s = M_{\tau(s)}$ and $\tilde{N}_s = N_{\tau(s)}$. Then, by optional stopping, $\tilde{M}$, $\tilde{N}$, $\tilde{M}^2 - s$, $\tilde{N}^2 - s$ and $\tilde{M}\tilde{N}$ are continuous local martingales for the filtration $(\tilde{\mathcal{F}}_s)_{s\geqslant 0}$, where $\tilde{\mathcal{F}}_s = \mathcal{F}_{\tau(s)}$. Hence, by Lévy's characterization of Brownian motion, $\tilde{Z} = \tilde{M} + i\tilde{N}$ is a complex $(\tilde{\mathcal{F}}_s)_{s\geqslant 0}$-Brownian motion starting from $z' = \Phi(z)$. Since $\tilde{T} = \inf\{t \geqslant 0 : \tilde{Z}_t \notin D'\}$ and $\tilde{B}_t = \tilde{Z}_t$ for $t \leqslant \tilde{T}$, this proves the claimed identity of distributions.

In the case where Φ fails to be C^2 in a neighbourhood of $\bar{D}$, choose a sequence of open sets $D_n \uparrow D$ with $\bar{D}_n \subseteq D$ for all n. Set $D'_n = \Phi(D_n)$ and set

$$T_n = \inf\{t \geqslant 0 : B_t \notin D_n\}, \quad T'_n = \inf\{t \geqslant 0 : B'_t \notin D'_n\}.$$

Set $\tilde{T}_n = \int_0^{T_n} |\Phi'(B_t)|^2 dt$. Then $\tilde{T}_n \uparrow \tilde{T}$ and $T'_n \uparrow T'$ almost surely as $n \to \infty$. Since Φ is C^2 in a neighbourhood of $\bar{D}_n$, we know that $(\tilde{T}_n, (\tilde{B}_t)_{t<\tilde{T}_n})$ and $(T'_n, (B'_t)_{t<T'_n})$ have the same distribution for all n, which implies the desired result on letting $n \to \infty$. □

3.4 First exit distributions and harmonic measure

The conformal invariance property provides an effective means to calculate the distribution of Brownian motion on its first exit from a simply connected proper domain D. Let B be a complex Brownian motion starting from $z \in D$ and set $T = T_D = \inf\{t \geqslant 0 : B_t \notin D\}$, as above. Then $T < \infty$ almost surely. In the case $D = \mathbb{D}$ and $z = 0$, we know that B_t converges in $\bar{\mathbb{D}}$ as $t \uparrow T$, with limit B_T uniformly distributed on the unit circle. We can choose a conformal isomorphism $\Phi : D \to \mathbb{D}$ taking z to 0. By conformal invariance, as $t \uparrow T$, B_t converges in $\hat{D}$ to a limit $B^*_T \in \partial D$. Denote by $h_D(z, .)$ the distribution of B^*_T. This *first exit distribution* is also called the *harmonic measure for D starting from z*. In the case where D is a Jordan domain, we have $B^*_T = B_T$ and, as we showed above, for every harmonic function u in D which extends continuously to $\bar{D}$, we can recover u from its boundary values by

$$u(z) = \mathbb{E}(u(B_T)) = \int_{\partial D} u(s) h_D(z, ds).$$

We can compute the harmonic measure as follows. By conformal invariance, for $s_1, s_2 \in \partial D$ and $\theta_1, \theta_2 \in [0, 2\pi)$ such that $\Phi(s_j) = e^{i\theta_j}$ for $j = 1, 2$, we have

$$\mathbb{P}(B^*_T \in [s_1, s_2]) = \mathbb{P}_0(B_{T_\mathbb{D}} \in [e^{i\theta_1}, e^{i\theta_2}]) = \frac{\theta_2 - \theta_1}{2\pi}.$$

We often fix a parametrization of ∂D by some interval $I \subseteq \mathbb{R}$ and then regard $h_D(z, .)$ as a measure on I. For good parametrizations the harmonic measure then has a density given by

$$h_D(z, t) = \frac{1}{2\pi} \frac{d\theta}{dt}.$$

For example, take $D = \mathbb{D}$ and parametrize the boundary as $(e^{it} : t \in [0, 2\pi))$. For $w = x + iy \in \mathbb{D}$, the Möbius transformation $\Phi_{0,w}$ takes w to 0. The boundary parametrizations are then related by $e^{i\theta} = (e^{it} - w)/(1 - \bar{w}e^{it})$, so

$$h_\mathbb{D}(w, t) = \frac{1}{2\pi} \frac{1 - |w|^2}{|e^{it} - w|^2} = \frac{1}{2\pi} \frac{1 - x^2 - y^2}{(\cos t - x)^2 + (\sin t - y)^2}, \quad 0 \leqslant t < 2\pi. \tag{2}$$

Or take $D = \mathbb{H}$ with the obvious parametrization of the boundary by $\mathbb{R}$. Fix $w = x + iy \in \mathbb{H}$ and define $\Phi : \mathbb{H} \to \mathbb{D}$ by $\Phi(z) = (z - w)/(z - \bar{w})$ so that $\Phi(w) = 0$. The boundary parametrizations are related by $e^{i\theta} = (t - w)/(t - \bar{w})$, so

$$h_{\mathbb{H}}(w,t) = \frac{1}{\pi} \operatorname{Im} \left(\frac{1}{t - w} \right) = \frac{y}{\pi((x-t)^2 + y^2)}, \quad t \in \mathbb{R}.$$

3.5 An estimate for harmonic functions

The following lemma allows us to bound the partial derivatives of a harmonic function in terms of its supremum norm.

Lemma 3.2. *Let u be a harmonic function in D and let $z \in D$. Then*

$$\left| \frac{\partial u}{\partial x}(z) \right| \leqslant \frac{4\|u\|_\infty}{\pi \operatorname{dist}(z, \partial D)}.$$

Proof. It will suffice to prove, for all $\varepsilon > 0$, that the estimate holds with an extra factor of $1 + \varepsilon$ on the right. Fix $\varepsilon > 0$. By affine transformation, we reduce to the case where $z = 0$ and $\operatorname{dist}(0, \partial D) = 1 + \varepsilon$. Then u is continuous on $\bar{\mathbb{D}}$ so, for $z \in \mathbb{D}$,

$$u(z) = \int_0^{2\pi} u(e^{i\theta}) h_{\mathbb{D}}(z, \theta) d\theta.$$

The formula (2) shows that $\nabla h_{\mathbb{D}}(., \theta)$ is bounded on a neighbourhood of 0, uniformly in θ, with

$$\nabla h_{\mathbb{D}}(0, \theta) = \frac{1}{\pi}(\cos\theta, \sin\theta).$$

Hence we may differentiate under the integral sign to obtain

$$\nabla u(0) = \frac{1}{\pi} \int_0^{2\pi} u(e^{i\theta})(\cos\theta, \sin\theta) d\theta$$

so that

$$\left| \frac{\partial u}{\partial x}(0) \right| \leqslant \frac{\|u\|_\infty}{\pi} \int_0^{2\pi} |\cos\theta| d\theta = \frac{4\|u\|_\infty}{\pi} = \frac{4(1+\varepsilon)\|u\|_\infty}{\pi \operatorname{dist}(0, \partial D)}.$$

□

4

We consider partitions $\mathbb{H} = H \cup K$ where H is a simply connected domain and K is bounded. Thus H is a neighbourhood of ∞ in $\mathbb{H}$, as in Subsection 1.3. We call the complementary set K a *compact* $\mathbb{H}$*-hull*. We shall identify a canonical choice of conformal isomorphism $g_K : H \to \mathbb{H}$ and derive some useful properties.

4.1 Existence and uniqueness of g_K

Proposition 4.1. *Let $x \in \mathbb{R}$ and let D be a neighbourhood of x in $\mathbb{H}$. Let $\Phi : D \to \mathbb{H}$ be a conformal isomorphism which is bounded near x. Then Φ extends analytically to a neighbourhood of x in $\mathbb{C}$ with $\Phi'(x) > 0$.*

Proof. It will suffice to consider the case $x = 0$. Fix $\varepsilon > 0$ and consider the set

$$\tilde{D} = \{z \in \mathbb{C} : z \in D \text{ or } \overline{z} \in D \text{ or } z \in (-\varepsilon, \varepsilon)\}.$$

We can and do choose ε sufficiently small so that $\tilde{D}$ is a (proper) simply connected domain. By the Riemann mapping theorem, there exists a unique conformal isomorphism $\Psi : \tilde{D} \to \mathbb{C} \setminus ((-\infty, -1] \cup [1, \infty))$ with $\Psi(0) = 0$ and $\Psi'(0) > 0$. By reflection symmetry and uniqueness, $\overline{\Psi(z)} = \Psi(\bar{z})$. Hence Ψ must map $(-\varepsilon, \varepsilon)$ to $(-1, 1)$ and, since $\Psi'(0) > 0$, Ψ must restrict to a conformal isomorphism from D to $\mathbb{H}$. Consider now any conformal isomorphism $\Phi : D \to \mathbb{H}$ which is bounded near 0. Then $f = \Phi \circ \Psi^{-1}$ is a Mobius transformation and f is bounded near 0, so f is holomorphic in a neighbourhood of 0 in $\mathbb{C}$ with $f'(0) > 0$. But $\Phi = f \circ \Psi$ and $\Psi(0) = 0$ so, by the chain rule, Φ extends analytically to a neighbourhood of 0 in $\mathbb{C}$ with $\Phi'(0) > 0$. □

Proposition 4.2. *Let K be a compact $\mathbb{H}$-hull. There exists a unique conformal isomorphism $g_K : H \to \mathbb{H}$ such that $g_K(z) - z \to 0$ as $z \to \infty$. Moreover, for some $a_K \in \mathbb{R}$, we have*

$$g_K(z) = z + \frac{a_K}{z} + O(|z|^{-2}), \quad z \to \infty.$$

Proof. Set $D = \{z \in \mathbb{C} : -1/z \in H\}$. Then D is a neighbourhood of 0 in $\mathbb{H}$. By the Riemann mapping theorem, there exists a conformal isomorphism $\Phi : D \to \mathbb{H}$ which is bounded near 0. Then, by Proposition 4.1, Φ extends analytically to a neighbourhood of 0 in $\mathbb{C}$ with $\Phi'(0) > 0$. By translation and rescaling if necessary, we may choose Φ so that $\Phi(0) = 0$ and $\Phi'(0) = 1$. Then, by Taylor's theorem, for some $b, c \in \mathbb{R}$,

$$\Phi(z) = z + bz^2 + cz^3 + O(|z|^4), \quad z \to 0.$$

Define g_K on H by $g_K(z) = -1/\Phi(-1/z) - b$. It is a straightforward exercise to check that g_K is a conformal isomorphism to $\mathbb{H}$ having the claimed expansion at ∞, with $a_K = b^2 - c$.

Finally, if $g : H \to \mathbb{H}$ is any conformal isomorphism such that $g(z) - z \to 0$ as $z \to \infty$, then $\Phi = g_K \circ g^{-1}$ is a conformal automorphism of $\mathbb{H}$ with $f(z) - z \to 0$ as $z \to \infty$. But then f is a Möbius tranformation by the Schwarz lemma, so we must have $f(z) = z$ for all z, showing that $g = g_K$. □

The last two propositions combine to give the following whole-plane perspective, which is sometimes helpful. Write H^* for the domain containing all points of H and their complex conjugates, together with all points $x \in \mathbb{R}$ such that H is a neighbourhood of x in $\mathbb{H}$. Then g_K extends uniquely to a conformal isomorphism g_K^* with domain H^*. We have $g_K^*(\bar{z}) = \overline{g_K^*(z)}$ for all $z \in H^*$ and $g_K^*(H^*) = \mathbb{C} \setminus S$ for some compact set $S \subseteq \mathbb{R}$. We abuse notation in writing $g_K(x)$ for $g_K^*(x)$ when $x \in H^* \cap \mathbb{R}$, whilst continuing to regard g_K as defined on H.

Let K and g_K be as in the last proposition, and fix $r \in (0, \infty)$ and $b \in \mathbb{R}$. Set $\tilde{K} = rK + b = \{z \in \mathbb{H} : (z - b)/r \in K\}$. Define $g : \tilde{K} \to \mathbb{H}$ by $g(z) = rg_K((z - b)/r)$. Then g is a conformal isomorphism and

$$g(z) = z + \frac{r^2 a_K}{z} + O(|z|^{-2}), \quad z \to \infty.$$

Hence, by uniqueness, we have $g = g_{rK+b}$ and $a_{rK+b} = r^2 a_K$.

Consider now two compact $\mathbb{H}$-hulls K and K_0, with $K_0 \subseteq K$. Set $\tilde{K} = g_{K_0}(K \setminus K_0)$ and consider the function $g = g_{\tilde{K}} \circ g_{K_0}$. Now g_{K_0} restricts to a conformal isomorphism $H \to \tilde{H}$ so g is a conformal isomorphism $H \to \mathbb{H}$. Also, as $|z| \to \infty$ we have

$$g(z) = g_{K_0}(z) + \frac{a_{\tilde{K}}}{g_{K_0}(z)} + O(|g_{K_0}(z)|^{-2}) = z + \frac{a_{K_0} + a_{\tilde{K}}}{z} + O(|z|^{-2}).$$

So, by uniqueness, we obtain $g_K = g_{\tilde{K}} \circ g_{K_0}$ and $a_K = a_{\tilde{K}} + a_{K_0}$.

4.2 Estimates on g_K

Proposition 4.3. *There is a constant $C < \infty$ with the following properties. For all $r \in (0, \infty)$ and all $\xi \in \mathbb{R}$, for any compact $\mathbb{H}$-hull $K \subseteq r\mathbb{D} + \xi$, we have for all $z \in H$*

$$|g_K(z) - z| \leqslant Cr \tag{3}$$

and

$$\left| g_K(z) - z - \frac{a_K}{z - \xi} \right| \leqslant \frac{Cr|a_K|}{|z - \xi|^2}, \quad |z - \xi| \geqslant 2r. \tag{4}$$

Proof. We shall prove the result in the case $r = 1$ and $\xi = 0$. The general case then follows by scaling and translation. Define for $\theta \in [0, \pi]$

$$a_K(\theta) = \mathbb{E}_{e^{i\theta}}(\mathrm{Im}(B_T)),$$

where B is a complex Brownian motion starting from $e^{i\theta}$ and $T = T_H = \inf\{t \geqslant 0 : B_t \notin H\}$. Set

$$a = \int_0^\pi a_K(\theta) p(\theta) d\theta \tag{5}$$

where $p(\theta) = 2\sin\theta/\pi$. Set $\phi(z) = \mathrm{Im}(z - g_K(z))$, then ϕ is bounded and harmonic on H. Note that $\mathrm{Im}(g_K(B_t)) \to 0$ as $t \uparrow T$, so, by optional stopping and the strong Markov property,

$$\phi(z) = \mathbb{E}_z(\mathrm{Im}(B_T)) = \int_0^\pi h_D(z,\theta) a_K(\theta) d\theta, \quad |z| > 1 \tag{6}$$

where $D = \mathbb{H} \setminus \bar{\mathbb{D}}$. Consider the conformal isomorphism $w = \Phi(z) = z + z^{-1} : D \to \mathbb{H}$ and note that $\Phi(e^{i\theta}) = 2\cos\theta$. Then, for $z \in D$,

$$h_D(z,\theta) = h_{\mathbb{H}}(w, 2\cos\theta) \frac{d}{d\theta} \Phi(e^{i\theta}) = \mathrm{Im}\left(\frac{1}{2\cos\theta - w}\right) \frac{2\sin\theta}{\pi}.$$

There is a constant $C < \infty$ such that, for all $|z| \geqslant 3/2$ and $\theta \in (0,\pi)$,

$$\left|\frac{1}{w - 2\cos\theta} - \frac{1}{z}\right| = \frac{|2\cos\theta - z^{-1}|}{|z||z + z^{-1} - 2\cos\theta|} \leqslant \frac{C}{|z|^2}.$$

Define

$$f(z) = u(z) + iv(z) = g_K(z) - z - a/z, \quad z \in H$$

where a is given by the integral in (5). Then f is holomorphic in D and $f(z) \to 0$ as $z \to \infty$. Also

$$v(z) = \mathrm{Im}\left(-\frac{a}{z}\right) - \phi(z) = \int_0^\pi \mathrm{Im}\left(\frac{1}{w - 2\cos\theta} - \frac{1}{z}\right) a_K(\theta) p(\theta) d\theta$$

so $|v(z)| \leqslant Ca/|z|^2$ whenever $|z| \geqslant 3/2$.

Since $v(x) = 0$ for $|x| \geqslant 3/2$, we can extend v by reflection as a harmonic function in $\{|z| > 3/2\}$, with the same bounds. Hence, for $|z| \geqslant 2$, we can apply Lemma 3.2 in the domain $\{w \in \mathbb{C} : |w| > (3/4)|z|\}$ to obtain, for a new constant $C < \infty$,

$$\left|\frac{\partial v}{\partial x}(z)\right|, \left|\frac{\partial v}{\partial y}(z)\right| \leqslant \frac{Ca}{|z|^3}.$$

So, for all $\rho \geqslant 2$ and $\theta \in (0,\pi)$,

$$|f(\rho e^{i\theta})| = \left|\int_{\rho e^{i\theta}}^\infty f'(z) dz\right| \leqslant \sqrt{2} Ca \int_\rho^\infty s^{-3} ds = \frac{Ca}{\sqrt{2}\rho^2}.$$

This forces $a = a_K$ and gives the estimate (4). The estimate (3) then follows immediately for $|z| \geqslant 3/2$, and then for all z, because we must have $g_K(H \cap \{|z| \leqslant 3/2\}) \subseteq \rho\mathbb{D}$, where $\rho = \sup\{|g_K(z)| : |z| = 3/2\}$. $\square$

4.3 Half-plane capacity

The constant a_K is called the *half-plane capacity* of K, written $\text{hcap}(K)$. From the representation (5), we now know that

$$\text{hcap}(K) = a_K \geqslant 0$$

with equality only if $K = \emptyset$. From Subsection 4.1, we have, for $r \in (0, \infty)$ and $b \in \mathbb{R}$,

$$\text{hcap}(rK + b) = r^2 \, \text{hcap}(K)$$

and, for $K_0 \subseteq K$,

$$\text{hcap}(K_0) \leqslant \text{hcap}(g_{K_0}(K \setminus K_0)) + \text{hcap}(K_0) = \text{hcap}(K)$$

with equality only if $K_0 = K$. For the slit $S = (0, i]$, we have $g_S(z) = \sqrt{z^2 + 1}$ and $\text{hcap}(S) = 1/2$. On the other hand, for the half-disc $A = \bar{\mathbb{D}} \cap \mathbb{H}$, $g_A(z) = z + z^{-1}$ and $\text{hcap}(A) = 1$. By comparison with A, we see that

$$\text{hcap}(K) \leqslant \text{rad}(K)^2$$

for all compact $\mathbb{H}$-hulls K, where $\text{rad}(K)$ is the radius of the smallest ball centred on the real axis and containing K. In the proof of Proposition 4.3, we saw that $y h_D(iy, \theta) \to p(\theta)$ uniformly in $\theta \in [0, \pi]$ as $y \to \infty$. So, from (5) and (6) we obtain

$$\text{hcap}(K) = \lim_{y \to \infty} y \mathbb{E}_{iy}(\text{Im}(B_{T_H})).$$

Thus, half-plane capacity may be considered to measure the average height of the boundary of a hull seen by a Brownian motion started at ∞.

4.4 Boundary deformation under g_K

The following result illustrates another way that Brownian motion provides a tool to understand conformal maps.

Proposition 4.4. *Let K be a compact $\mathbb{H}$-hull and let $\xi \in \mathbb{R}$. Suppose that $H = \mathbb{H} \setminus K$ is a neighbourhood of ξ in $\mathbb{H}$. Then g_K extends analytically to a neighbourhood of ξ in $\mathbb{C}$, with $g_K'(\xi) \in (0, 1]$.*

Proof. Since g_K is bounded near ξ, the possibility of extending g_K analytically near ξ, with $g_K'(\xi) > 0$, is established in Proposition 4.1. Let B be a complex Brownian motion starting from $z = x + iy$ and consider

$$T_H = \inf\{t \geqslant 0 : B_t \notin H\}, \quad T_{\mathbb{H}} = \inf\{t \geqslant 0 : B_t \notin \mathbb{H}\}.$$

Note that, for $a, b \in \mathbb{R}$ with $a < b$, in the limit $y \to \infty$ with $x/y \to 0$, we have

$$\pi y \, \mathbb{P}_z(B_{T_{\mathbb{H}}} \in [a, b]) = \int_a^b \frac{y^2}{y^2 + (t - x)^2} dt \to b - a.$$

Suppose that H is a neighbourhood of $[a,b]$ in $\mathbb{H}$ and that $z \in H$, then by conformal invariance of Brownian motion

$$\mathbb{P}_{g_K(z)}(B_{T_{\mathbb{H}}} \in [g_K(a), g_K(b)]) = \mathbb{P}_z(B_{T_H} \in [a,b]) \leqslant \mathbb{P}_z(B_{T_{\mathbb{H}}} \in [a,b]).$$

Set $z = iy$ and write $g_K(z) = u + iv$. Note that, as $y \to \infty$, we have $v/y \to 1$, $v \to \infty$ and $u/v \to 0$. Hence on multiplying the preceding inequality by πy and letting $y \to \infty$ we obtain

$$g_K(b) - g_K(a) \leqslant b - a.$$

The bound $g_K'(\xi) \leqslant 1$ then follows by the mean value theorem. □

5

We now discuss two results which are fundamental to the theory of Schramm–Loewner evolutions. The first, due to Loewner, establishes a one-to-one correspondence between continuous real-valued paths $(\xi_t)_{t\geqslant 0}$ and increasing families $(K_t)_{t\geqslant 0}$ of compact $\mathbb{H}$-hulls having a certain local growth property. The null path $\xi_t \equiv 0$ corresponds to $K_t = (0, 2i\sqrt{t}]$. For smooth paths $(\xi_t)_{t\geqslant 0}$ starting from 0, we have $K_t = \{\gamma_s : 0 < s \leqslant t\}$ for some continuous path $(\gamma_t)_{t\geqslant 0}$ in $\bar{\mathbb{H}}$ starting from 0 and such that $\gamma_t \in \mathbb{H}$ for all $t > 0$. More generally, it may be the case that $(K_t)_{t\geqslant 0}$ is *generated by* a continuous path $(\gamma_t)_{t\geqslant 0}$ in $\bar{\mathbb{H}}$, meaning that H_t is the unbounded component of $\mathbb{H} \setminus \{\gamma_s : 0 < s \leqslant t\}$. The second key result, due (for $\kappa \neq 8$) to Rohde and Schramm, tells us that, when $(\xi_t)_{t\geqslant 0}$ is a Brownian motion, of any diffusivity $\kappa \in (0,\infty)$, the corresponding compact $\mathbb{H}$-hulls $(K_t)_{t\geqslant 0}$ are almost surely generated by such a path γ.

5.1 Local growth property and Loewner transform

Let $(K_t)_{t\geqslant 0}$ be a family of compact $\mathbb{H}$-hulls. Say that $(K_t)_{t\geqslant 0}$ is *increasing* if, for all $s,t \geqslant 0$ with $s < t$, we have $K_s \subseteq K_t$ and $K_t \setminus K_s \neq \emptyset$. Assume that $(K_t)_{t\geqslant 0}$ is increasing and set $K_{s,t} = g_{K_s}(K_t \setminus K_s)$. Say that $(K_t)_{t\geqslant 0}$ has the *local growth property* if, for all $T \geqslant 0$,

$$\sup_{s,t\in[0,T],\, 0<t-s\leqslant h} \operatorname{rad}(K_{s,t}) \to 0 \quad \text{as} \quad h \downarrow 0. \tag{7}$$

For such a family, by compactness, there is, for each $t \geqslant 0$, a unique $\xi_t \in \mathbb{C}$ such that $\xi_t \in \overline{K_{t,t+h}}$ for all $h > 0$. Moreover, $\xi_t \in \mathbb{R}$ and, using the estimate (3), we can show that ξ_t depends continuously on t. The process $(\xi_t)_{t\geqslant 0}$ is called the *Loewner transform* of $(K_t)_{t\geqslant 0}$. Note that we have, for each $t \geqslant 0$, as $h \downarrow 0$,

$$\operatorname{hcap}(K_{t+h}) - \operatorname{hcap}(K_t) = \operatorname{hcap}(K_{t,t+h}) \leqslant \operatorname{rad}(K_{t,t+h})^2 \to 0.$$

If we assume that $K_0 = \emptyset$ and $\operatorname{rad}(K_t) \to \infty$ as $t \to \infty$, then the map $t \mapsto \operatorname{hcap}(K_t)$ will be a homeomorphism of $[0,\infty)$. Then, by a time-reparametrization, we may if we wish assume that

$$\operatorname{hcap}(K_t) = 2t \quad \text{ for all } t \geqslant 0. \tag{8}$$

5.2 Loewner's differential equation

Let $(\xi_t)_{t\geqslant 0}$ now be any continuous real-valued function and consider the open set $U = \{(t,z) \in [0,\infty) \times \mathbb{C} : z \neq \xi_t\}$. Define a time-dependent, holomorphic vector field $b : U \to \mathbb{C}$ by

$$b(t,z) = \frac{2}{z - \xi_t} = \frac{2(x - \xi_t - iy)}{|z - \xi_t|^2}.$$

Define $U_n = \{(t,z) \in U : |z - \xi_t| > 1/n\}$ and note that, for all $(t,z), (t,z') \in U_n$,

$$|b(t,z) - b(t,z')| \leqslant 2n^2 |z - z'|.$$

By standard results in the theory of ordinary differential equations, for each $z \in \mathbb{C} \setminus \{\xi_0\}$, there is a unique $\zeta(z) \in (0, \infty)$ and a unique continuous map $t \mapsto z_t : [0, \zeta(z)) \to \mathbb{C}$ such that, for all $t \in [0, \zeta(z))$, we have $(t, z_t) \in U$ with

$$z_t = z + \int_0^t \frac{2}{z_s - \xi_s} ds \tag{9}$$

and, if $\zeta(z) < \infty$, then $|z_t - \xi_t| \to 0$ as $t \uparrow \zeta(z)$. Then $(z_t)_{t<\zeta(z)}$ is the *maximal solution* to *Loewner's differential equation* $\dot{z}_t = 2/(z_t - \xi_t)$ starting from z. Restricting to the upper half-plane, set $H_t = \{z \in \mathbb{H} : \zeta(z) > t\}$ and define $g_t : H_t \to \mathbb{C}$ by $g_t(z) = z_t$. We call the map $\zeta : \mathbb{C} \setminus \{\xi_0\} \to (0, \infty]$ the *lifetime* and we call the family of maps $(g_t)_{t \geqslant 0}$ the *Loewner flow* for driving function $(\xi_t)_{t \geqslant 0}$.

5.3 Loewner's theorem

Theorem 5.1. *Let $(\xi_t)_{t \geqslant 0}$ be a continuous real-valued function. Let ζ be the lifetime and $(g_t)_{t \geqslant 0}$ the Loewner flow for driving function $(\xi_t)_{t \geqslant 0}$. Set $K_t = \{z \in \mathbb{H} : \zeta(z) \leqslant t\}$ and $H_t = \mathbb{H} \setminus K_t$. Then $(K_t)_{t \geqslant 0}$ is an increasing family of compact $\mathbb{H}$-hulls having the local growth property, and parametrized so that* $\operatorname{hcap}(K_t) = 2t$ *for all t. Moreover, we obtain all such families of compact $\mathbb{H}$-hulls in this way. Moreover, $(K_t)_{t \geqslant 0}$ has Loewner transform $(\xi_t)_{t \geqslant 0}$. Moreover, for all $t \geqslant 0$, g_t is the unique conformal isomorphism $H_t \to \mathbb{H}$ such that $g_t(z) - z \to 0$ as $|z| \to \infty$. Moreover, for all $x \in \mathbb{R} \setminus \{\xi_0\}$, H_t is a neighbourhood of x in $\mathbb{H}$ if and only if $\zeta(x) > t$.*

Proof. Define $H_t^\dagger = \{z \in \mathbb{C} \setminus \{0\} : \zeta(z) > t\}$ and define $g_t^\dagger : H_t^\dagger \to \mathbb{C}$ by $g_t^\dagger(z) = z_t$, where $(z_t)_{t<\zeta(z)}$ is the maximal solution to Loewner's equation starting from z. By standard results for differential equations, the set $H_t^\dagger$ is open and the map $g_t^\dagger$ is holomorphic and injective, for all $t \geqslant 0$. By taking the complex conjugate in Loewner's equation, we see that $(\bar{z}_t)_{t<\zeta(z)}$ is the maximal solution starting from $\bar{z}$. Thus, $H_t^\dagger$ is closed under conjugation and $g_t^\dagger(\bar{z}) = \overline{g_t^\dagger(z)}$. We deduce that $H_t = H_t^\dagger \cap \mathbb{H}$ is open, that g_t is holomorphic and injective, and that, since $g_t^\dagger$ is injective, we must have $g_t(H_t) \subseteq \mathbb{H}$. We see also that H_t is a neighbourhood of x in $\mathbb{H}$ whenever $\zeta(x) > t$.

Note that $\operatorname{Im}(b(t, z)) < 0$ for all $(t, z) \in U$ with $z \in \mathbb{H}$. So, given $t \geqslant 0$ and $z \in \mathbb{H}$, we can solve the differential equation $\dot{z}_s = b(s, z_s)$ in $\mathbb{H}$, backwards in time, with terminal point $z_t = z$, to find a point $z_0 \in \mathbb{H}$ such that $g_t(z_0) = z$. Hence g_t maps H_t onto $\mathbb{H}$ and so is a conformal isomorphism from H_t to $\mathbb{H}$. This implies in particular that H_t is simply connected.

In order to establish the remaining claims about $(g_t)_{t \geqslant 0}$ and $(K_t)_{t \geqslant 0}$, we need some simple estimates for the Loewner flow. Fix $T \geqslant 0$ and set $r = \sup_{t \leqslant T} |\xi_t - \xi_0| \vee \sqrt{T}$. Fix $R \geqslant 4r$ and $z \in \mathbb{H}$ with $|z - \xi_0| \geqslant R$ and write z_t for $g_t(z)$ as usual. Define

$$\tau = \inf\{t \in [0, \zeta(z)) : |z_t - z| = r\} \wedge T.$$

Then $\tau < \zeta(z)$ and $|z_t - \xi_t| = |(z_t - z) + (z - \xi_0) + (\xi_0 - \xi_t)| \geqslant R - 2r$ for all $t \leqslant \tau$. Now

$$z_t - z = \int_0^t \frac{2}{z_s - \xi_s} ds, \quad z(z_t - z) - 2t = 2\int_0^t \frac{z - z_s - \xi_s}{z_s - \xi_s} ds.$$

so, for $t \leqslant \tau$,

$$|z_t - z| \leqslant \frac{2t}{R - 2r} \leqslant \frac{t}{r} \leqslant r, \quad |z(z_t - z) - 2t| \leqslant \frac{(4r + 2|\xi_0|)t}{R - 2r}.$$

The first estimate implies that $\tau = T$ (or $|z_\tau - z| \leqslant \tau/r < T/r \leqslant r$, a contradiction) and then $\zeta(z) > T$ so $z \in H_T$. Hence

$$|z - \xi_0| \leqslant 4r \text{ for all } z \in K_T. \tag{10}$$

and so K_T is a compact $\mathbb{H}$-hull. Then from the second estimate we deduce that, for all $t \geqslant 0$, we have $z(g_t(z) - z) \to 2t$ as $|z| \to \infty$. In particular $g_t(z) - z \to 0$ as $|z| \to \infty$, so $g_t = g_{K_t}$ and then $\operatorname{hcap}(K_t) = 2t$ for all t.

Fix $s \geqslant 0$. Define for $t \geqslant 0$

$$\tilde{\xi}_t = \xi_{s+t}, \quad \tilde{H}_t = g_s(H_{s+t}), \quad \tilde{K}_t = \mathbb{H} \setminus \tilde{H}_t, \quad \tilde{g}_t = g_{s+t} \circ g_s^{-1}.$$

Then $(\tilde{g}_t)_{t \geqslant 0}$ is the Loewner flow driven by $(\tilde{\xi}_t)_{t \geqslant 0}$, $\tilde{H}_t$ is the domain of $\tilde{g}_t$, and $\tilde{K}_t = g_s(K_{s+t} \setminus K_s) = K_{s,s+t}$. The estimate (10) applies to give

$$|z - \xi_s| \leqslant 4\left(\sup_{s \leqslant u \leqslant s+t} |\xi_u - \xi_s| \vee \sqrt{t}\right) \text{ for all } z \in K_{s,s+t}.$$

Hence $(K_t)_{t \geqslant 0}$ has the local growth property and has Loewner transform $(\xi_t)_{t \geqslant 0}$.

Suppose now that $(K_t)_{t \geqslant 0}$ is any increasing family of compact $\mathbb{H}$-hulls having the local growth property, parametrized so that $\operatorname{hcap}(K_t) = 2t$ for all t. Set $g_t = g_{K_t}$ and take $(\xi_t)_{t \geqslant 0}$ to be the Loewner transform of $(K_t)_{t \geqslant 0}$. We know that $(\xi_t)_{t \geqslant 0}$ is continuous and now show that $(g_t)_{t \geqslant 0}$ is the Loewner flow driven by $(\xi_t)_{t \geqslant 0}$.

Fix $z \in \mathbb{H}$ and define $\zeta(z) = \inf\{t \geqslant 0 : z \in K_t\}$. If $\zeta(z) < \infty$ then for $s < \zeta(z) < t$ we have $z \in K_t \setminus K_s$ so $g_s(z) \in K_{s,t}$ and so $|g_s(z) - \xi_s| \leqslant 2\operatorname{rad}(K_{s,t})$; hence by the local growth property $|g_s(z) - \xi_s| \to 0$ as $s \uparrow \zeta(z)$.

Fix $s, t \geqslant 0$ with $s \leqslant t$ and $z \in H_t$. Recall that $\operatorname{hcap}(K_{s,t}) = 2(t - s)$ and $g_{K_{s,t}}(g_s(z)) = g_t(z)$. Apply the estimates of Proposition 4.3 to the compact $\mathbb{H}$-hull $K_{s,t}$, taking $\xi = \xi_s$, $w = g_s(z)$ and $r = 2\operatorname{rad}(K_{s,t})$, to obtain

$$|g_t(z) - g_s(z)| \leqslant 2C\operatorname{rad}(K_{s,t}) \tag{11}$$

and

$$\left| g_t(z) - g_s(z) - \frac{2(t-s)}{g_s(z) - \xi_s} \right| \leqslant \frac{4C\operatorname{rad}(K_{s,t})(t-s)}{|g_s(z) - \xi_s|^2}$$

provided $|g_s(z) - \xi_s| \geqslant 4\,\mathrm{rad}(K_{s,t})$. Then, by the local growth property, the function $s \mapsto g_s(z)$ is continuous on $[0,t]$. Hence $|g_s(z) - \xi_s|$ is uniformly positive on $[0,t]$ and so, by the local growth property, $s \mapsto g_s(z)$ is differentiable on $[0,t]$ with

$$\dot{g}_s(z) = \frac{2}{g_s(z) - \xi_s}.$$

Hence $H_t = \{z \in \mathbb{H} : \zeta(z) > t\}$ and $(g_t)_{t \geqslant 0}$ is the Loewner flow driven by $(\xi_t)_{t \geqslant 0}$. Hence all such families of compact $\mathbb{H}$-hulls can be obtained by a Loewner flow.

Finally, fix $x \in \mathbb{R}$ and suppose that H_t is a neighbourhood of x in $\mathbb{H}$. Since $K_t = \cap_{s>t} K_s$, there exists $s > t$ such that H_s is a neighbourhood of x in $\mathbb{H}$. Write g_t^* for the extension of g_t as a conformal map on the reflected domain H_t^*. Then $x \in H_s^*$ so $g_t^*(x) \in g_t^*(H_s^*)$. On the other hand $\xi_t \in g_t^*(H_t^* \setminus H_s^*)$, so $g_t(x) \neq \xi_t$. On letting $z \to x$ in (11), we obtain $|g_t^*(x) - g_s^*(x)| \leqslant 2C\,\mathrm{rad}(K_{s,t})$, so $s \mapsto |g_s^*(x) - \xi_s|$ is continuous and hence uniformly positive on $[0,t]$. Now we can pass to the limit $z \to x$ in (the integrated form of) Loewner's equation to see that $(g_s^*(x) : s \leqslant t)$ is a solution, and hence we must have $\zeta(x) > t$. □

5.4 Rohde–Schramm theorem

Theorem 5.2. *Let $\kappa \in [0,\infty)$ and let $(\xi_t)_{t \geqslant 0}$ be a real Brownian motion of diffusivity κ. Let $(K_t)_{t \geqslant 0}$ be the family of compact $\mathbb{H}$-hulls with Loewner transform $(\xi_t)_{t \geqslant 0}$, given by Theorem 5.1. Then there exists a unique continuous random process $(\gamma_t)_{t \geqslant 0}$ in $\overline{\mathbb{H}}$ which generates $(K_t)_{t \geqslant 0}$.*

The importance of the family of such processes $\gamma = (\gamma_t)_{t \geqslant 0}$ was first recognised by Schramm. We call γ a *Schramm–Loewner evolution of parameter* κ or $SLE(\kappa)$ for short.

5.5 Scaling and Markov properties of SLE

Proposition 5.3. *Let γ be an $SLE(\kappa)$ for some $\kappa \in [0,\infty)$. Fix $r \in (0,\infty)$ and define a rescaled process $\tilde{\gamma} = (\tilde{\gamma}_t)_{t \geqslant 0}$ by*

$$\tilde{\gamma}_t = r^{-1}\gamma_{r^2 t}.$$

Then $\tilde{\gamma}$ is also an $SLE(\kappa)$.

Proof. Define for $t \geqslant 0$ and $z \in \mathbb{H}$

$$\tilde{\xi}_t = r^{-1}\xi_{r^2 t}, \quad \tilde{\zeta}(z) = r^{-2}\zeta(rz)$$

and for $t < \tilde{\zeta}(z)$ define

$$\tilde{g}_t(z) = r^{-1} g_{r^2 t}(rz).$$

Then $(\tilde{\xi}_t)_{t \geqslant 0}$ is a Brownian motion of diffusivity κ. Also, $\tilde{g}_0(z) = z$ and from Loewner's equation for $(g_t)_{t \geqslant 0}$ we obtain

$$\dot{\tilde{g}}_t(z) = \frac{2}{\tilde{g}_t(z) - \tilde{\xi}_t}, \quad t < \tilde{\zeta}(z)$$

with $\tilde{g}_t(z)-\tilde{\xi}_t \to 0$ as $t \uparrow \tilde{\zeta}(z)$ for all $z \in \mathbb{H}$. So $(\tilde{g}_t)_{t\geqslant 0}$ is the Loewner flow driven by $(\tilde{\xi}_t)_{t\geqslant 0}$. Define $\tilde{K}_t = r^{-1}K_{r^2 t}$. Then $\tilde{K}_t = \{z \in \mathbb{H} : \tilde{\zeta}(z) \leqslant t\}$ so $(\tilde{K}_t)_{t\geqslant 0}$ is the family of compact $\mathbb{H}$-hulls with Loewner transform $(\tilde{\xi}_t)_{t\geqslant 0}$. Now $\tilde{\gamma}$ generates $(\tilde{K}_t)_{t\geqslant 0}$ so $\tilde{\gamma}$ is an $SLE(\kappa)$. □

It is simplest to frame the Markov property in terms of the compact $\mathbb{H}$-hulls $(K_t)_{t\geqslant 0}$ rather than the path γ.

Proposition 5.4. *Let $(K_t)_{t\geqslant 0}$ be (the compact $\mathbb{H}$-hulls of) an $SLE(\kappa)$ for some $\kappa \in [0,\infty)$. Write $(g_t)_{t\geqslant 0}$ and $(\xi_t)_{t\geqslant 0}$ for the associated Loewner flow and transform. Fix $s \geqslant 0$ and for $t \geqslant 0$ set $\tilde{K}_t = K_{s,s+t} = g_s(K_{s+t} \setminus K_s)$. Then the family $(\tilde{K}_t - \xi_s)_{t\geqslant 0}$ is also an $SLE(\kappa)$ and is moreover independent of $(K_u)_{0\leqslant u\leqslant s}$.*

Proof. Note that $\tilde{g}_{\tilde{K}_t} \circ g_s = g_{s+t}$ so

$$\tilde{K}_{t,t+h} = \tilde{g}_{\tilde{K}_t}(g_s(K_{s+t+h} \setminus K_s) \setminus g_s(K_{s+t} \setminus K_s)) = g_{s+t}(K_{s+t+h} \setminus K_{s+t}) = K_{s+t+h,s+t}.$$

Hence $(\tilde{K}_t)_{t\geqslant 0}$ has Loewner transform $(\xi_{s+t})_{t\geqslant 0}$ and so $(\tilde{K}_t - \xi_s)_{t\geqslant 0}$ has Loewner transform $(\tilde{\xi}_t)_{t\geqslant 0}$, where $\tilde{\xi}_t = \xi_{s+t} - \xi_s$. The desired conclusions now follow because $(\tilde{\xi}_t)_{t\geqslant 0}$ is a Brownian motion of diffusivity κ independent of $(\xi_u)_{0\leqslant u\leqslant s}$. □

6

The restriction to the real line of the Loewner flow associated to $SLE(\kappa)$ is a simple transformation of a flow of Bessel processes. For these processes, which have scaling properties like Brownian motion and SLE, it is possible to do some explicit calculations. Once translated back in terms of the Loewner flow, we can deduce probabilities for certain events relating to the SLE path γ.

6.1 Hitting probabilities at 0 for the Bessel flow

Let B be a real Brownian motion starting from 0 and fix $a \in (0,\infty)$. Consider for each $x \in \mathbb{R} \setminus \{0\}$ the integral equation

$$X_t(x) = x + B_t + \int_0^t \frac{a}{X_s(x)} ds. \tag{12}$$

By standard results for ordinary differential equations, there exists $\zeta(x) \in (0,\infty)$ and a continuous path $(X_t(x))_{t<\zeta(x)}$ such that (12) holds for all $t \in [0,\zeta(x))$ and, if $\zeta(x) < \infty$, then $X_t(x) \to 0$ as $t \uparrow \zeta(x)$. Moreover, $\zeta(x)$ and $(X_t(x))_{t<\zeta(x)}$ are uniquely determined by these properties. Furthermore, for $x, y \in (0,\infty)$ with $x < y$, we have $\zeta(x) \leqslant \zeta(y)$ and $X_t(x) < X_t(y)$ for all $t < \zeta(x)$. (This can be shown by reversing time and using uniqueness of solutions for ordinary differential equations.)

Each of the processes $(X_t(x))_{t<\zeta(x)}$ is a Bessel process. Note that they are all constructed from a single Brownian motion. We call the whole family of processes the *Bessel flow*. In cases where the lifetime $\zeta(x)$ is finite, the solution $X_t(x)$ hits the point 0 where the drift is singular at time $\zeta(x)$, otherwise $X_t(x)$ has infinite lifetime and never hits 0. The following *scaling property* may be established by the same argument used for Proposition 5.3: fix $r \in (0,\infty)$ and set $\tilde{\zeta}(x) = r^{-2}\zeta(rx)$ and $\tilde{X}_t(x) = r^{-1}X_{r^2t}(rx)$, then the family of processes $(\tilde{X}_t(x))_{t<\tilde{\zeta}(x)}$ for $x \in \mathbb{R} \setminus \{0\}$ is also a Bessel flow.

Proposition 6.1. *Let $x, y \in (0,\infty)$ with $x < y$. Then*

(a) for $a \in (0,1/4]$, we have

$$\mathbb{P}(\zeta(x) < \zeta(y) < \infty) = 1;$$

(b) for $a \in (1/4,1/2)$, we have

$$\mathbb{P}(\zeta(x) < \infty) = 1, \quad \mathbb{P}(\zeta(x) < \zeta(y)) = \phi\left(\frac{y-x}{y}\right)$$

where ϕ is given by

$$\phi(\theta) = c\int_0^\theta \frac{du}{u^{2-4a}(1-u)^{2a}}, \quad \phi(1) = 1;$$

(c) for $a \in [1/2, \infty)$, we have

$$\mathbb{P}(\zeta(x) < \infty) = 0$$

and indeed, for $a \in (1/2, \infty)$,

$$\mathbb{P}\left(\inf_{t \geqslant 0} X_t(x) > 0\right) = 1.$$

Proof. Fix $x > 0$ and write $X_t = X_t(x)$ and $\zeta = \zeta(x)$. For $r \in (0, \infty)$ define a stopping time

$$T(r) = \inf\{t \in [0, \zeta) : X_t = r\}.$$

Fix $r, R \in (0, \infty)$ and assume that $0 < r < x < R$. Write $S = T(r) \wedge T(R)$. Note that $T(r) < \zeta$ on $\{\zeta < \infty\}$. Also, $X_t \geqslant B_t + x$ for all $t < \zeta$, so $T(R) < \infty$ almost surely on $\{\zeta = \infty\}$. In particular, $S < \infty$ almost surely. Assume for now that $a \neq 1/2$. Set $M_t = X_t^{1-2a}$ for $t < \zeta$. Note that M^S is uniformly bounded. By Itô's formula

$$dM_t = (1-2a)X_t^{-2a}dX_t - a(1-2a)X_t^{-2a-1}dt = (1-2a)X_t^{-2a}dB_t.$$

Hence M^S is a bounded martingale and by optional stopping

$$x^{1-2a} = M_0 = \mathbb{E}(M_S) = r^{1-2a}\mathbb{P}(X_S = r) + R^{1-2a}\mathbb{P}(X_S = R). \tag{13}$$

Note that as $r \downarrow 0$ we have $\{X_S = R\} \uparrow \{T(R) < \zeta\}$ and so $\mathbb{P}(X_S = R) \to \mathbb{P}(T(R) < \zeta)$. Similarly, $\mathbb{P}(X_S = r) \to \mathbb{P}(T(r) < \infty)$ as $R \to \infty$. For $a \in (0, 1/2)$, we can let $r \to 0$ in (13) to obtain

$$\mathbb{P}(T(R) < \zeta) = (x/R)^{1-2a}.$$

Then, letting $R \to \infty$, we deduce that $\mathbb{P}(\zeta = \infty) = 0$. For $a \in (1/2, \infty)$, we can let $R \to \infty$ in (13) to obtain

$$\mathbb{P}(T(r) < \infty) = (r/x)^{2a-1}$$

which implies $\mathbb{P}\left(\inf_{t \geqslant 0} X_t(x) > 0\right) = 1$ and hence $\mathbb{P}(\zeta = \infty) = 1$. In the case $a = 1/2$, we instead set $M_t = \log X_t$ and argue as above to obtain

$$\log x = \mathbb{P}(X_S = r)\log r + \mathbb{P}(X_S = R)\log R.$$

This forces $\mathbb{P}(X_S = r) \to 0$ as $r \to 0$ and so

$$\mathbb{P}(T(R) < \zeta) = 1.$$

Since $T(R) \to \infty$ as $R \to \infty$, it follows that $\mathbb{P}(\zeta = \infty) = 1$.

Assume from now on that $a \in (0, 1/2)$. It remains to show for $0 < x < y$ that

$$\mathbb{P}(\zeta < \zeta(y)) = \begin{cases} 1, & \text{if } a \leqslant 1/4 \\ \phi(\frac{y-x}{y}), & \text{if } a > 1/4. \end{cases}$$

Define for $\theta \in [0,1]$

$$\chi(\theta) = \int_\theta^1 \frac{du}{u^{2-4a}(1-u)^{2a}}.$$

Note that χ is continuous on $[0,1]$ as a map into $[0,\infty]$, with $\chi(0) < \infty$ for $a \in (1/4, 1/2)$ and $\chi(0) = \infty$ for $a \in (0, 1/4]$. Note also that χ is C^2 on $(0,1)$, with

$$\chi''(\theta) + 2\left(\frac{1-2a}{\theta} - \frac{a}{1-\theta}\right)\chi'(\theta) = 0.$$

Fix $y > x$ and write $Y_t = X_t(y)$. For $t < \zeta$, define $R_t = Y_t - X_t$, $\theta_t = R_t/Y_t$ and $N_t = \chi(\theta_t)$. By Itô's formula

$$dR_t = -\frac{aR_t dt}{X_t Y_t}, \quad d\theta_t = \left(\frac{\theta_t}{Y_t}\right)^2 \left(\frac{1-2a}{\theta_t} - \frac{a}{1-\theta_t}\right) dt - \frac{\theta_t}{Y_t} dB_t$$

so

$$dN_t = \chi'(\theta_t) d\theta_t + \frac{1}{2}\chi''(\theta_t) d\theta_t d\theta_t = -\frac{\chi'(\theta_t)\theta_t dB_t}{Y_t}.$$

Hence $(N_t : t < \zeta)$ is a local martingale.

Consider the random variables

$$A(x) = \int_0^\zeta \frac{1}{X_t^2} dt, \quad A_n(x) = \int_{T(2^{-n+1}x)}^{T(2^{-n}x)} \frac{1}{X_t^2} dt, \quad n \geqslant 1.$$

By the strong Markov property (of the driving Brownian motion), the random variables $(A_n(x) : n \in \mathbb{N})$ are independent. By the scaling property, they all have the same distribution. Hence, since $A_1(x) > 0$ almost surely, we must have $A(x) = \infty$ almost surely.

Every continuous local martingale is a time-change of Brownian motion. Hence, since N is non-negative, both N_t and the quadratic variation

$$[N]_t = \int_0^t \frac{\chi'(\theta_s)^2 \theta_s^2}{Y_s^2} ds$$

converge to a finite limit almost surely as $t \uparrow \zeta$. Hence θ_t must also converge as $t \uparrow \zeta$. If $\zeta < \zeta(y)$, then $\theta_\zeta = 1$ so $N_\zeta = 0$. If $\zeta = \zeta(y)$, then the conjunction of $A(y) = \infty$ and $[N]_\zeta < \infty$ forces $\theta_t \to 0$ as $t \uparrow \zeta$. In the case $a \in (0, 1/4]$, this would imply that $N_t = \chi(\theta_t) \to \infty$ as $t \uparrow \zeta$, a contradiction, so $\mathbb{P}(\zeta < \zeta(y)) = 1$. On the other hand, for $a \in (1/4, 1/2)$, the process N^ζ is a bounded martingale so by optional stopping

$$\chi\left(\frac{y-x}{y}\right) = N_0 = \mathbb{E}(N_\zeta) = \chi(0)\mathbb{P}(\zeta = \zeta(y)).$$

□

A variation of the calculation for $\mathbb{P}(\zeta(x) < \zeta(y))$ allows us to compute $\mathbb{P}(\zeta(x) < \zeta(-y))$.

Proposition 6.2. *Let $x, y \in (0, \infty)$. Then for $a \in (0, 1/2)$ we have*

$$\mathbb{P}(\zeta(x) < \zeta(-y)) = \psi\left(\frac{y}{x+y}\right)$$

where ψ is given by

$$\psi(\theta) = c\int_0^\theta \frac{du}{u^{2a}(1-u)^{2a}}, \quad \psi(1) = 1.$$

Proof. Note that ψ is continuous and increasing on $[0, 1]$ with $\psi(0) = 0$ and $\psi(1) = 1$. Also ψ is C^2 on $(0, 1)$ with

$$\psi''(\theta) + 2a\left(\frac{1}{\theta} - \frac{1}{1-\theta}\right)\psi'(\theta) = 0.$$

Write $X_t = X_t(x)$ and $Y_t = X_t(-y)$ and set $T = \zeta(x) \wedge \zeta(-y)$. For $t \leqslant T$ set $R_t = X_t + Y_t$ and $\theta_t = Y_t/R_t$. Define a process $Q = (Q_t)_{t\geqslant 0}$ by setting $Q_t = \psi(\theta_{T\wedge t})$. Then Q is continuous and uniformly bounded. Note that $Q_T = \theta_T$ and that $\theta_T = 1$ if $\zeta(x) < \zeta(-y)$ and $\theta_T = 0$ if $\zeta(-y) < \zeta(x)$. By Itô's formula, for $t \leqslant T$,

$$dR_t = \frac{aR_t}{X_tY_t}dt, \quad d\theta_t = \frac{a}{R_t^2}\left(\frac{1}{\theta_t} - \frac{1}{1-\theta_t}\right)dt - \frac{dB_t}{R_t}$$

so

$$dQ_t = \psi'(\theta_t)d\theta_t + \frac{1}{2}\psi''(\theta_t)d\theta_t d\theta_t = -\frac{\psi'(\theta_t)dB_t}{R_t}.$$

Hence Q is a bounded martingale. By optional stopping

$$\mathbb{P}(\zeta(x) < \zeta(-y)) = \mathbb{P}(\theta_T = 1) = \mathbb{E}(Q_T) = Q_0 = \psi(\theta_0) = \psi\left(\frac{y}{x+y}\right).$$

□

6.2 Hitting probabilities for $SLE(\kappa)$ on the real line

Fix $\kappa \in [0, \infty)$ and a real Brownian motion B starting from 0. Set $\xi_t = -\sqrt{\kappa}B_t$. Then $\xi = (\xi_t)_{t\geqslant 0}$ is a Brownian motion of diffusivity κ. Recall that ξ determines by Loewner's theorem a flow of conformal isomorphisms $g_t : H_t \to \mathbb{H}$ and that by the Rohde–Schramm theorem there is a continuous random process γ in $\mathbb{H}$ starting from 0 such that H_t is the unbounded connected component of $\mathbb{H} \setminus \gamma(0, t]$ for all t. Then γ is a realization of $SLE(\kappa)$. Recall also that each map g_t extends analytically to the reflected domain H_t^* and that for all $z \in \mathbb{C} \setminus \{0\}$ we have

$$\dot{g}_t(z) = \frac{2}{g_t(z) - \xi_t}, \quad t < \zeta(z)$$

where $\zeta(z) = \inf\{t \geqslant 0 : z \notin H_t^*\}$. Moreover if $\zeta(z) < \infty$ then $g_t(z) - \xi_t \to 0$ as $t \to \zeta(z)$.

For $x \in \mathbb{R} \setminus \{0\}$, set $\zeta^*(x) = \zeta(x\sqrt{\kappa})$ and define for $t < \zeta^*(x)$

$$X_t(x) = \frac{g_t(x\sqrt{\kappa}) - \xi_t}{\sqrt{\kappa}}.$$

Set $a = 2/\kappa$. Then

$$X_t(x) = x + B_t + \int_0^t \frac{a}{X_s(x)} ds$$

and $X_t(x) \to 0$ as $t \to \zeta^*(x)$ on $\{\zeta^*(x) < \infty\}$. So the family of processes $(X_t(x))_{t<\zeta^*(x)}$ is the Bessel flow of parameter a driven by B. The following result is a direct translation of Propositions 6.1 and 6.2. (Note that, since $\zeta(x) = \zeta^*(x/\sqrt{\kappa})$, the appearance of the ratio $y/(x+y)$ below is the result of a cancellation by $1/\sqrt{\kappa}$.)

Proposition 6.3. *Let $x, y \in (0, \infty)$. Then*

(a) for $\kappa \in [0, 4]$, we have

$$\mathbb{P}(\zeta(x) < \infty) = 0$$

and indeed, for $\kappa < 4$,

$$\mathbb{P}\left(\inf_{t\geqslant 0}(g_t(x) - \xi_t) > 0\right) = 1;$$

(b) for $\kappa \in (4, 8)$, we have

$$\mathbb{P}(\zeta(x) < \infty) = 1$$

and

$$\mathbb{P}(\zeta(x) < \zeta(x+y)) = \phi\left(\frac{y}{x+y}\right), \quad \mathbb{P}(\zeta(x) < \zeta(-y)) = \psi\left(\frac{y}{x+y}\right)$$

where ϕ and ψ are defined in Propositions 6.1 and 6.2;

(c) for $\kappa \in [8, \infty)$, we have

$$\mathbb{P}(\zeta(x) < \zeta(x+y) < \infty) = 1.$$

For $x \in (0, \infty)$ and $t \geqslant 0$, we have $t < \zeta(x)$ if and only if H_t is a neighbourhood of x in $\mathbb{H}$. This is true in turn if and only if $\gamma[0,t] \cap [x, \infty) = \emptyset$. Hence

$$\zeta(x) = \inf\{t \geqslant 0 : \gamma_t \in [x, \infty)\}$$

and, for $y \in (0, \infty)$, γ hits $[x, x+y)$ if and only if $\zeta(x) < \zeta(x+y)$. We make a second translation of Propositions 6.1 and 6.2 in terms of the $SLE(\kappa)$ path γ itself.

Proposition 6.4. *Let γ be an $SLE(\kappa)$. Then*

(a) for $\kappa \in [0, 4]$, we have $\gamma[0, \infty) \cap \mathbb{R} = \{0\}$ almost surely;

(b) for $\kappa \in (4,8)$ *and all* $x, y \in (0,\infty)$*,* γ *hits both* $[x,\infty)$ *and* $(-\infty,-y]$ *almost surely, and*

$$\mathbb{P}(\gamma \text{ hits } [x, x+y)) = \phi\left(\frac{y}{x+y}\right), \quad \mathbb{P}(\gamma \text{ hits } [x,\infty) \text{ before } (-\infty,-y]) = \psi\left(\frac{y}{x+y}\right);$$

(c) for $\kappa \in [8,\infty)$*, we have* $\mathbb{R} \subseteq \gamma[0,\infty)$ *almost surely.*

Proof. For $\kappa \in [0,4]$, we know that $\gamma_0 = 0$ and γ does not hit $(-\infty,-1/n]$ or $[1/n,\infty)$ almost surely, for all $n \in \mathbb{N}$. This implies (a). Statement (b) is a direct translation of the corresponding statement in Proposition 6.3. For $\kappa \in (8,\infty)$, we know that almost surely, for all rationals $x, y \in (0,\infty)$, $\gamma_t \in [x, x+y)$ for some $t \geqslant 0$. Since γ is continuous, this implies that $[0,\infty) \subseteq \gamma[0,\infty)$ almost surely, and (c) follows by symmetry. □