On differentiability of a flow for an SDE with discontinuous drift

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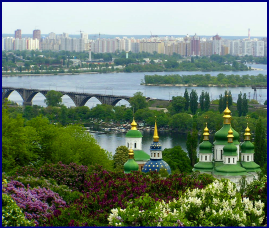


Problem

Consider an SDE

$$\begin{cases} d\varphi_t(x) = a(\varphi_t(x))dt + \sigma(\varphi_t(x))dw(t), \\ \varphi_0(x) = x, \end{cases}$$

where $x \in \mathbb{R}^d$, $(w(t))_{t\geq 0}$ is a d-dimensional Wiener process,

Problem

Let t, ω be fixed. We study the differentiability of the mapping $x \mapsto \varphi_t(x, \omega)$ from $\mathbb{R}^d$ to $\mathbb{R}^d$.

Ordinary differential equations

Example 1

Let $a = (a^1, \dots, a^d) \in C_b^1(\mathbb{R}^d)$

$$\begin{cases} d\varphi_t(x) = a(\varphi_t(x))dt, \\ \varphi_0(x) = x, \end{cases}$$

where $x \in \mathbb{R}^d$. Then for fixed t, the mapping $x \mapsto \varphi_t(x)$ is a diffeomophism.

The derivative satisfies the differential equation

$$\nabla\varphi_t(x) = E + \int_0^t \nabla a(\varphi_s(x))\nabla\varphi_s(x)ds,$$

where E is a $d \times d$-identity matrix, $\nabla a(x)$ is a $d \times d-$ matrix and

$$(\nabla a(x))^{ij} = \frac{\partial a^i(x)}{\partial x_j}.$$

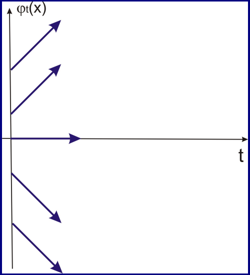
Ordinary differential equations

Example 2

Let $d = 1$. Consider $a(x) = \operatorname{sign} x$. Then

$$\varphi_t(x) = x + \int_0^t \operatorname{sign} \varphi_s(x) ds.$$

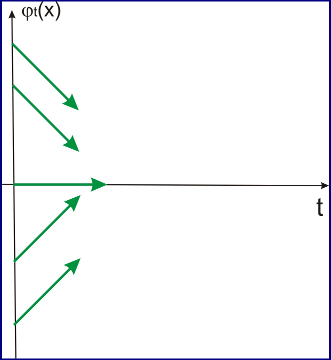
For all $t > 0$, the mapping $x \mapsto \varphi_t(x)$ is discontinuous at the point $x_0 = 0$ but it is an injection.



Ordinary differential equations

Example 3

Let $a(x) = -\operatorname{sign} x$. Then for all $t > 0$, the mapping $x \mapsto \varphi_t(x)$ is continuous but it is not an injection.



Stochastic differential equations

$$\begin{cases} d\varphi_t(x) = a(\varphi_t(x))dt + \sigma(\varphi_t(x))dw(t), \\ \varphi_0(x) = x. \end{cases}$$

- $\{a, \sigma\} \subset C_b^{1,\varepsilon}$. There exists a flow of diffeomorphisms. The derivative $\nabla\varphi_t(x) =: \psi_t(x)$ is a solution of equation

$$d\psi_t(x) = \nabla a(\varphi_t(x))\psi_t(x)dt + \nabla\sigma(\varphi_t(x))\psi_t(x)dw(t).$$

- $\{a, \sigma\} \subset Lip$.
  - there exists a flow of homeomorphisms [Kunita, 1990];
  - there exists a generalized derivative [Bouleau & Hirsch, 1991].
- if $\sigma \in C^2$ and it is non-degenerate, and $a \in H\ddot{o}l$ then there exists a flow of diffeomorphisms [Flandoli et al., 2010].

One-dimensional case

$$\begin{cases} d\varphi_t(x) = a(\varphi_t(x))dt + dw(t), \\ \varphi_0(x) = x, \end{cases} \tag{1}$$

where $a(x),\ x \in \mathbb{R}$, is measurable bounded function, $(w(t))_{t\geq 0}$ is a one-dimensional Wiener process.

For each $x \in \mathbb{R}$ there exists a unique strong solution to (1) (c. f. [Zvonkin, 1974]).

Heuristic approach

$$\begin{cases} d\varphi_t(x) = a(\varphi_t(x))dt + dw(t), \\ \varphi_0(x) = x. \end{cases}$$

Suppose that $a \in C_b^1$. Then

$$\psi_t(x) = (\varphi_t(x))'_x$$

satisfies the equation

$$d\psi_t(x) = a'(\varphi_t(x))\psi_t(x)dt.$$

Heuristic approach

$$\psi_t(x) = \exp\left\{\int_0^t a'(\varphi_s(x))ds\right\}.$$

By the occupation times formula

$$\psi_t(x) = \exp\left\{\int_{\mathbb{R}} L_z^{\varphi(x)}(t)a'(z)dz\right\} = \exp\left\{\int_{\mathbb{R}} L_z^{\varphi(x)}(t)da(z)\right\},$$

where $L_z^{\varphi(x)}(t)$ is a local time of the process $(\varphi_t(x))_{t\geq 0}$ at the point $z \in \mathbb{R}$ that is defined by the formula

$$L_z^{\varphi(x)}(t) = \lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon}\int_0^t \mathbb{1}_{[z,z+\varepsilon)}(\varphi_s(x))ds, \ t \geq 0.$$

$$\begin{cases} d\varphi_t(x) = a(\varphi_t(x))dt + dw(t), \\ \varphi_0(x) = x. \end{cases}$$

(A) a has bounded variation on each interval of $\mathbb{R}$;

(B) for all $x \in \mathbb{R}$

$$|a(x)|^2 \le C(1 + |x|^2);$$

Let $W^1_{p,loc}(\mathbb{R})$ be a set of functions on $\mathbb{R}$ belonging to $W^1_p([c, d]), c < d$.

Theorem 1 [Aryasova & Pilipenko, 2012]

1) *For all* $t \geq 0$,

$$P\{\forall p \geq 1 \ : \ \varphi_t(\cdot) \in W^1_{p,loc}(\mathbb{R})\} = 1.$$

2) *For all* $t \geq 0$ *a generalized derivative* $\nabla \varphi_t(x)$ *is defined by the formula*

$$\nabla \varphi_t(x) = \exp\left\{ \int_{\mathbb{R}} L_z^{\varphi(x)}(t) da(z) \right\},$$

where $(L_z^{\varphi(x)}(t))_{t \geq 0}$ *is a local time of the process* $\varphi(x)$ *at the point* z.

3) *For any fixed* t, ω, *the mapping* $x \mapsto \varphi_t(x)$ *is a homeomorphism. Moreover, for all* $\{x_1, x_2\} \in \mathbb{R}, \ x_1 \neq x_2$,

$$P\left\{\varphi_t(x_1) \neq \varphi_t(x_2), \ t \geq 0\right\} = 1.$$

Example 2

$$\varphi_t(x) = x + \int_0^t \operatorname{sign} \varphi_s(x) ds + w(t).$$

Example 3

$$\varphi_t(x) = x - \int_0^t \operatorname{sign} \varphi_s(x) ds + w(t).$$

For fixed (t, ω), the mapping $x \mapsto \varphi_t(x, \omega)$ is continuous and it is an injection.

Multidimensional case

$$\begin{cases} d\varphi_t(x) = a(\varphi_t(x))dt + dw(t), \\ \varphi_0(x) = x. \end{cases} \tag{2}$$

where $x \in \mathbb{R}^d$, $(w(t))_{t\geq 0}$ is a $d-$dimensional Wiener process, $a = (a^1, \dots, a^d)$ is a bounded measurable mapping from $\mathbb{R}^d$ to $\mathbb{R}^d$.

According to [Veretennikov, 1981] there exists a unique strong solution to equation (2).

Case of continuous drift vector

Let for all $i = \overline{1, d}$, $a^i \in C_b^1(\mathbb{R}^d)$. Then

$$d\nabla\varphi_t(x) = \nabla a(\varphi_t(x))\nabla\varphi_t(x)dt,$$

where

$$\nabla f(x) = \left\|(\nabla f(x))^{ij}\right\|_{i,j=\overline{1,d}}, \quad (\nabla f(x))^{ij} = \frac{\partial f^i}{\partial x_j}(x).$$

$$\nabla\varphi_t(x) = E + \int_0^t dA_s(\varphi(x))\nabla\varphi_t(x)dt, \tag{3}$$

where E is a $d \times d$-identity matrix, $A_t(\varphi(x)) = \|A_t^{ij}\|_{i,j=\overline{1,d}}$,

$$A_t^{ij}(\varphi(x)) = \int_0^t \nabla a^{ij}(\varphi_s(x))ds$$

is a *homogeneous additive continuous functional* of the process $(\varphi_t(x))_{t\geq 0}$.

Theory of W-functionals by [Dynkin, 1963]

Definition 1

A non-negative function $A_t(\xi) : [0, \infty) \to \mathbb{R}$ is called a **W-functional** of the process $(\xi(t))_{t \geq 0}$ if it is

- adapted to the filtration $\mathcal{N}_t = \sigma\{\xi(s) : 0 \leq s \leq t\}$;
- continuous in t;
- homogeneous additive, i.e. for all $t \geq 0, \ s > 0, \ x \in \mathbb{R}$,

$$A_{t+s}(\xi) = A_s(\xi) + \theta_s A_t(\xi) \quad P_x - \text{almost surely},$$

where θ is a shift operator.

Theory of W-functionals by [Dynkin, 1963]

Definition 2

For the process $(\xi(t))_{t\geq 0}$ the function

$$f_t(x) = \sup_{x\in\mathbb{R}^d} \mathbb{E}_x A_t(\xi)$$

is called the characteristic of W-functional $A_t(\xi)$.

Theory of W-functional

Example

$$A_t(\xi) = \int_0^t a(\xi(s))ds,$$

where a is a non-negative measurable function.

$$f_t(x) = \mathbb{E}_x \int_0^t a(\xi(s))ds = \int_{\mathbb{R}^d} \left(\int_0^t p(s,x,y)ds \right) a(y)dy = \int_{\mathbb{R}^d} k(t,x,y)\mu(dy),$$

where $p(t,x,y)$ is the transition density of the process $(\xi(t))_{t\geq 0}$.

$$\mu(dy) = a(y)dy,$$

$$k(t,x,y) = \int_0^t p(s,x,y)ds.$$

Theory of W-functional by [Dynkin, 1963]

Theorem 1

The W-functional is defined uniquely by its characteristic.

Theory of W-functional by [Dynkin, 1963]

Theorem 2

Let $A_{n,t}(\xi),\ n \geq 1$ be non-negative functionals of the process ξ and $f_{n,t}(x) = \mathbb{E}_x A_{n,t}(\xi)$ be their characteristics. Suppose that for each $t > 0$, a function $f_t(x)$ satisfies the condition

$$\lim_{n\to\infty} \sup_{0\leq u\leq t} \sup_{x\in\mathbb{R}} |f_{n,u}(x) - f_u(x)| = 0.$$

Then $f_t(x)$ is the characteristic of a non-negative W-functional $A_t(\xi)$. Moreover,

$$A_t(\xi) = \underset{n\to\infty}{\mathrm{l.i.m.}}\, A_{n,t}(\xi).$$

Let $A_{n,t}(\xi) = \int_0^t a_n(\xi(s))ds$. Then

$$f_t(x) = \int_{\mathbb{R}^d} k(t,x,y)a_n(y)dy,$$

where

$$k(t,x,y) = \int_0^t p(s,x,y)ds,$$

$p(t,x,y)$ is a transition density of the process $(\xi(t))_{t\geq 0}$.
If

$$a_n(x)dx \to \mu(dx),\ n\to\infty,\ \text{in some sense},$$

then we can expect the convergence of corresponding characteristics.

Example

Let $d = 1$, $\mu(dy) = \delta_0(y)$, $a_n(x) = \frac{1}{2n}\mathbb{1}_{[-\frac{1}{n},\frac{1}{n}]}(x)$ Then

$$A_t(\xi) = \lim_{n\to\infty} \frac{1}{2n} \int_0^t \mathbb{1}_{[-\frac{1}{n},\frac{1}{n}]}(\xi(s))ds$$

is a local time of the process $(\xi(t))_{t\geq 0}$ at the point 0.

Which functionals can be approximated ?

Condition A

The transition density of the process satisfies the inequality

$$C_1 \exp\left\{-\beta_1 \frac{\|x-y\|^2}{t}\right\} \leq p(t,x,y) \leq C_2 \exp\left\{-\beta_2 \frac{\|x-y\|^2}{t}\right\}$$

in every domain of the form $t \in [0,T]$, $x \in \mathbb{R}^d$, $y \in \mathbb{R}^d$.
The measure μ is such that

$$\lim_{t\to 0} \sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} k(t,x,y)\mu(dy) = 0.$$

Theorem 3[Aryasova & Pilipenko, 2013]

Let $a : \mathbb{R}^d \to \mathbb{R}^d$ be such that for all $1 \le i, j \le d$, its generalized derivatives $\mu^{ij} = \frac{\partial a^i}{\partial x^j}$ are signed measures. Let $\mu^{ij} = \mu^{ij,+} - \mu^{ij,-}$ be its Hahn-Jordan decomposition. Suppose that measures $\mu^{ij,+}, \mu^{ij,-}$ satisfy Condition A. Then for any $p \ge 1$ the solution to equation (2) is Sobolev differentiable with respect to initial data, and

$$P\left\{\forall t > 0 : \varphi_t(\cdot) \in W^1_{p,loc}(\mathbb{R}^d, \mathbb{R}^d)\right\} = 1.$$

The matrix of derivatives $Y_t(x) := \left\| \frac{\partial \varphi^i_t(x)}{\partial x^j} \right\|_{i,j=\overline{1,d}}$ satisfies the following integral equation

$$Y_t(x) = E + \int_0^t (dA_s(\varphi(x)))\, Y_s(x), \tag{4}$$

where E is a $d \times d$-identity matrix.

Remark

$$A_t(\varphi(x)) = \int_0^t \frac{d\mu}{dx}(\varphi(s))ds$$

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