On differentiability of a flow for an SDE with discontinuous drift

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(joint work with Andrey Pilipenko)

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Consider an SDE

\[
\begin{aligned}
\frac{d\varphi_t(x)}{dt} &= a(\varphi_t(x))dt + \sigma(\varphi_t(x))dw(t), \\
\varphi_0(x) &= x,
\end{aligned}
\]

where \( x \in \mathbb{R}^d \), \((w(t))_{t \geq 0}\) is a \( d \)-dimensional Wiener process,

Problem

Let \( t, \omega \) be fixed. We study the differentiability of the mapping 
\( x \mapsto \varphi_t(x, \omega) \) from \( \mathbb{R}^d \) to \( \mathbb{R}^d \).
Example 1

Let $a = (a^1, \ldots, a^d) \in C^1_b(\mathbb{R}^d)$

\[
\begin{align*}
    d\varphi_t(x) &= a(\varphi_t(x))dt, \\
    \varphi_0(x) &= x,
\end{align*}
\]

where $x \in \mathbb{R}^d$. Then for fixed $t$, the mapping $x \mapsto \varphi_t(x)$ is a diffeomorphism.

The derivative satisfies the differential equation

\[
\nabla \varphi_t(x) = E + \int_0^t \nabla a(\varphi_s(x)) \nabla \varphi_s(x) ds,
\]

where $E$ is a $d \times d$-identity matrix, $\nabla a(x)$ is a $d \times d$- matrix and

\[
(\nabla a(x))^{ij} = \frac{\partial a^i(x)}{\partial x_j}.
\]
Example 2

Let $d = 1$. Consider $a(x) = \text{sign } x$. Then

$$\varphi_t(x) = x + \int_0^t \text{sign } \varphi_s(x) \, ds.$$ 

For all $t > 0$, the mapping $x \mapsto \varphi_t(x)$ is discontinuous at the point $x_0 = 0$ but it is an injection.
Example 3

Let \( a(x) = -\text{sign} x \). Then for all \( t > 0 \), the mapping \( x \mapsto \varphi_t(x) \) is continuous but it is not an injection.
Stochastic differential equations

\[
\begin{cases}
    d\varphi_t(x) = a(\varphi_t(x))dt + \sigma(\varphi_t(x))dw(t), \\
    \varphi_0(x) = x.
\end{cases}
\]

- \(\{a, \sigma\} \subset C_b^{1,\varepsilon}\). There exists a flow of diffeomorphisms. The derivative \(\nabla \varphi_t(x) =: \psi_t(x)\) is a solution of equation
  \[d\psi_t(x) = \nabla a(\varphi_t(x))\psi_t(x)dt + \nabla \sigma(\varphi_t(x))\psi_t(x)dw(t).\]

- \(\{a, \sigma\} \subset Lip\).
  - there exists a flow of homeomorphisms [Kunita, 1990];
  - there exists a generalized derivative [Bouleau & Hirsch, 1991].

- if \(\sigma \in C^2\) and it is non-degenerate, and \(a \in Höl\) then there exists a flow of diffeomorphisms [Flandoli et al., 2010].
One-dimensional case

\[
\begin{aligned}
\begin{cases}
\frac{d\varphi_t(x)}{dt} &= a(\varphi_t(x))dt + dw(t), \\
\varphi_0(x) &= x,
\end{cases}
\end{aligned}
\tag{1}
\]

where \(a(x), x \in \mathbb{R}\), is measurable bounded function, \((w(t))_{t \geq 0}\) is a one-dimensional Wiener process.

For each \(x \in \mathbb{R}\) there exists a unique strong solution to (1) (c. f. [Zvonkin, 1974]).
Heuristic approach

\[
\begin{aligned}
    d\varphi_t(x) &= a(\varphi_t(x))dt + dw(t), \\
    \varphi_0(x) &= x.
\end{aligned}
\]

Suppose that \( a \in C^1_b \). Then

\[
\psi_t(x) = (\varphi_t(x))'_x
\]

satisfies the equation

\[
d\psi_t(x) = a'(\varphi_t(x))\psi_t(x)dt.
\]
Heuristic approach

\[ \psi_t(x) = \exp \left\{ \int_0^t a'(\varphi_s(x)) ds \right\}. \]

By the occupation times formula

\[ \psi_t(x) = \exp \left\{ \int_{\mathbb{R}} L_z^{\varphi(x)}(t) a'(z) dz \right\} = \exp \left\{ \int_{\mathbb{R}} L_z^{\varphi(x)}(t) da(z) \right\}, \]

where \( L_z^{\varphi(x)}(t) \) is a local time of the process \((\varphi_t(x))_{t \geq 0}\) at the point \( z \in \mathbb{R} \) that is defined by the formula

\[ L_z^{\varphi(x)}(t) = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^t 1_{[z, z+\varepsilon)}(\varphi_s(x)) ds, \quad t \geq 0. \]
\begin{align*}
\left\{ \begin{array}{l}
d\varphi_t(x) = a(\varphi_t(x))dt + dw(t), \\
\varphi_0(x) = x.
\end{array} \right.
\end{align*}

(A) $a$ has bounded variation on each interval of $\mathbb{R}$;

(B) for all $x \in \mathbb{R}$
\[|a(x)|^2 \leq C(1 + |x|^2);\]

Let $W_{p,loc}^1(\mathbb{R})$ be a set of functions on $\mathbb{R}$ belonging to $W_p^1([c, d]), c < d$. 

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Theorem 1 [Aryasova & Pilipenko, 2012]

1) For all \( t \geq 0 \),

\[
P\left\{ \forall p \geq 1 : \varphi_t(\cdot) \in W_{p,loc}^1(\mathbb{R}) \right\} = 1.
\]

2) For all \( t \geq 0 \) a generalized derivative \( \nabla \varphi_t(x) \) is defined by the formula

\[
\nabla \varphi_t(x) = \exp \left\{ \int_{\mathbb{R}} L_{\varphi(x)}(t) da(z) \right\},
\]

where \( (L_{\varphi(x)}(t))_{t \geq 0} \) is a local time of the process \( \varphi(x) \) at the point \( z \).

3) For any fixed \( t, \omega \), the mapping \( x \mapsto \varphi_t(x) \) is a homeomorphism. Moreover, for all \( \{x_1, x_2\} \in \mathbb{R}, x_1 \neq x_2 \),

\[
P \left\{ \varphi_t(x_1) \neq \varphi_t(x_2), t \geq 0 \right\} = 1.
\]
Example 2

\[
\varphi_t(x) = x + \int_0^t \text{sign} \varphi_s(x) \, ds + w(t).
\]

Example 3

\[
\varphi_t(x) = x - \int_0^t \text{sign} \varphi_s(x) \, ds + w(t).
\]

For fixed \((t, \omega)\), the mapping \(x \mapsto \varphi_t(x, \omega)\) is continuous and it is an injection.
Multidimensional case

\[
\begin{aligned}
d\varphi_t(x) &= a(\varphi_t(x))dt + dw(t), \\
\varphi_0(x) &= x.
\end{aligned}
\]

where \( x \in \mathbb{R}^d \), \((w(t))_{t \geq 0}\) is a \( d \)--dimensional Wiener process, 
a = (a^1, \ldots, a^d) is a bounded measurable mapping from \( \mathbb{R}^d \) to \( \mathbb{R}^d \).

According to [Veretennikov, 1981] there exists a unique strong solution to equation (2).
Case of continuous drift vector

Let for all \( i = 1, d \), \( a^i \in C^1_b(\mathbb{R}^d) \). Then

\[
d\nabla \varphi_t(x) = \nabla a(\varphi_t(x)) \nabla \varphi_t(x) dt,
\]

where

\[
\nabla f(x) = \left\| (\nabla f(x))^i_j \right\|_{i,j=1,d}, \quad (\nabla f(x))^i_j = \frac{\partial f^i}{\partial x_j}(x).
\]

\[
\nabla \varphi_t(x) = E + \int_0^t dA_s(\varphi(x)) \nabla \varphi_t(x) dt, \quad (3)
\]

where \( E \) is a \( d \times d \)-identity matrix, \( A_t(\varphi(x)) = \left\| A^i_j \right\|_{i,j=1,d}, \)

\[
A^i_j(\varphi(x)) = \int_0^t \nabla a^i_j(\varphi_s(x)) ds
\]

is a \textbf{homogeneous additive continuous functional} of the process \((\varphi_t(x))_{t \geq 0}\).
Theory of W-functionals by [Dynkin, 1963]

Definition 1

A non-negative function \( A_t(\xi) : [0, \infty) \to \mathbb{R} \) is called a **W-functional** of the process \( (\xi(t))_{t \geq 0} \) if it is

- adapted to the filtration \( \mathcal{N}_t = \sigma\{\xi(s) : 0 \leq s \leq t\} \);
- continuous in \( t \);
- homogeneous additive, i.e. for all \( t \geq 0, \ s > 0, \ x \in \mathbb{R} \),

\[
A_{t+s}(\xi) = A_s(\xi) + \theta_s A_t(\xi) \quad P_x - \text{almost surely},
\]

where \( \theta \) is a shift operator.
Theory of W-functionals by [Dynkin, 1963]

Definition 2

For the process \((\xi(t))_{t \geq 0}\) the function

\[ f_t(x) = \sup_{x \in \mathbb{R}^d} \mathbb{E}_x A_t(\xi) \]

is called the characteristic of W-functional \(A_t(\xi)\).
Theory of W-functional

Example

\[ A_t(\xi) = \int_0^t a(\xi(s)) ds, \]
where \( a \) is a non-negative measurable function.

\[ f_t(x) = \mathbb{E}_x \int_0^t a(\xi(s)) ds = \int_{\mathbb{R}^d} \left( \int_0^t p(s, x, y) ds \right) a(y) dy = \int_{\mathbb{R}^d} k(t, x, y) \mu(dy), \]
where \( p(t, x, y) \) is the transition density of the process \((\xi(t))_{t \geq 0}\).

\[ \mu(dy) = a(y) dy, \]
\[ k(t, x, y) = \int_0^t p(s, x, y) ds. \]
Introduction

One-dimensional case

General case

Theory of W-functional by [Dynkin, 1963]

Theorem 1

The W-functional is defined uniquely by its characteristic.
Theorem 2

Let $A_{n,t}(\xi)$, $n \geq 1$ be non-negative functionals of the process $\xi$ and $f_{n,t}(x) = \mathbb{E}_x A_{n,t}(\xi)$ be their characteristics. Suppose that for each $t > 0$, a function $f_t(x)$ satisfies the condition

$$\lim_{n \to \infty} \sup_{0 \leq u \leq t} \sup_{x \in \mathbb{R}} |f_{n,u}(x) - f_u(x)| = 0.$$ 

Then $f_t(x)$ is the characteristic of a non-negative W-functional $A_t(\xi)$. Moreover,

$$A_t(\xi) = \lim_{n \to \infty} A_{n,t}(\xi).$$
Let $A_{n,t}(\xi) = \int_0^t a_n(\xi(s))ds$. Then

$$f_t(x) = \int_{\mathbb{R}^d} k(t, x, y)a_n(y)dy,$$

where

$$k(t, x, y) = \int_0^t p(s, x, y)ds,$$

$p(t, x, y)$ is a transition density of the process $(\xi(t))_{t \geq 0}$.

If

$$a_n(x)dx \rightarrow \mu(dx), \ n \rightarrow \infty, \ \text{in some sense},$$

then we can expect the convergence of corresponding characteristics.
Example

Let $d = 1$, $\mu(dy) = \delta_0(y)$, $a_n(x) = \frac{1}{2n} \mathbf{1}_{[-\frac{1}{n}, \frac{1}{n}]}(x)$. Then

$$A_t(\xi) = \lim_{n \to \infty} \frac{1}{2n} \int_0^t \mathbf{1}_{[-\frac{1}{n}, \frac{1}{n}]}(\xi(s)) ds$$

is a local time of the process $(\xi(t))_{t \geq 0}$ at the point 0.
Which functionals can be approximated?

Condition A

The transition density of the process satisfies the inequality

\[ C_1 \exp \left\{ -\beta_1 \frac{\|x - y\|^2}{t} \right\} \leq p(t, x, y) \leq C_2 \exp \left\{ -\beta_2 \frac{\|x - y\|^2}{t} \right\} \]

in every domain of the form \( t \in [0, T], x \in \mathbb{R}^d, y \in \mathbb{R}^d \).

The measure \( \mu \) is such that

\[
\lim_{t \to 0} \sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} k(t, x, y) \mu(dy) = 0.
\]
Theorem 3[Aryasova & Pilipenko, 2013]

Let \( a : \mathbb{R}^d \to \mathbb{R}^d \) be such that for all \( 1 \leq i, j \leq d \), its generalized derivatives \( \mu^{ij} = \frac{\partial a^i}{\partial x^j} \) are signed measures. Let \( \mu^{ij} = \mu^{ij,+} - \mu^{ij,-} \) be its Hahn-Jordan decomposition. Suppose that measures \( \mu^{ij,+}, \mu^{ij,-} \) satisfy Condition A. Then for any \( p \geq 1 \) the solution to equation (2) is Sobolev differentiable with respect to initial data, and

\[
P \{ \forall t > 0 : \varphi_t(\cdot) \in W^{1}_{p,loc}(\mathbb{R}^d, \mathbb{R}^d) \} = 1.
\]

The matrix of derivatives \( Y_t(x) := \left\| \frac{\partial \varphi_t^i(x)}{\partial x^j} \right\|_{i,j=1,d} \) satisfies the following integral equation

\[
Y_t(x) = E + \int_0^t (dA_s(\varphi(x))) Y_s(x),
\]

where \( E \) is a \( d \times d \)-identity matrix.
Remark

\[ A_t(\varphi(x)) = \int_0^t \frac{d\mu}{dx}(\varphi(s)) ds \]


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Danke für Ihre Aufmerksamkeit