Scaling limits of large random trees

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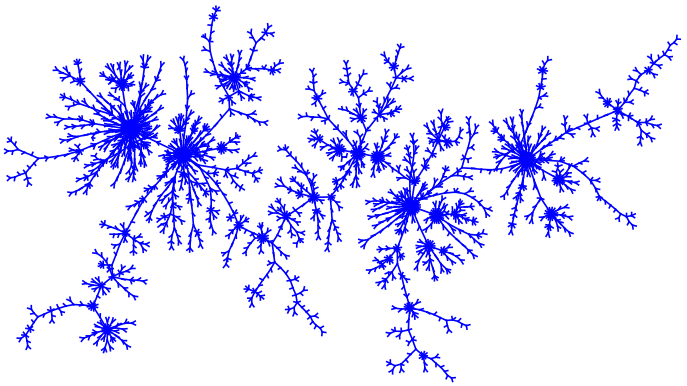

based on joint works with Grégory MIERMONT, Jim PITMAN, Robin STEPHENSON and Matthias WINKEL

Outline

1. Introduction: scaling limits of Galton-Watson trees
2. Scaling limits of Markov branching trees
3. Applications: Galton-Watson trees, combinatorial trees, sequences of random trees built recursively

Scaling limits: a basic example

I.i.d sequence of *centered* random variables $X_i \in \{-1, 1\}$:

$$-1,\ 1,\ 1,\ 1,\ 1,\ -1,\ -1,\ -1,\ 1,\ -1,\ -1,\ 1,\ 1,\ 1,\ 1, ...$$

Centered random walk: $S_n = X_1 + ... + X_n$

How does S_n behave when n is large ?

(1) what is the growth rate ?

(2) what is the limit after rescaling ?

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Central limit theorem:

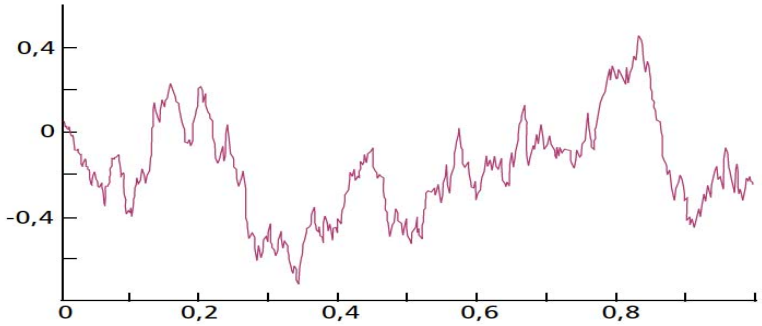
$$\frac{S_n}{\sqrt{n}} \overset{\text{law}}{\to} \mathcal{N}(0, 1)$$

Functional version:

DONSKER'S theorem (51):

$$\left(\frac{S_{[nt]}}{\sqrt{n}}, t \in [0, 1] \right) \overset{\text{law}}{\to} (B(t), t \in [0, 1])$$

where B is a standard Brownian motion



Scaling limits: a basic example

More generally:

- invariance principle: random walks with i.i.d. centered increments X_i, $i \geq 1$ with finite variance σ^2 converge at speed $\sigma\sqrt{n}$ towards a standard Brownian motion

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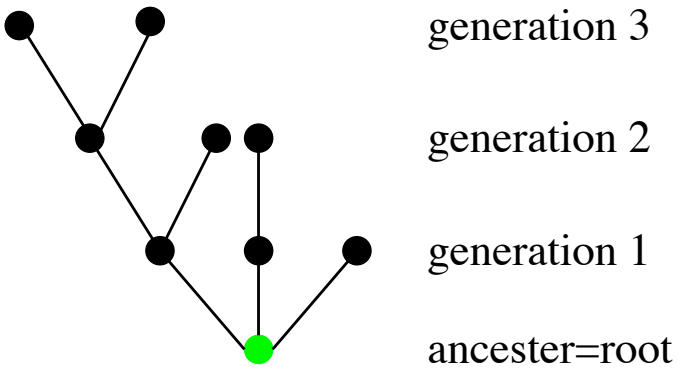
- however: random walks with i.i.d. increments $X_i, i \geq 1$ such that

$$\mathbb{P}(X_1 > x) \sim C_1 x^{-\alpha} \quad \text{and} \quad \mathbb{P}(X_1 < -x) \sim C_2 x^{-\alpha} \quad \text{for some } \alpha \in (0,2)$$

  with $C_1 + C_2 > 0$ (and centered if $\alpha \in [1,2)$) converge at speed $n^{1/\alpha}$ towards an $\alpha-$stable Lévy process.

Scaling limits: the example of Galton-Watson trees

Galton-Watson processes are introduced in 1873 to study the extinction of family names



η: proba. on $Z_+ = \{0, 1, 2, ...\}$ (offspring distribution), such that $\eta(1) < 1$, with mean m

Extinction probability $= 1$ in subcritical ($m < 1$) and critical ($m = 1$) cases

$\in [0, 1)$ in supercritical cases ($m > 1$)

Large Galton-Watson trees

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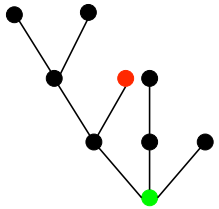
$$\frac{H_n^{(U)}}{\sqrt{n}} \xrightarrow{\text{law}} \frac{R}{\sigma},$$



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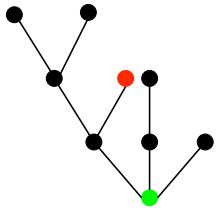

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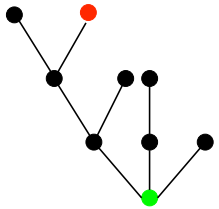

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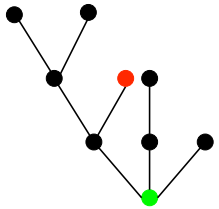

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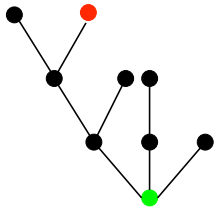

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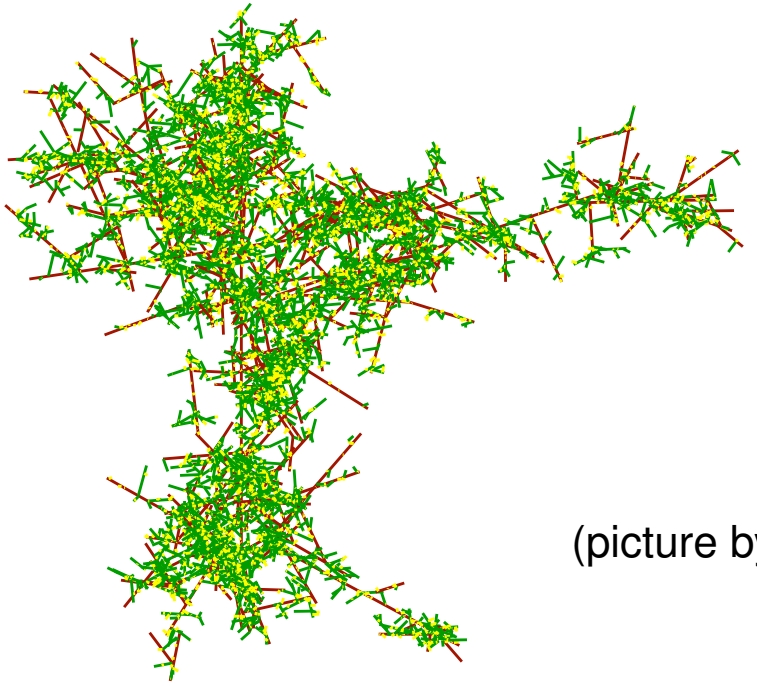
With length 1 on each edge, what does the tree look like when $n \to \infty$?

Universal limit: the Brownian tree $\mathcal{T}_{\mathrm{Br}}$



(picture by G. Miermont)

ALDOUS 93:

$$\frac{T_n^{\mathrm{GW}}}{\sqrt{n}} \xrightarrow{\text{law}} \frac{2}{\sigma}\mathcal{T}_{\mathrm{Br}}$$

Compact (random) *real tree*, i.e. compact metric space with the tree property:

$$\forall x, y \in \mathcal{T}_{\mathrm{Br}} \ \ \exists! \text{ path from } x \text{ to } y$$

almost surely: - binary tree

- self-similar
- Hausdorff dimension $= 2$
- the set of leaves is dense in the tree

Topology on the set of compact metric spaces

For A, B compact subsets of a metric space (Z, d_Z)

$$d_{\text{Hausdorff}}(A, B) = \inf\{\epsilon > 0 : A \subset B_\epsilon \text{ and } B \subset A_\epsilon\},$$

where $A_\epsilon = \{x \in Z : d_Z(A, x) \leq \epsilon\}$

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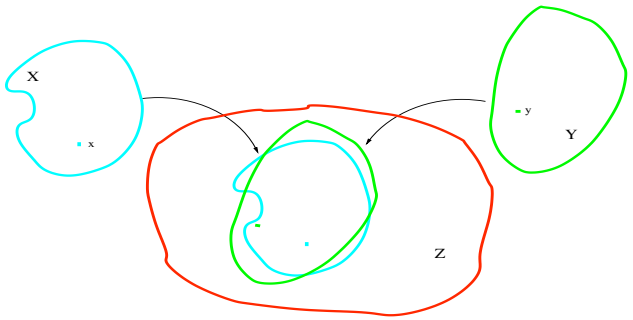
<u>Gromov-Hausdorff distance</u>: let (X, x), (Y, y) be compact, pointed, metric spaces

$$d_{\text{GH}}((X, x), (Y, y)) := \inf\left(d_{\text{Hausdorff}}(\varphi_1(X), \varphi_2(Y)) \vee d_Z(\varphi_1(x), (\varphi_2(y)))\right)$$

the infimum being on all isometric embeddings $\varphi_1 : X \hookrightarrow Z$ and $\varphi_2 : Y \hookrightarrow Z$ into a same metric space (Z, d_Z).



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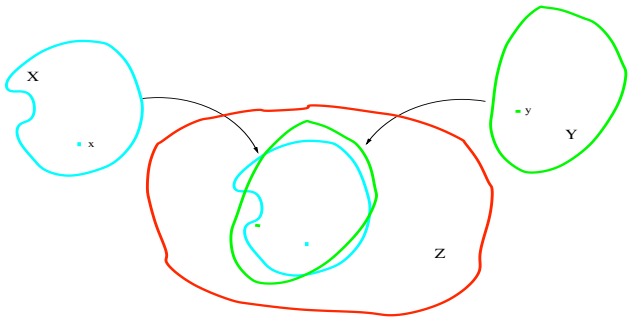

(X, x) and (Y, y) are *equivalent* if $\exists\, \varphi$ isometry: $Y = \varphi(X)$, $y = \varphi(x)$

d_{GH}: distance on the set of equivalence classes

<u>Ref. :</u> GROMOV 81 EVANS-PITMAN-WINTER 06 EVANS-WINTER 06

Topology on the set of compact metric space

ALDOUS 93:

$$\frac{T_n^{\mathrm{GW}}}{\sqrt{n}} \underset{\mathrm{GH}}{\overset{\mathrm{law}}{\to}} \frac{2}{\sigma}\mathcal{T}_{\mathrm{Br}},$$

jointly with the convergence of the uniform probability on the nodes of T_n^{GW} towards a probability measure on the leaves of $\mathcal{T}_{\mathrm{Br}}$

Examples: scaling limits of combinatorial trees

- $T_n^{(\mathrm{ord})}$: uniform among *ordered* trees with n nodes, rooted

$$\frac{T_n^{(\mathrm{ord})}}{\sqrt{n}} \xrightarrow{\mathrm{law}} \mathcal{T}_{\mathrm{Br}}$$

since $T_n^{(\mathrm{ord})} \sim T_n^{\mathrm{GW}}$ with a Geometric(1/2) offspring distribution

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- $T_n^{(\mathrm{lab})}$: uniform among *labelled* trees with n nodes, rooted

$$\frac{T_n^{(\mathrm{lab})}}{\sqrt{n}} \xrightarrow{\mathrm{law}} 2\mathcal{T}_{\mathrm{Br}}$$

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  since $T_n^{(\mathrm{lab})} \sim T_n^{\mathrm{GW}}$ with a Poisson(1) offspring distribution

- $T_n^{(\mathrm{P})}$: uniform among *non-ordered, non-labelled* trees with n nodes, rooted.
  Problem: it is not a conditioned Galton-Watson tree !

Large Galton-Watson trees: when the variance is infinite

Assume: $\eta(k) = \mathbb{P}(\text{ to have } k \text{ children}) \underset{k\to\infty}{\sim} Ck^{-\beta-1}$, $1 < \beta < 2$

Then (DUQUESNE 03):

$$\frac{T_n^{\mathrm{GW}}}{n^{1-1/\beta}} \underset{\mathrm{GH}}{\overset{\mathrm{law}}{\to}} C^{-1/\beta}\mathcal{T}_\beta$$

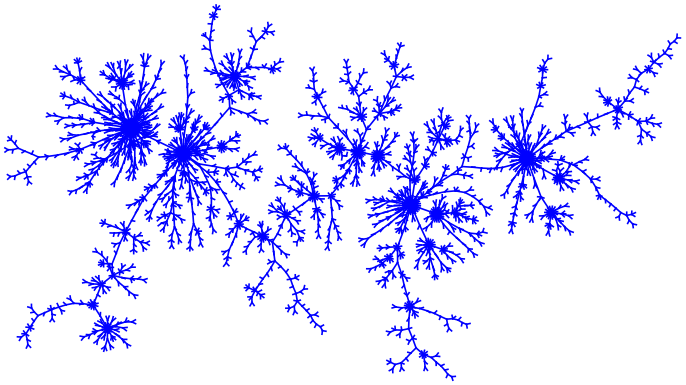
jointly with the convergence of the uniform probability on the nodes of T_n^{GW} towards a probability measure on the leaves of $\mathcal{T}_\beta$

- "smaller" trees
- $\mathcal{T}_\beta$ belongs to the family of stable Lévy trees (introduced by DUQUESNE, LE GALL, LE JAN 98):

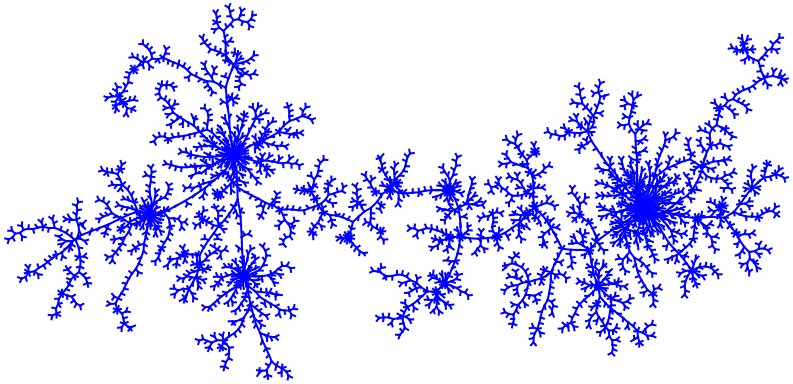
  - each branching vertex branches in an infinite, countable number of subtrees

  - self-similar, with Hausdorff dimension $= 1 + \dfrac{1}{\beta - 1}$ (DUQUESNE-LE GALL 05, H.-MIERMONT 04)

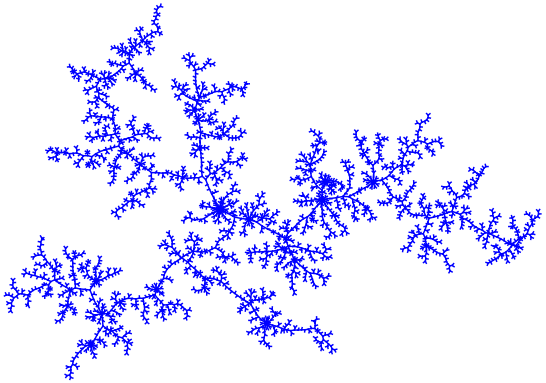
Simulations



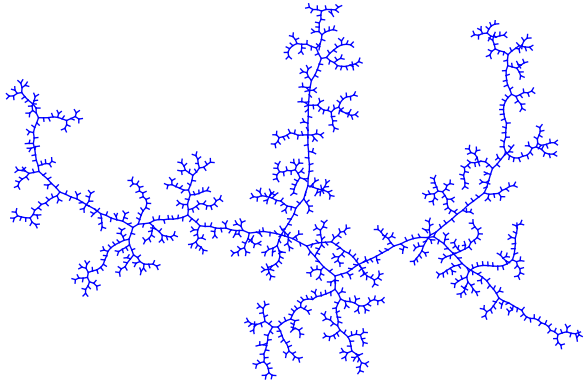
$\beta = 1.2$



$\beta = 1.5$



$\beta = 1.9$



$\beta = 2$ (Brownian tree)

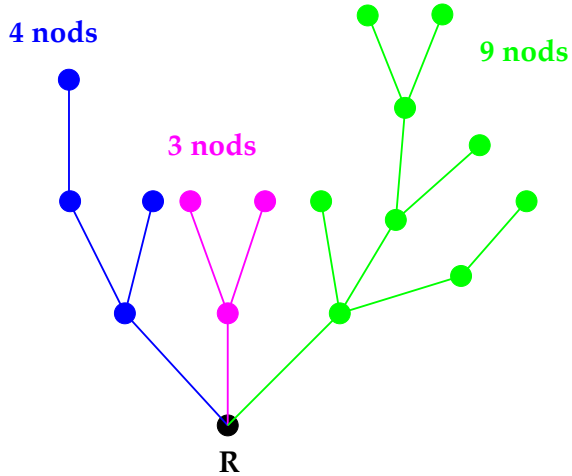
(simulations by N. Curien)

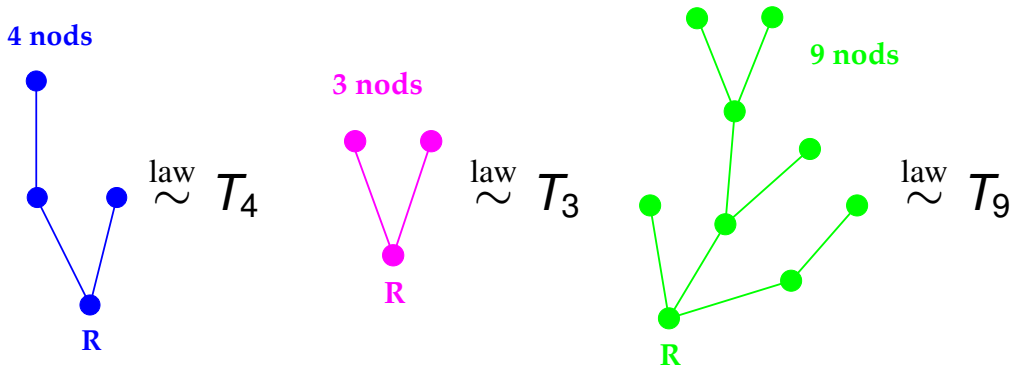
Markov branching trees

$(T_n, n \geq 1)$: T_n random rooted tree with n nodes

Markov branching property:

T_n :
$(n = 17)$





Conditional on "the root of T_n has k children-trees with size $n_1 \geq \ldots \geq n_k$", these trees are independent, with respective distributions those of $T_{n_1}, \ldots, T_{n_k}$

Similar definition for sequences of trees indexed by the number of leaves

Ex.: Galton-Watson trees conditioned to have n nodes (respectively n leaves); sequences of trees built recursively

Scaling limits of Markov branching trees

$(T_n, n \geq 1)$ Markov Branching sequence indexed by *leaves*

Behavior when n is large ?

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Let $q_n(n_1, ..., n_p) := \mathbb{P}$(the root of T_n has p children-trees with $n_1 \geq ... \geq n_p$ leaves)

and $\mathcal{S}^{\downarrow} = \left\{ s_1 \geq s_2 \geq ... \geq 0 : \sum_{i \geq 1} s_i = 1 \right\}$

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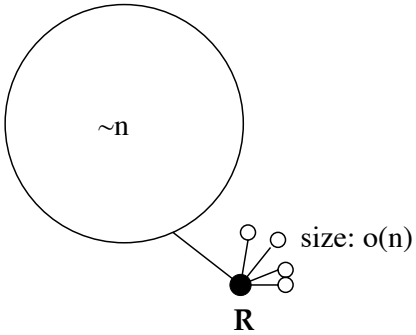
Assumption: $\forall$ bounded continuous $f : \mathcal{S}^{\downarrow} \to \mathbb{R}$,

$$n^{\gamma} \sum_{(n_1,\ldots,n_p) \text{ partition of } n} q_n(n_1, ..., n_p)\Big(1 - \frac{n_1}{n}\Big) f\Big(\frac{n_1}{n}, ..., \frac{n_p}{n}, 0, ..\Big) \underset{n\to\infty}{\to} \int_{\mathcal{S}^{\downarrow}} (1 - s_1) f(\mathbf{s}) \nu(\mathrm{d}\mathbf{s}),$$

with $\gamma > 0$ and ν a non-trivial $\sigma-$finite measure on $\mathcal{S}^{\downarrow}$ such that $\int_{\mathcal{S}^{\downarrow}} (1 - s_1) \nu(\mathrm{d}\mathbf{s}) < \infty$.

Informally: macroscopic branchings are rare:

with proba.~ 1 :



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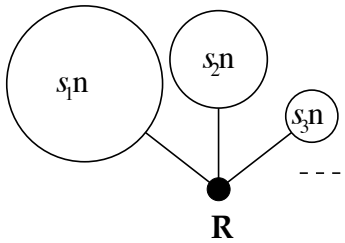

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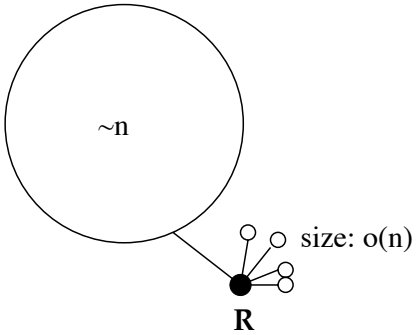

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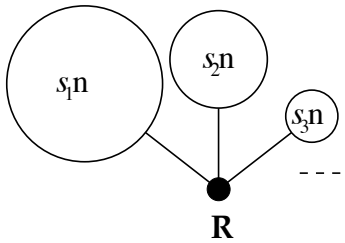

BROUTIN-DEVROYE-MCLEISH-DE LA SALLE 08: macroscopic branchings at each step: height of $T_n \sim c\log(n)$ + no interesting scaling limits

Scaling limits of Markov branching trees

$(T_n, n \geq 1)$ Markov branching indexed by *leaves*

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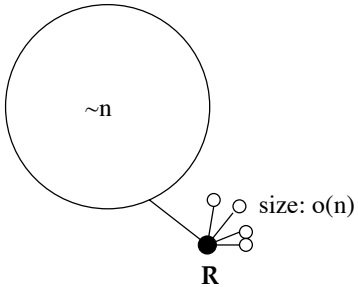

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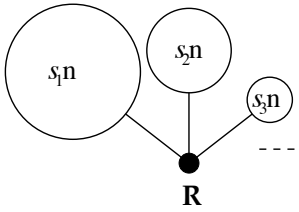

Theorem (H.-Miermont 11)

Under the previous assumption, $\exists$ random compact real tree $\mathcal{T}_{\gamma,\nu}$ s.t.

$$\frac{T_n}{n^\gamma} \underset{\mathrm{GH}}{\overset{\mathrm{law}}{\to}} \mathcal{T}_{\gamma,\nu},$$

jointly with the convergence of the uniform probability on the leaves of T_n towards a probability measure on the leaves of $\mathcal{T}_{\gamma,\nu}$.

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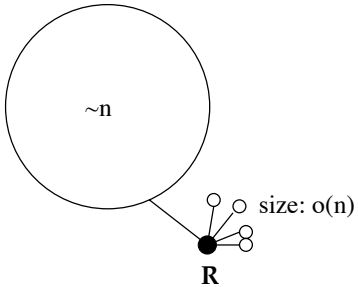

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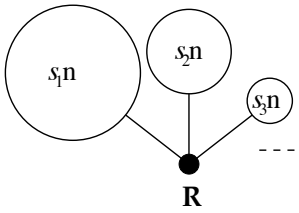

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Similar result for sequences of trees indexed by the number of nodes, with $\gamma \in (0, 1]$.

Outline of proof:

1. height of a random leaf
2. scaling limit of the tree spanned by k random leaves (finite dimensional cv)
3. tightness criterion

Markov branching trees: the limiting tree

- $\mathcal{T}_{\gamma,\nu}$ is self-similar, with Hausdorff dimension $\max(1, 1/\gamma)$ (H.-MIERMONT 04)

  γ: index of self-similarity

  ν: distribution of relative masses in the subtrees of a branching vertex

Markov branching trees: the limiting tree

- $\mathcal{T}_{\gamma,\nu}$ is self-similar, with Hausdorff dimension $\max(1, 1/\gamma)$ (H.-MIERMONT 04)

  γ: index of self-similarity

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- we recover the Brownian tree $\mathcal{T}_{\mathrm{Br}}$ when $\gamma = 1/2$ and $\nu(s_1 + s_2 < 1) = 0$,

$$\nu(s_1 \in \mathrm{d}x) = \frac{\sqrt{2}}{\sqrt{\pi} x^{3/2}(1-x)^{3/2}}, \quad 1/2 < x < 1$$

  and all stable trees $\mathcal{T}_\beta$, $1 < \beta < 2$ when $\gamma = 1 - 1/\beta$ and $\nu = \nu_\beta$

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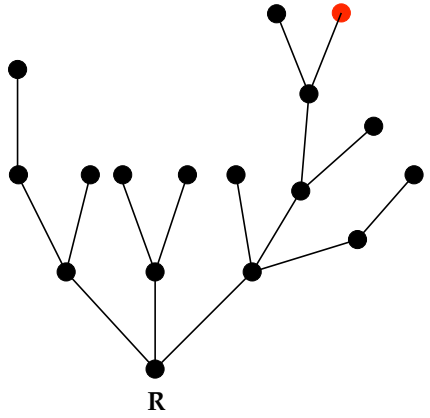
- the height of a typical leaf in $\mathcal{T}_{\gamma,\nu}$ writes

$$\int_0^\infty \exp(-\gamma \xi_r) \mathrm{d}r,$$

  for ξ a subordinator (increasing Lévy process) with Laplace exponent
  $\phi(\lambda) = \int_{\mathcal{S}^\downarrow} \sum_i (1 - s_i^\lambda) s_i \nu(\mathrm{d}\mathbf{s})$ $\qquad (\mathbb{E}[\exp(-\lambda \xi_t)] = \exp(-t\phi(\lambda)))$

Proof: a Markov chain in the Markov branching sequence of trees

First step: height of a leaf chosen uniformly at random



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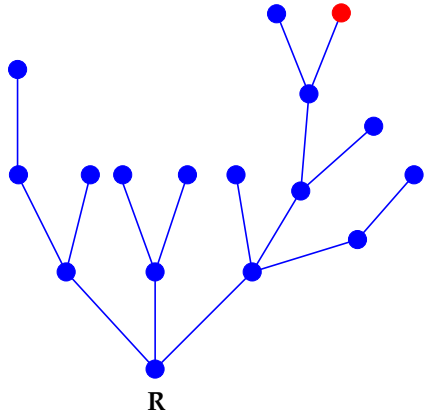

$X_n(0) = 9$

$X_n(k)$: size of the sub-tree above generation k containing the marked leaf

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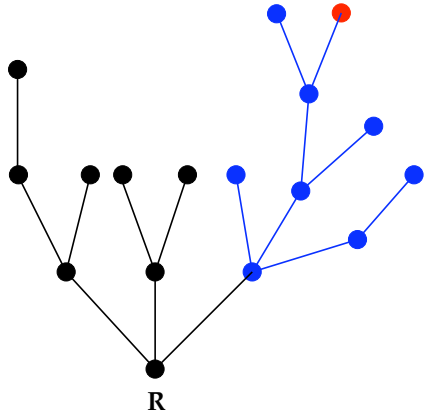

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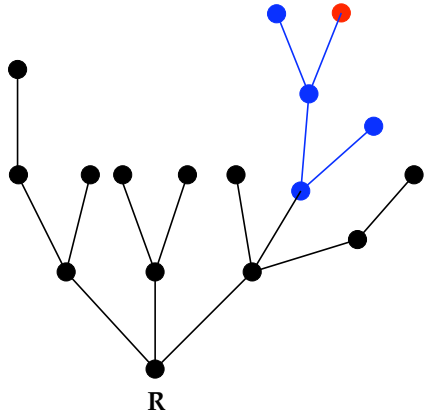

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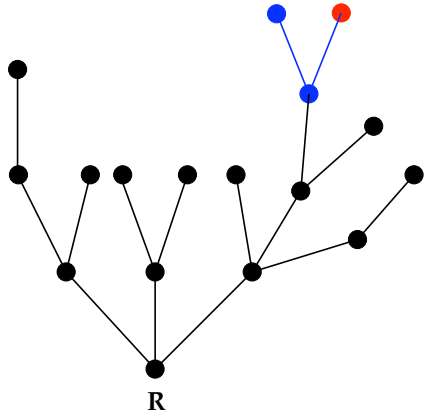

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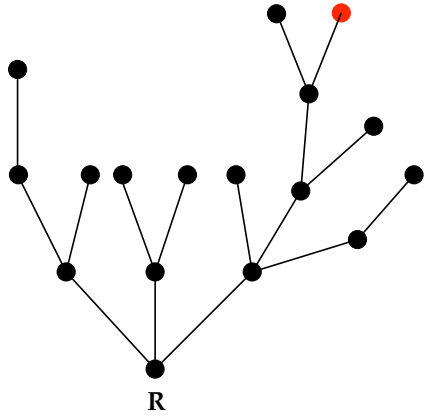

$X_n(0) = 9, X_n(1) = 5, X_n(2) = 3, X_n(3) = 2, X_n(4) = 1$

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It is a Markov chain!

A_n = absorption time at 1= height of the marked leaf

Proof: a Markov chain in the Markov branching sequence of trees

Let:

- ξ be a subordinator with Laplace exponent

$$\phi(\lambda) = \int_{\mathcal{S}^{\downarrow}} \sum_{i \geq 1} (1 - s_i^{\lambda}) s_i \nu(\mathrm{d}\mathbf{s}), \quad \lambda \geq 0,$$

- ρ be the acceleration of time: $\rho(t) = \inf\left\{u \geq 0 : \int_0^u \exp(-\gamma \xi_r)\mathrm{d}r \geq t\right\}$

- $X_\infty(t) = \exp(-\xi_{\rho(t)}), \quad t \geq 0$ absorbed at 0 at time $\int_0^\infty \exp(-\gamma \xi_r)\mathrm{d}r$

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Proposition (H.-Miermont 11)

Under our assumption of rare macroscopic branchings,

$$\left(\frac{X_n([n^\gamma t])}{n}, t \geq 0\right) \overset{\text{law}}{\to} (X_\infty(t), t \geq 0),$$

for the Skorokhod topology on the set $\mathbb{D}([0,\infty),[0,\infty))$, and

$$\frac{A_n}{n^\gamma} \overset{\text{law}}{\to} \int_0^\infty \exp(-\gamma \xi_r)\mathrm{d}r.$$

Application 1: Galton-Watson trees

- we recover Aldous' results on critical Galton-Watson trees conditioned by nodes

$$\frac{T_n^{\mathrm{GW}}}{\sqrt{n}} \xrightarrow{\mathrm{law}} \frac{2}{\sigma}\mathcal{T}_{\mathrm{Br}} \quad \text{if } \sigma^2 < \infty$$

  and its extension by Duquesne to infinite variance cases

- RIZZOLO 11: $T_n^{\mathrm{GW,L}}$ critical GW with finite variance σ^2, n leaves

$$\frac{T_n^{\mathrm{GW,L}}}{\sqrt{n}} \xrightarrow{\mathrm{law}} \frac{2}{\sigma\eta(0)}\mathcal{T}_{\mathrm{Br}}$$

  where $\eta(0)$: proba. of having 0 child (see also KORTCHEMSKI 11)

Application 2: combinatorial trees

- $T_n^{(\mathrm{ord})}$: uniform among *ordered* trees with n nods, rooted

$$\frac{T_n^{(\mathrm{ord})}}{\sqrt{n}} \xrightarrow{\mathrm{law}} \mathcal{T}_{\mathrm{Br}}$$

- $T_n^{(\mathrm{lab})}$: uniform among *labelled*, trees with n nods, rooted

$$\frac{T_n^{(\mathrm{lab})}}{\sqrt{n}} \xrightarrow{\mathrm{law}} 2\mathcal{T}_{\mathrm{Br}}$$

- $T_n^{(\mathrm{P})}$: uniform among *non-ordered, non-labelled* trees with n nods, rooted
  BROUTIN-FLAJOLET 08 : height DRMOTA-GITTENBERGER 10 : profile

Corollary (H.-MIERMONT 11) :

$$\frac{T_n^{(\mathrm{P})}}{\sqrt{n}} \xrightarrow{\mathrm{law}} c_{\mathrm{P}}\mathcal{T}_{\mathrm{Br}}, \quad c_{\mathrm{P}} \sim 1.491.$$

Analog result for a uniform rooted tree with n non-ordered, non-labelled leaves and with at most m children per node (up to a constant c_m)

Application 3: sequences of trees built recursively

- Rémy's algorithm (85): $T_{\mathrm{R}}(n)$ rooted, binary, n labelled leaves $n \geq 1$

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start with $T_{\mathrm{R}}(1)$:

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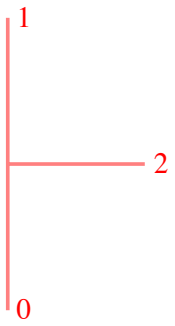

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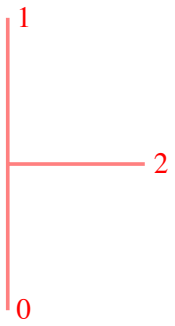

at each step :
- choose a leaf at random
- attach in its "middle" a new edge

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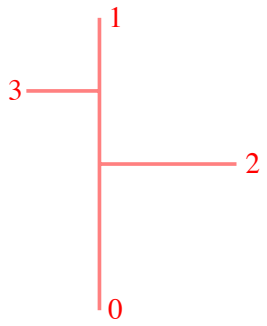

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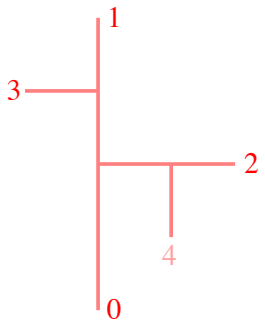

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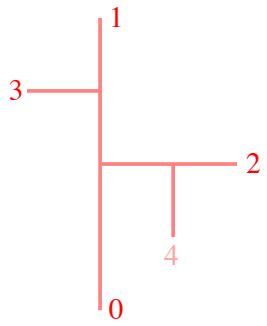

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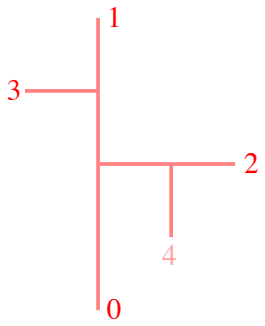

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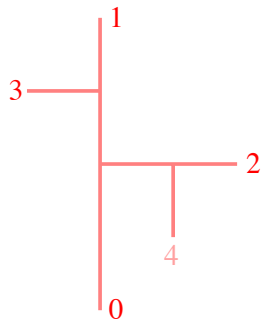

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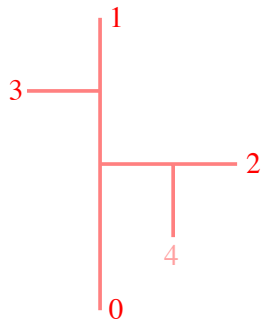

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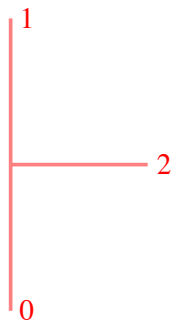

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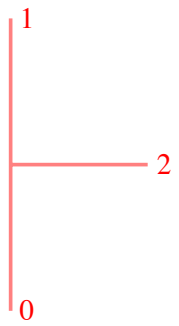

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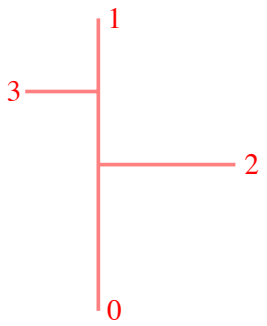

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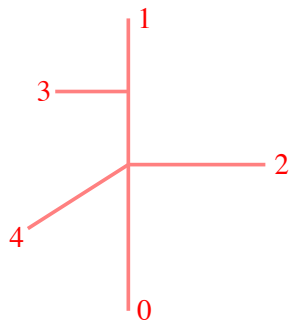

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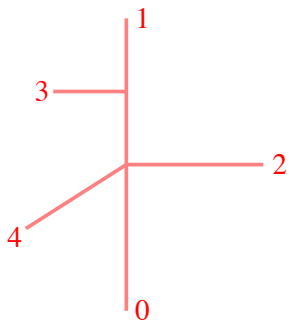

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CURIEN- H. 12: almost sure cv.

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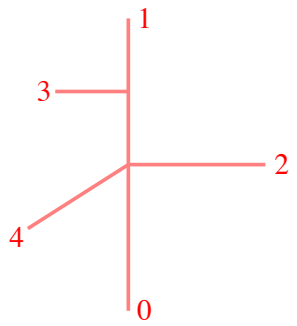

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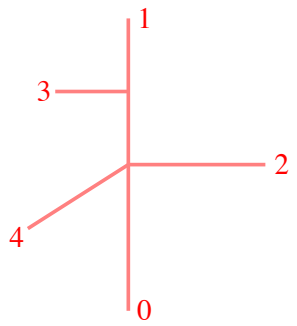

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• k-ary trees (work in progress with R. Stephenson): at each step, add $k-1$ edges (instead of 1) in Rémy's algorithm $\rightarrow$ $T_k(n)$ rooted, 1+(n-1)(k-1) leaves

$$\frac{T_k(n)}{n^{1/k}} \xrightarrow{\text{Proba}} \mathcal{T}_{1/k,\nu_k}$$

where $\nu_k(\mathrm{d}\mathbf{s}) = \frac{(k-1)!}{k(\Gamma(\frac{1}{k}))^{(k-1)}} \prod_{i=1}^k s_i^{-(1-1/k)} \left(\sum_{i=1}^k \frac{1}{1-s_i} \right) \mathbf{1}_{\{s_k = 1 - s_1 - \ldots - s_{k-1}\}} \mathrm{d}s_1 \ldots \mathrm{d}s_{k-1}$.