Self-intersection local time for Gaussian processes

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The object of investigation

x is a random process in $\mathbb{R}^2$, $\Delta_k = \{0 \le t_1 \le \ldots \le t_k \le 1\}$

Self-intersection local time for x of multiplicity k

$$T_k^x = \int_{\Delta_k} \prod_{i=1}^{k-1} \delta_0(x(t_{i+1}) - x(t_i)) d\vec{t}$$

Approximation

$$T_{\varepsilon,k}^x = \int_{\Delta_k} \prod_{i=1}^{k-1} f_\varepsilon(x(t_{i+1}) - x(t_i)) d\vec{t},$$

$$f_\varepsilon(x) = \frac{1}{2\pi\varepsilon} e^{-\frac{\|x\|^2}{2\varepsilon}}, \ \varepsilon > 0, \ x \in \mathbb{R}^2$$

History and preliminary results

1. Existence of the self-intersections for Wiener process (A. Dvoretzky, P. Erdos and S. Kakutani, 1954)
2. Renormalization (E.B. Dynkin, 1987, J. Rosen, 1986)
3. Asymptotic expansion of an area of Wiener sausage (Le Gall, 1990)
4. Application of self-intersection local times
   E. Bolthausen Large deviation and interacting random walks // Lectures on probability theory and statistics, 2002, 1-124
   J. Westwater On Edward's model for polymer chains.II. Self-consistent potential // Comm. Math. Phys., 1981, V.79, 1, 53-73

Ito-Wiener expansion

$$T^w_{\varepsilon,k} = \sum_{n=0}^{\infty} \sum_{n_1^1+n_2^1+\ldots+n_1^{k-1}+n_2^{k-1}=n} \int_{\Delta_k} \prod_{j=1}^{k-1} \frac{1}{n_1^j!}\frac{1}{n_2^j!} I^1_{n_1^j}\left(\left(\frac{1_{[s_j;s_{j+1}]}}{\sqrt{s_{j+1}-s_j+\varepsilon}}\right)^{\otimes n_1^j}\right) H_{n_1^j}(0)$$

$$I^2_{n_2^j}\left(\left(\frac{1_{[s_j;s_{j+1}]}}{\sqrt{s_{j+1}-s_j+\varepsilon}}\right)^{\otimes n_2^j}\right) H_{n_2^j}(0) f_{s_{j+1}-s_j+\varepsilon}(0)d\vec{s}.$$

The renormalization can be provided by the subtraction of the first terms which blow up in the Ito–Wiener expansion.

1. P. Imkeller, V. Perez-Abrew, J. Vives Chaos expansions of double intersection local time of Brownian motion in and renormalization // Stoch. Roc. Appl., 1995, V.56,1, 1-34.
2. A. Dorogovtsev, V. Bakun Random mappings and a Generalized additive functional of a Wiener process. – Theory of Probability and its Applications, 2004, V.48, 1, 63–79.

Description of Gaussian processes appropriate for white noise analysis

Continuous in square mean Gaussian process can be represented as follows

$$x(t) = ((g(t), \xi_1), (g(t), \xi_2)),\ \ t \in [0;1]$$

ξ_1, ξ_2 are independent Gaussian white noises in $L_2([0;1])$, $g \in C([0;1], L_2([0;1]))$ (for the fixed $t \in [0;1]$ $g(t)$ is an element of $L_2([0;1])$).

Fourier-Wiener transform

α is square integrable, measurable with respect to white noise (ξ_1, ξ_2) random variable
Fourier-Wiener transform

Definition

$$\mathcal{T}(\alpha)(h_1, h_2) = E\alpha \exp\{(h_1, \xi_1) + (h_2, \xi_2) - \frac{1}{2}(\|h_1\|^2 + \|h_2\|^2)\},$$

$h_1, h_2 \in L_2([0;1])$.

α is uniquely recovered from its Fourier-Wiener transform.

1. T. Hida Brownian motion, Appl.of Math. 11 (Springer–Verlag, Michigan university, 1980), p.325.

Fourier-Wiener transform of self-intersection local time

$$T_k^x = \int_{\Delta_k} \prod_{i=1}^{k-1} \delta_0(x(t_{i+1}) - x(t_i)) d\vec{t}$$

$$\mathcal{T}(T_k^x)(h_1, h_2) =$$

$$= \int_{\Delta_k} \frac{1}{\Gamma_{t_1 \ldots t_k}} e^{-\frac{1}{2}(\|P_{t_1 \ldots t_k} h_1\|^2 + \|P_{t_1 \ldots t_k} h_2\|^2)} d\vec{t}$$

$\Gamma_{t_1 \ldots t_k}$ is the Gram determinant constructed of $\Delta g(t_1), \ldots, \Delta g(t_{k-1})$,
$P_{t_1 \ldots t_k}$ is a projection on $LS\{\Delta g(t_1), \ldots, \Delta g(t_{k-1})\}$.

Renormalization of T_k^x is equivalent to regularization of $\mathcal{T}(T_k^x)(h_1, h_2)$.

Compactly perturbed Wiener processes

S is a compact operator in $L_2([0;1])$ and I is an identity operator.

Definition. The process

$$x(t) = (((I+S)1_{[0;t]}, \xi_1), ((I+S)1_{[0;t]}, \xi_2)), \ t \in [0;1]$$

is said to be the compactly perturbed Wiener process.

Renormalization

$g(t) = (I+S)1_{[0;t]}$, $\widetilde{\Delta g}(t_1), \ldots, \widetilde{\Delta g}(t_{k-1})$ is the orthonormal system,

$M \subset \{1, \ldots, k-1\}, k \geq 2$, P_M is the projection on $LS\{\widetilde{\Delta g}(t_i), i \in M\}$

Theorem. For an arbitrary $h \in L_2([0;1])$ the following integral converges

$$\int_{\Delta_k} \frac{1}{\Gamma_{t_1 \ldots t_k}} \left(\sum_{M \subset \{1,\ldots,k-1\}} (-1)^{|M|} e^{-\frac{1}{2}\|P_M h\|^2} \right) d\vec{t},$$

A. Dorogovtsev, O. Izyumtseva, 2011

1. A. Dorogovtsev, O. Izyumtseva Self-intersection local times for Gaussian processes, Germany: LAP LAMBERT Academic Publishing, 2011, 152 p. (Russian)
2. A. Dorogovtsev, O. Izyumtseva On regularization of the formal Fourier-Wiener transform of the self-intersection local time of a planar Gaussian process // Theory Stoch. Proc., 2011., V.17, 33, 1, 28-38.

Characteristics of singularities of the compact operator

$\varphi(t) = \sup_{b-a\le t} \|SQ_{a,b}\|,\ Q_{a,b}$ is an operator of multiplication on $1_{[a;b]}$ in $L_2([0;1])$
$\varphi(t) \to 0,\ t \to 0$

Example. P_i is a projection on $1_{\Delta_i},\ \Delta_i = \left[\frac{1}{i+1}; \frac{1}{i}\right],\ i = 1, 2, \ldots$
$S = \sum_{i=1}^{\infty} \lambda_i P_i,\ \lambda_i \to 0,\ i \to \infty$
Taking different λ_i one can get various behaviour of φ ($\lambda_i = \frac{1}{i} \implies \varphi(t) \sim \sqrt{t},\ t \to 0$)

Theorem. Let $\varphi(t) = O(\sqrt{t}),\ t \to 0$. Then

1. $$\overline{\lim_{t\to 0}} \frac{x(t)}{\sqrt{2t\ lln\frac{1}{t}}} = 1, \quad a.s.$$

2. $$\overline{\lim_{h\to 0}} \sup_{0\le t\le 1-h} \frac{|x(t+h)-x(t)|}{\sqrt{2h\ln\frac{1}{h}}} = 1, \quad a.s.$$

3. $$P\left\{\sup_{[0;t]}|x(s)| < \sqrt{\frac{1}{lln|t|}}\right\} \sim \frac{4}{\pi}|\ln t|^{-\frac{\pi^2}{8}},\ t\to 0$$

A. Dorogovtsev, O. Izyumtseva, 2011

A. Dorogovtsev, O. Izyumtseva Asymptotic and geometric properties of compactly perturbed Wiener process and self-intersection local time // Communications on Stochastic Analysis Serials Publications, 2013, V.7, 2, 337-348.

Almost uniform distribution on small circle

$\tau_r^x = \inf\{t : \|x(t)\| = r\}$, ($\tau_r^x = 1$ if $\max_{[0;1]} \|x(t)\| < r$)

Theorem. Let $\varphi(t) = o(\frac{1}{(\ln t)^3}), t \to 0$. Then the distribution of random vector $\frac{1}{r}x(\tau_r^x)$ converges weakly to uniform distribution on the unit circle, when $r \to 0$, **A. Dorogovtsev, O. Izyumtseva, 2013**

1. A. Dorogovtsev, O. Izyumtseva Asymptotic and geometric properties of compactly perturbed Wiener process and self-intersection local time // Communications on Stochastic Analysis Serials Publications, 2013, V.7, 2, 337-348.

Integrators

Definition. If there exists the constant $c > 0$ such that the process y for any $n \geq 1$, $a_1, ..., a_n \in \mathbb{R}$ and $0 = t_0 < t_1 < ... < t_n = 1$ satisfies the following relation

$$E\left(\sum_{k=1}^{n-1} a_k(y(t_{k+1}) - y(t_k))\right)^2 \leq c\sum_{k=1}^{n-1} a_k^2 \Delta t_k$$

then the process y is said to be an integrator.

1. A. Dorogovtsev Stochastic integration and one class of Gaussian random processes // Ukrainian mathematical journal, 1998, V.50, 4, 485-495.
2. A. Dorogovtsev Smoothing Problem in Anticipating Scenario // Ukrainian mathematical journal, 2005, V.57, 9, 1424-1441.

Description of integrators

Lemma. The Gaussian process x can be represented as follows

$$x(t) = ((A1_{[0;t]}, \xi_1), (A1_{[0;t]}, \xi_2)), \ \ t \in [0;1]$$

with some linear continuous operator A in $L_2([0;1])$ iff x is an integrator

The comparing of the smoothness of Gaussian integrator and the growth of its approximations of self-intersection local time

$x(t) = (A1_{[0;t]}, \xi),\ t \in [0;1],\ A$ is a continuous linear operator in $L_2([0;1]),\ Q_{a,b}$ is an operator of multiplication on $1_{[a;b]}$, $\varphi(t) = \sup_{b-a\leq t} \|AQ_{a,b}\|$,
$d(s,t) \leq \varphi(|t-s|)\sqrt{|t-s|}$. Put $\omega(\delta) = \varphi(\delta)\sqrt{\delta}\sqrt{\ln\frac{1}{\delta}} + \int_0^\delta \frac{\varphi(u)}{\sqrt{u\ln\frac{1}{u}}}du$.

Theorem (uniform modulus of continuity).

$$\overline{\lim_{\delta\to 0}} \sup_{u,v\in[0;1],|u-v|\leq\delta} \frac{|x(u)-x(v)|}{\omega(\delta)} \leq C \quad a.s.$$

The comparing of the smoothness of Gaussian integrator and the growth of its approximations of self-intersection local time

$\phi(t) := \varphi(t)^2 t, \ (\varphi(t) = \sup_{b-a \leq t} \|AQ_{a,b}\|)$

Lemma. Suppose that there exists $\alpha > 1$ such that $\phi(t) \leq ct^{\alpha}$, then $ET^{x}_{\varepsilon,2} \geq C\ \varepsilon^{\frac{1}{\alpha}-1}$

For more smooth integrator $ET^{x}_{\varepsilon,2}$ has more fast growth!

Thank you!