# Analytic Properties of the Survival Probabilities in Risk Models with Investments and Their Applications

Elena Ragulina

Donetsk National University

Junior Female Researchers in Probability
Berlin and Potsdam
October 10-11, 2013

# Outline

# 1 Analytic Properties of the Survival Probabilities

## 1.1 The Classical Risk Model

All random variables and processes are defined on a filtered probability space $\left(\Omega, \mathcal{F}, \left(\mathcal{F}_t\right)_{t\geq 0}, \mathbf{P}\right)$.

**Assumptions of the classical risk model:**

- an insurance company has an initial surplus $x \geq 0$;
- premiums arrive with constant rate $c > 0$;
- the number of claims $N_t$ in the time interval $[0,t]$ is a homogeneous Poisson process with intensity $\lambda > 0$;
- claim sizes $Y_i$, $i \geq 1$, are nonnegative i.i.d. random variables, independent of $N_t$, with d.f. $F(y) = \mathbf{P}\{Y_i \leq y\}$, $\mathbf{E}Y_i = \mu < \infty$.

**The surplus can be invested in two assets:**

- **Risk-free asset B.** The price of the risk-free asset at time $t$ equals

$$B_t = B_0 e^{rt},$$

where

$B_0 > 0$ is the price at time $t = 0$,

$r > 0$ is a risk-free interest rate.

- **Risky asset S.** The price of the risky asset at time $t$ equals

$$S_t = S_0 e^{r_{st} t + \sum_{i=1}^{N_t^{st}} Y_i^{st}},$$

where

$S_0 > 0$ is the price at time $t = 0$,

$r_{st} > r$,

$N_t^{st}$ is a homogeneous Poisson process with intensity $\lambda_{st} > 0$,

$Y_i^{st}$, $i \geq 1$, are i.i.d. random variables, with d.f. $F_{st}(y) = \mathbf{P}\left\{Y_i^{st} \leq y\right\}$, $0 < F_{st}(0) < 1$, $\mathbf{E}Y_i^{st} = 0$.

Let $\tau_i^{st}$, $i \geq 1$, be the time of the $i$-th jump of $N_t^{st}$.

All random variables and processes above are independent.

At every time $t \geq 0$ the insurance company invests

- part $\alpha \in (0,1]$ of its surplus in the risky asset,
- part $1-\alpha$ of its surplus in the risk-free asset.

Let $\overline{r} = \alpha r_{st} + (1-\alpha) r$.

Let $X_t(x)$ be a surplus of the insurance company at time $t$ if its initial surplus is $x$. The surplus process follows equation

$$X_t(x) = x + \int_0^t \left(\overline{r} X_s(x) + c\right) ds + \alpha \sum_{i=1}^{N_t^{st}} X_{\tau_i^{st}-}(x)\left(e^{Y_i^{st}} - 1\right) - \sum_{i=1}^{N_t} Y_i . \tag{1}$$

**Definition 1.** The infinite-horizon survival probability $\varphi(x)$ is a function of the initial surplus $x$ that is defined as

$$\varphi(x) = \mathbf{P}\left\{X_s(x) \geq 0 \ \ \forall s \geq 0\right\}.$$

**Definition 2.** The finite-horizon survival probability $\varphi(x,t)$ is a function of the initial surplus $x$ and the time horizon $t$ that is defined as

$$\varphi(x,t) = \mathbf{P}\left\{X_s(x) \geq 0 \ \ \forall s \in [0,t]\right\}.$$

**Theorem 1** *(an upper bound for the infinite-horizon survival probability).* If the surplus process follows equation (1), there exist constants $b_1 > 0$ and $b_2 > 0$ such that

$$\varphi(x) \le 1 - b_1 x^{-b_2}$$

when $x$ is large enough.

**Theorem 2** *(a lower bound for the infinite-horizon survival probability).* Let the surplus process follow equation (1), and the set $Z^*$ be defined as

$$Z^* = \left\{ z:\ \mathbf{E}\left(1-\alpha+\alpha e^{Y_1^{st}}\right)^z < +\infty,\ \lambda - \bar{r}z - \lambda_{st}\left(\mathbf{E}\left(1-\alpha+\alpha e^{Y_1^{st}}\right)^z - 1\right) > 0 \right\}.$$

If there exists constant $z_0 \in (0,1] \cap Z^*$ such that

$$\lambda_{st}\left(\mathbf{E}\left(1-\alpha+\alpha e^{Y_1^{st}}\right)^{-z_0} - 1\right) - \bar{r}z_0 < 0,$$

then there exists constant $b_0 > 0$ such that

$$\varphi(x) \ge 1 - b_0 x^{-z_0}$$

for all $x > 0$.

**Theorem 3.** Let the surplus process follow equation (1). Then $\varphi(x)$ is continuous on $[0,+\infty)$. If in addition the conditions of Theorem 2 hold, the d.f. $F(y)$ is the sum of a continuous component $F^c(y)$ and a discrete component $F^d(y)$, and points of increase $y_1, y_2, \ldots$ of $F^d(y)$ are such that $0 < y_1 < y_2 < \ldots < y_n < \ldots$, $\lim_{n\to\infty} y_n = +\infty$ (if the set of such points is countable), $F^d(y_n) - F^d(y_n -) = p_n$, $n \geq 1$, then

1) $\varphi(x)$ has a continuous derivative on $[0, y_1]$, $[y_1, y_2]$, …, $[y_{n-1}, y_n]$, … (in the extreme points we imply one-sided derivatives), a derivative of $\varphi(x)$ does not exist at $y_1$, $y_2$, …, and

$$\varphi'(y_n -) - \varphi'(y_n +) = \frac{\lambda p_n \varphi(0)}{\bar{r} y_n + c} > 0, \quad n \geq 1;$$

2) $\varphi(x)$ satisfies the integro-differential equation

$$(\bar{r}x + c)\varphi'(x) = (\lambda + \lambda_{st})\varphi(x) - \lambda \int_0^x \varphi(x-y)\,dF(y) - \lambda_{st} \int_{-\infty}^{+\infty} \varphi\left(\left((1-\alpha) + \alpha e^y\right)x\right) dF_{st}(y)$$

on $[0,+\infty)$ (at $y_1, y_2, \ldots$ we imply right-hand derivatives of $\varphi(x)$) with boundary condition $\lim_{x\to+\infty} \varphi(x) = 1$;

3) for all $x_0 \geq 0$ we have

$$\sup_{x\in[x_0,+\infty)} \left|\varphi'(x)\right| \leq \frac{\lambda + \lambda_{st}}{\bar{r}x_0 + c}.$$

**Theorem 4.** Let the surplus process follow equation (1). If $Y_i$, $i \geq 1$, have a density function $f(y)$ which is continuous on $[0,+\infty)$, and $Y_i^{st}$, $i \geq 1$, have a density function $f_{st}(y)$ which is continuous on $(-\infty,+\infty)$, then $\varphi(x,t)$ is continuous on $(0,+\infty)\times[0,+\infty)$ as a function of two variables. If $f(y)$ has a derivative $f'(y)$ on $[0,+\infty)$ such that $|f'(y)|$ is integrable and bounded on this interval, and $f_{st}(y)$ has a derivative $f'_{st}(y)$ on $(-\infty,+\infty)$ such that $|f'_{st}(y)|$, $f_{st}(y)e^{-y}$, and $|f'_{st}(y)|e^{-y}$ are integrable and bounded on this interval, then

1) there exist partial derivatives of $\varphi(x,t)$ w.r.t. $x$ and $t$ on $(0,+\infty)\times[0,+\infty)$, which are continuous as functions of two variables;

2) $\varphi(x,t)$ satisfies the partial integro-differential equation

$$\frac{\partial\varphi(x,t)}{\partial t}-(\bar{r}x+c)\frac{\partial\varphi(x,t)}{\partial x}+(\lambda+\lambda_{st})\varphi(x,t)-\lambda\int_0^x\varphi(x-y,t)\,dF(y)-\lambda_{st}\int_{-\infty}^{+\infty}\varphi\left(\left((1-\alpha)+\alpha e^y\right)x,t\right)dF_{st}(y)=0$$

on $(0,+\infty)\times[0,+\infty)$ with boundary conditions $\varphi(x,0)=1$, $\lim\limits_{x\to+\infty}\varphi(x,t)=1$;

3) for all $x_0>0$ and $T>0$ we have

$$\sup_{\substack{x\in[x_0,+\infty),\\ t\in[0,T]}}\left|\frac{\partial\varphi(x,t)}{\partial x}\right|\leq C_0\left(C_1+\frac{C_2}{x_0}\right),$$

where

$$C_0=\begin{cases}\dfrac{1-e^{(\bar{r}-\lambda-\lambda_{st})T}}{\lambda+\lambda_{st}-\bar{r}} & \text{if } \lambda+\lambda_{st}\neq\bar{r},\\ T & \text{if } \lambda+\lambda_{st}=\bar{r},\end{cases}\qquad C_1=\lambda\left(f(0)+\int_0^{+\infty}|f'(y)|\,dy\right),$$

$$C_2=\frac{\lambda_{st}}{\alpha}\left((1-\alpha)\int_{-\infty}^{+\infty}e^{-y}f_{st}(y)\,dy+\int_{-\infty}^{+\infty}\left((1-\alpha)e^{-y}+\alpha\right)|f'_{st}(y)|\,dy\right).$$

## 1.2 The Risk Model with Stochastic Premiums

**In contrast to the classical risk model**

- the number of premiums $N_t^{pr}$ in the time interval $[0,t]$ is a homogeneous Poisson process with intensity $\lambda_{pr} > 0$;
- premium sizes $Y_i^{pr}$, $i \geq 1$, are nonnegative i.i.d. random variables, independent of $N_t^{pr}$, with d.f. $F_{pr}(y) = \mathbf{P}\{Y_i^{pr} \leq y\}$, $\mathbf{E}Y_i^{pr} = \mu_{pr} < \infty$.

The surplus is invested similarly. All random variables and processes in this model are independent.

The surplus process follows equation

$$X_t(x) = x + \bar{r}\int_0^t X_s(x)\,ds + \alpha\sum_{i=1}^{N_t^{st}} X_{\tau_i^{st}-}(x)\left(e^{Y_i^{st}} - 1\right) + \sum_{i=1}^{N_t^{pr}} Y_i^{pr} - \sum_{i=1}^{N_t} Y_i\,. \tag{2}$$

Theorems 1 and 2 are true for this model.

**Theorem 5.** Let the surplus process follow equation (2). Then $\varphi(x)$ is continuous on $(0,+\infty)$. If in addition the conditions of Theorem 2 hold, the d.f. $F(y)$ is the sum of a continuous component $F^c(y)$ and a discrete component $F^d(y)$, and points of increase $y_1, y_2, \ldots$ of $F^d(y)$ are such that $0 < y_1 < y_2 < \ldots < y_n < \ldots$, $\lim_{n\to\infty} y_n = +\infty$ (if the set of such points is countable), $F^d(y_n) - F^d(y_n -) = p_n$, $n \geq 1$, then

1) $\varphi(x)$ is continuous on $[0,+\infty)$ and has a continuous derivative on $(0, y_1]$, $[y_1, y_2]$, …, $[y_{n-1}, y_n]$, … (in the extreme points we imply one-sided derivatives), a derivative of $\varphi(x)$ does not exist at $y_1$, $y_2$, …, and

$$\varphi'(y_n -) - \varphi'(y_n +) = \frac{\lambda p_n \varphi(0)}{\bar{r} y_n} > 0, \quad n \geq 1;$$

2) $\varphi(x)$ satisfies the integro-differential equation

$$\bar{r} x \varphi'(x) = (\lambda_{pr} + \lambda + \lambda_{st})\varphi(x) - \lambda_{pr} \int_0^{+\infty} \varphi(x+y)\, dF_{pr}(y) - \lambda \int_0^x \varphi(x-y)\, dF(y) - \lambda_{st} \int_{-\infty}^{+\infty} \varphi\left(\left((1-\alpha) + \alpha e^y\right) x\right) dF_{st}(y)$$

on $(0,+\infty)$ (at $y_1$, $y_2$, … we imply right-hand derivatives of $\varphi(x)$) with boundary condition $\lim_{x\to+\infty} \varphi(x) = 1$, and its value at $x = 0$ is given by

$$\varphi(0) = \frac{\lambda_{pr}}{\lambda_{pr} + \lambda} \int_0^{+\infty} \varphi(y)\, dF_{pr}(y);$$

3) for all $x_0 > 0$ we have

$$\sup_{x \in [x_0, +\infty)} |\varphi'(x)| \leq \frac{\lambda_{pr} + \lambda + \lambda_{st}}{\bar{r} x_0}.$$

**Theorem 6.** Let the surplus process follow equation (2). If $Y_i^{pr}$ and $Y_i$, $i \geq 1$, have density functions $f_{pr}(y)$ and $f(y)$ correspondingly which are continuous on $[0,+\infty)$, and $Y_i^{st}$, $i \geq 1$, have a density function $f_{st}(y)$ which is continuous on $(-\infty,+\infty)$, then $\varphi(x,t)$ is continuous on $(0,+\infty)\times[0,+\infty)$ as a function of two variables. If $f_{pr}(y)$ and $f(y)$ have derivatives $f'_{pr}(y)$ and $f'(y)$ correspondingly on $[0,+\infty)$ such that $|f'_{pr}(y)|$ and $|f'(y)|$ are integrable and bounded on this interval, and $f_{st}(y)$ has a derivative $f'_{st}(y)$ on $(-\infty,+\infty)$ such that $|f'_{st}(y)|$, $f_{st}(y)e^{-y}$, and $|f'_{st}(y)|e^{-y}$ are integrable and bounded on this interval, then

1) there exist partial derivatives of $\varphi(x,t)$ w.r.t. $x$ and $t$ on $(0,+\infty)\times[0,+\infty)$, which are continuous as functions of two variables;

2) $\varphi(x,t)$ satisfies the partial integro-differential equation

$$\frac{\partial\varphi(x,t)}{\partial t}-\overline{r}x\frac{\partial\varphi(x,t)}{\partial x}+\left(\lambda_{pr}+\lambda+\lambda_{st}\right)\varphi(x,t)-\lambda_{pr}\int_0^{+\infty}\varphi(x+y,t)\,dF_{pr}(y)-$$

$$-\lambda\int_0^{x}\varphi(x-y,t)\,dF(y)-\lambda_{st}\int_{-\infty}^{+\infty}\varphi\left(\left((1-\alpha)+\alpha e^{y}\right)x,t\right)dF_{st}(y)=0$$

on $(0,+\infty)\times[0,+\infty)$ with boundary conditions $\varphi(x,0)=1$, $\lim\limits_{x\to+\infty}\varphi(x,t)=1$;

3) for all $x_0>0$ and $T>0$ we have

$$\sup_{\substack{x\in[x_0,+\infty),\\ t\in[0,T]}}\left|\frac{\partial\varphi(x,t)}{\partial x}\right|\leq C_0'\left(C_1'+\frac{C_2}{x_0}\right),$$

where

$$C_0' = \begin{cases} \dfrac{1 - e^{\left(\overline{r} - \lambda_{pr} - \lambda - \lambda_{st}\right)T}}{\lambda_{pr} + \lambda + \lambda_{st} - \overline{r}} & \text{if} \quad \lambda_{pr} + \lambda + \lambda_{st} \neq \overline{r}, \\ T & \text{if} \quad \lambda_{pr} + \lambda + \lambda_{st} = \overline{r}, \end{cases}$$

$$C_1' = \left|\lambda_{pr} f_{pr}(0) - \lambda f(0)\right| + \lambda_{pr} \int_0^{+\infty} \left| f_{pr}'(y)\right| dy + \lambda \int_0^{+\infty} \left| f'(y)\right| dy,$$

$$C_2 = \frac{\lambda_{st}}{\alpha} \left( (1-\alpha) \int_{-\infty}^{+\infty} e^{-y} f_{st}(y)\, dy + \int_{-\infty}^{+\infty} \left( (1-\alpha) e^{-y} + \alpha \right) \left| f_{st}'(y)\right| dy \right).$$

# 2 Uniform Statistical Estimates for the Survival Probabilities

### The finite-horizon survival probability in the classical risk model

The surplus process follows equation (1).
$\varphi(x,T)$ is defined on $[0,+\infty)$ for any fixed $T>0$.
$x_* > 0$ and $x^* > x_*$ are arbitrary.

Our aim is to derive a formula that relates accuracy and reliability of the uniform approximation of $\varphi(x,T)$ with its statistical estimate for all $x \in \left[x_*, x^*\right]$.

$h>0$ is an arbitrary small enough number such that $h \ll x^* - x_*$.
Let

$$h_1 = h, \qquad x_1 = x_* + h_1,$$

$$h_i = h\frac{C_1 x_0 x_{i-1} + C_2 x_{i-1}}{C_1 x_0 x_{i-1} + C_2 x_0}, \quad x_i = x_{i-1} + h_i, \qquad 2 \le i \le K-1,$$

$$h_K = x^* - x_{K-1},$$

where $K = K\left(x_*, x^*, h, C_1, C_2\right)$ is such that $x_{K-1} < x^* \le x_K$.

We simulate paths of processes $X_t(x_*)$, $X_t(x_i)$, $1 \le i \le K-1$, and $X_t\left(x^*\right)$ on $[0,T]$ $N$ times. In the issue the survival occurs $\nu_{x_*}(N)$, $\nu_{x_i}(N)$, and $\nu_{x^*}(N)$ times correspondingly.

We define $\hat{\varphi}_{h,x_*,x^*}(x,T)$ for $x\in\left[x_*,x^*\right]$ as follows:

$$\hat{\varphi}_{h,x_*,x^*}(x_*,T)=\frac{\nu_{x_*}(N)}{N},\qquad \hat{\varphi}_{h,x_*,x^*}(x_i,T)=\frac{\nu_{x_i}(N)}{N},\ \ i=\overline{1,K-1},\qquad \hat{\varphi}_{h,x_*,x^*}(x^*,T)=\frac{\nu_{x^*}(N)}{N};$$

$$\hat{\varphi}_{h,x_*,x^*}(x,T)=\hat{\varphi}_{h,x_*,x^*}(x_*,T)+\frac{x-x_*}{x_1-x_*}\left(\hat{\varphi}_{h,x_*,x^*}(x_1,T)-\hat{\varphi}_{h,x_*,x^*}(x_*,T)\right),\ \ x\in(x_*,x_1);$$

$$\hat{\varphi}_{h,x_*,x^*}(x,T)=\hat{\varphi}_{h,x_*,x^*}(x_{i-1},T)+\frac{x-x_{i-1}}{x_i-x_{i-1}}\left(\hat{\varphi}_{h,x_*,x^*}(x_i,T)-\hat{\varphi}_{h,x_*,x^*}(x_{i-1},T)\right),\ \ x\in(x_{i-1},x_i),\ \ i=\overline{2,K-1};$$

$$\hat{\varphi}_{h,x_*,x^*}(x,T)=\hat{\varphi}_{h,x_*,x^*}(x_{K-1},T)+\frac{x-x_{K-1}}{x^*-x_{K-1}}\left(\hat{\varphi}_{h,X_0,X}(x^*,T)-\hat{\varphi}_{h,x_*,x^*}(x_{K-1},T)\right),\ \ x\in(x_{K-1},x^*).$$

**Theorem 7.** Let the surplus process follow equation (1), and conditions of Theorem 4 which are necessary for existence of partial derivatives of $\varphi(x,t)$ hold. Then for any $X_0>0$, $X>X_0$, $h\in(0,X-X_0)$ i $\varepsilon>0$ we have

$$\mathbf{P}\left\{\sup_{x\in\left[x_*,x^*\right]}\left|\varphi(x,T)-\hat{\varphi}_{h,x_*,x^*}(x,T)\right|\le\varepsilon+C_0h\left(C_1+\frac{C_2}{x_*}\right)\right\}\ge 1-2(K+1)e^{-2N\varepsilon^2}.$$

# 3 A Problem of Optimal Control by Franchise Amount

In the classical risk model we assume that $c > \lambda\mu$ (the net profit condition).

If the insurance company uses the expected value principle for premium calculation, then the safety loading is defined as $\theta = c/\lambda\mu - 1$.

The insurance company has an opportunity to choose a franchise amount continuously.

A franchise is a provision in the insurance policy whereby the insurer does not pay unless damage exceeds the franchise amount.

Let $d_t$ be a franchise amount at time $t \geq 0$. We assume that

- $0 \leq d_t \leq d_{\max}$, where $d_{\max}$ is a maximum allowed franchise amount, $0 < F_{cl}\left(d_{\max}\right) < 1$;
- every admissible strategy $\left(d_t\right)$ of franchise amount choice is a predictable process w.r.t. the natural filtration generated by $N_t$ and $Y_i$, $1 \leq i \leq N_t$.

Let $\theta > 0$ be constant. The premium rate is given by

$$c\left(d_t\right) = \lambda\left(1+\theta\right)\int_{d_t}^{+\infty} y\, dF\left(y\right).$$

Let $X_t^{(d_t)}(x)$ be a surplus at time $t$ if the initial surplus is $x$ and the strategy $(d_t)$ is used. The surplus process follows equation

$$X_t^{(d_t)}(x) = x + \int_0^t c(d_s)\,ds - \sum_{i=1}^{N_t} Y_i \mathbf{I}\{Y_i > d_{\tau_i}\}, \tag{3}$$

where $\mathbf{I}\{\cdot\}$ is an indicator of event, $\tau_i$ is the time of the $i$-th jump of $N_t$.

$\tau_x^{(d_t)} = \inf\{t \geq 0:\ X_t^{(d_t)}(x) < 0\}$ is the ruin time under the admissible strategy $(d_t)$.
$\varphi^{(d_t)}(x) = \mathbf{P}\{\tau_x^{(d_t)} = \infty\}$ is the corresponding infinite-horizon survival probability.

Our aim is to maximize the survival probability over all admissible strategies $(d_t)$, i.e. to find

$$\varphi^*(x) = \sup_{(d_t)} \varphi^{(d_t)}(x),$$

and show that there exists an optimal strategy $(d_t^*)$ such that $\varphi^*(x) = \varphi^{(d_t^*)}(x)$ for all $x \geq 0$.

The function $\varphi^*(x)$ satisfies the Hamilton-Jacobi-Bellman equation

$$\left(\varphi^*(x)\right)' = \inf_{d \in [0, d_{\max}]} \left( \frac{\left(1 - F(d)\right)\varphi^*(x) - \int_d^{d \vee x} \varphi^*(x-y)\,dF(y)}{(1+\theta)\int_d^{+\infty} y\,dF(y)} \right). \tag{4}$$

**Theorem 8.** *(existence theorem)* If $Y_i$, $i \geq 1$, have a density function $f(y)$, then there exists a solution $V(x)$ of (4) which is nondecreasing and continuously differentiable on $[0,+\infty)$, with $V(0)=\theta/(1+\theta)$, and $\theta/(1+\theta) \leq \lim_{x\to+\infty} V(x) \leq 1$.

**Theorem 9.** (*verification theorem*) Let the surplus process $X_t^{(d_t)}(x)$ be defined by (3), and $V(x)$ be the solution of (4) that satisfies conditions of Theorem 8. Then for any $x \geq 0$ and arbitrary admissible strategy $(d_t)$

$$\varphi^{(d_t)}(x) \leq \frac{V(x)}{\lim_{x\to+\infty} V(x)}, \tag{5}$$

and equality in (5) is attained under the strategy $(d_t^*) = \left(d_t^*\left(X_{t-}^{(d_t^*)}(x)\right)\right)$, where $(d_t^*(x))$ minimizes the right-hand side of (4), i.e.

$$\varphi^*(x) = \varphi^{(d_t^*)}(x) = \frac{V(x)}{\lim_{x\to+\infty} V(x)}.$$

## Exponentially distributed claim sizes

**Theorem 10.** Let the surplus process $X_t^{(d_t)}(x)$ be defined by (3), claim sizes be exponentially distributed with mean $\mu$, and $d_{\max} = \mu$. Then the strategy $d_t(x) \equiv 0$ is not optimal.

**Example.** If claim sizes are exponentially distributed with mean $\mu = 10$, $d_{\max} = \mu$, and $\theta = 0.1$, then

$$\varphi(x) \approx 1 - 0.9090909 e^{-x/110}, \quad x \geq 0,$$

$$\varphi^*(x) \approx \begin{cases} 0.111048767 e^{x/22} & \text{if } x \leq 8.93258, \\ 1 - 0.90382792 e^{-x/110} & \text{if } x > 8.93258, \end{cases}$$

$$d_t^*(x) = \begin{cases} 10 & \text{if } x \leq 8.93258, \\ 0 & \text{if } x > 8.93258. \end{cases}$$

**Thank you for your attention!**