Representation of Gaussian Isotropic Spin Random Fields

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Plan of the talk

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Examples

Example (Isotropic random field on $\mathbb{S}^2$)

A random field T on the sphere $\mathbb{S}^2$ is a collection of random variables $(T_x)_{x\in\mathbb{S}^2}$ indexed by the points of $\mathbb{S}^2$.
Loosely speaking we say that T is **isotropic** if, for each rotation of $\mathbb{S}^2$, the "rotated" field and T have the same law.

Example (Isotropic spin -1 random field on $\mathbb{S}^2$)

A random field T in the tangent bundle of $\mathbb{S}^2$ is a collection of random vectors $(T_x)_{x\in\mathbb{S}^2}$ indexed by the points of $\mathbb{S}^2$ where for each x, T_x belongs to the tangent plane to $\mathbb{S}^2$ at x.
We say that T is **isotropic** if its law is invariant w.r.t. the action of $SO(3)$ on the tangent bundle (more details later...).

Spin random fields and CMB

Why do we study spin random fields?

- We are interested in them from a theoretical point of view.
- They find applications in Cosmology e.g., for the modeling of the Cosmic Microwave Background data.
  Actually the temperature of this radiation is seen as a single realization of an isotropic random field on $\mathbb{S}^2$ and the modeling of its polarization requires the theory of (isotropic) spin random fields.

For more details on this topic: Newman and Penrose (1966), Geller and Marinucci (2011), Malyarenko (2011), Leonenko and Sakhno (2012) e.g.

Random fields on homogeneous spaces of compact groups

$G =$ a compact group, $\mathscr{X} =$ a G-homogeneous space, $\mathscr{B}(\mathscr{X}) =$ the Borel σ-field of $\mathscr{X}$, $(\Omega, \mathscr{F}, \mathbb{P}) =$ a probability space.
Therefore G acts transitively on $\mathscr{X}$ with an action $x \to gx, g \in G$.

Definition

A random field T on the G-homogeneous space $\mathscr{X}$ is a collection of (complex-valued) r.v.'s $(T_x)_{x \in \mathscr{X}}$ on $(\Omega, \mathscr{F}, \mathbb{P})$ such that the map

$$T : \Omega \times \mathscr{X} \longrightarrow \mathbb{C}, \qquad (\omega, x) \mapsto T_x(\omega) \tag{2.1}$$

is $\mathscr{F} \otimes \mathscr{B}(\mathscr{X})$-measurable.

We work with **a.s. square integrable** random fields T on $\mathscr{X}$, i.e. such that the functions $x \mapsto T_x$ belong to $L^2(\mathscr{X})$ a.s.

For such random fields we give the definition of **isotropy**. G acts on $L^2(\mathscr{X})$ with the left regular representation L, i.e. for $g \in G$

$$L_g : L^2(\mathscr{X}) \longrightarrow L^2(\mathscr{X}), \qquad L_g f(\cdot) := f(g^{-1}\cdot) \tag{2.2}$$

If $f \in L^2(\mathscr{X})$, we can consider the r.v.

$$T(f) := \int_{\mathscr{X}} T_x \overline{f(x)}\, dx = \langle T, f \rangle_{L^2(\mathscr{X})}$$

Definition (Isotropic random field on $\mathscr{X}$)

The a.s. square integrable r.f. T on $\mathscr{X}$ is isotropic if the joint laws of the random vectors

$$(T(f_1), \ldots, T(f_m)) \text{ and } (T(L_g f_1), \ldots, T(L_g f_m)) \tag{2.3}$$

coincide for every $g \in G$ and $f_1, f_2, \ldots, f_m \in L^2(\mathscr{X})$.

Isotropy and positive definite functions

Assume that the r.f. T on $\mathscr{X}$ is (i) a.s. square integrable, (ii) isotropic and (iii) second order ($T_x \in L^2(\mathbb{P})$, $\forall x \in \mathscr{X}$).
From now on we fix $x_0 \in \mathscr{X}$ and denote K the isotropy group of x_0, i.e. the subgroup of G formed by the elements fixing x_0 (so that $\mathscr{X} \cong G/K$).
The covariance kernel of T is completely characterized (by isotropy) by the function on G given by

$$\phi(g) := \mathrm{Cov}\,(T_{gx_0}, T_{x_0}),\ g \in G \tag{2.4}$$

Remark

Under the above assumption (i)-(ii)-(iii) on T, ϕ is

- *positive definite;*
- *continuous, as T is mean square continuous (*Marinucci and Peccati (2013)*);*
- *bi-K-invariant, i.e. $\phi(k_1 g k_2) = \phi(g), g \in G, k_1, k_2 \in K$.*

Construction of Gaussian isotropic random fields on $\mathscr{X}$

Let us consider a **white noise** S on $\mathscr{X}$, i.e. an isometry between $L^2(\mathscr{X})$ and $L^2(\mathbb{P})$ which moreover respects the **real** character: if $f \in L^2(\mathscr{X})$ is real, then $S(f) \in L^2(\mathbb{P})$ is a real r.v.

• To each $f \in L^2(\mathscr{X})$ **bi-K-invariant**, we can associate the r.f. T^f on $\mathscr{X}$ defined as follows

$$T^f_x := S(L_g f),\ gx_0 = x\ . \tag{2.5}$$

For a suitable white noise S, T^f is complex-valued Gaussian and isotropic. Its law is completely characterized by the functions on G

$$\phi^f(g) := \mathbb{E}[\, T^f_{gx_0}\, \overline{T^f_{x_0}}\,] = \langle L_g f, f\rangle_{L^2(\mathscr{X})} \tag{2.6}$$

$$\zeta^f(g) := \mathbb{E}[T^f_{gx_0}\, T^f_{x_0}] = \langle L_g f, \overline{f}\,\rangle_{L^2(\mathscr{X})}\ . \tag{2.7}$$

Remark that $\mathbb{E}[T^f_x] = \textit{const}$ as T^f is isotropic.

Can any Gaussian isotropic random field on $\mathscr{X}$ be represented in this way?

The answer is: NO in general.

Theorem

Let T a centered **real** *Gaussian isotropic random field on $\mathscr{X}$. Then there exist a real function $f \in L^2(\mathscr{X})$ bi-K-invariant s.t.*

$$T \overset{law}{=} T^f$$

Sketch of the proof.

Let ϕ the associated positive definite function of T. It is enough to show that there exists a **real** function $f \in L^2(\mathscr{X})$ bi-K-invariant s.t.

$$\phi(g) = \langle L_g f, f \rangle_{L^2(\mathscr{X})}, \ g \in G \tag{2.8}$$

We use the key lemma:
Every continuous and positive definite function on G has its Fourier series converging uniformly on G (Gangolli (1967)). □

The function f is not unique.

Example

P.Lévy' spherical Brownian Motion

Consider the case $\mathscr{X} = \mathbb{S}^n$, $G = SO(n+1)$. Let us choose as a particular instance of the previous representation formula for real Gaussian isotropic r.f.'s

$$f = c1_{H_o}$$

the indicator function of the half sphere centered at the north pole o, where c is some suitable constant. Then $\mathbb{E}[|T_x^f - T_y^f|^2] = d(x,y)$, d being the distance on $\mathbb{S}^n$. The random field W on $\mathbb{S}^n$ defined as

$$W_x := T_x^f - T_o^f, \quad x \in \mathscr{X}$$

is the **P.Lévy' spherical Brownian Motion** (Lévy (1959)). Actually W is centered, $W_o = 0$ and

$$\mathrm{Cov}\,(W_x, W_y) = \tfrac{1}{2}\Big(d(x,o) + d(y,o) - d(x,y)\Big)\ .$$

Spin line bundles on $\mathbb{S}^2$

The sphere $\mathbb{S}^2$ is the homogeneous space of $SO(3)$ and

$$\mathbb{S}^2 \cong SO(3)/K \tag{3.1}$$

where $K \cong SO(2)$ is the isotropy group of the north pole o.
For each $s \in \mathbb{Z}$, let χ_s be the s-th linear character of K: if $k \in K$ is a rotation by an angle γ, then $\chi_s(k) = \exp(is\gamma)$. K acts on $SO(3) \times \mathbb{C}$ by

$$k(g, z) := (gk, \chi_s(k^{-1})z) \tag{3.2}$$

Denote $\theta(g, z)$ the orbit of (g, z) and $\mathscr{E}_s$ the space of the orbits

$$\mathscr{E}_s := \{\theta(g, z) : (g, z) \in SO(3) \times \mathbb{C}\}$$

$SO(3)$ acts on $\mathscr{E}_s$ as $h\theta(g, z) := \theta(hg, z)$, $h \in SO(3)$.

Definition

$$\xi_s = \left(\mathscr{E}_s \xrightarrow{\pi_s} \mathbb{S}^2\right), \qquad \pi_s(\,\theta(g, z)\,) = gK \tag{3.3}$$

ξ_s is the spin $-s$ line bundle (or s-homogeneous line bundle).

Sections of ξ_s and functions on $SO(3)$ of type s

A section u of ξ_s is a map $u : \mathbb{S}^2 \longrightarrow \mathscr{E}_s$ such that $\pi_s \circ u = id_{\mathbb{S}^2}$.

A function $f : SO(3) \to \mathbb{C}$ is of type s if

$$f(gk) = \chi_s(k^{-1}) f(g), \ g \in SO(3), k \in K \tag{3.4}$$

Proposition

Given a function f of type s (3.4), we define the section u of ξ_s by

$$u(x) := \theta(g, f(g)), \ x = gK \tag{3.5}$$

Vice versa for each section u of ξ_s, there exists a unique function f of type s (3.4) such that u has the form (3.5). f is called the **pullback** *of u.*
There exists an **isometry**

$$L^2(\xi_s) \leftrightarrow L^2_s(SO(3))$$

$L^2(\xi_s)$ =the space of square integrable sections of ξ_s, $L^2_s(SO(3))$ =the space of square integrable functions of type s.

The action L of $SO(3)$ on $L^2_s(SO(3))$ given by

$$L_h f(g) := f(h^{-1}g), \qquad h \in SO(3) \tag{3.6}$$

(also called the *representation of* $SO(3)$ *induced by* χ_s), can be **equivalently** realized on $L^2(\xi_s)$ by

$$U_h u(x) = hu(h^{-1}x), \ h \in SO(3) \ . \tag{3.7}$$

We have

$$U_h u(gK) = hu(h^{-1}gK) =$$
$$h\theta(h^{-1}g, f(h^{-1}g)) = \theta(g, f(h^{-1}g)) = \theta(g, L_h f(g))$$

so that, thanks to the uniqueness of the pullback function:

Proposition

If f is the pullback function of the section u then $L_h f$ is the pullback of the section $U_h u$.

Spin random fields

A spin random field is a random section of a spin line bundle on $\mathbb{S}^2$, more precisely:

Definition (Spin random field)

A spin $(-s)$ random field T is a $\mathscr{F} \otimes \mathscr{B}(\mathbb{S}^2)$-measurable map

$$T : \Omega \times \mathbb{S}^2 \longrightarrow \mathscr{E}_s \qquad (\omega, x) \mapsto T_x(\omega) \tag{3.8}$$

such that for each ω, $x \mapsto T_x(\omega)$ is a section of the line bundle ξ_s In other words for each $x \in \mathbb{S}^2$, $T_x \in \pi_s^{-1}(x) =$ the fiber on x $(\cong \mathbb{C})$.

We work with **a.s. square integrable** spin random fields, i.e. whose sample paths $x \mapsto T_x$ belong to the space of square integrable sections $L^2(\xi_s)$ a.s.

The spin r.f. T is said to be second order if $\mathbb{E}[\| T_x \|^2_{p^{-1}(x)}] < +\infty$ for each $x \in \mathbb{S}^2$.

Isotropy for spin random fields

Given an a.s. square integrable spin $(-s)$ random field T and $u \in L^2(\xi_s)$, define the r.v. $T(u) := \langle T, u \rangle_{L^2(\xi_s)}$.

Definition

T is said to be isotropic if for every choice of square integrable sections $u_1, u_2, \dots, u_m$ of ξ_s, the random vectors

$$\big(T(u_1), \dots, T(u_m)\big) \text{ and } \big(T(U_g u_1), \dots, T(U_g u_m)\big) \tag{3.9}$$

have the same law for every $g \in SO(3)$.

- ...what about the mean square continuity for T? The naive approach

$$\lim_{y \to x} \mathbb{E}[\| T_x - T_y \|^2] = 0$$

does not work as $T_x, T_y \in$ different spaces, i.e. the fibers, even if isomorphic.

How can we overcome this difficulty and make easier the study of spin r.f.'s?

The answer is the pullback random field

Let T be a random section of the spin-$(-s)$-line bundle ξ_s and for each ω, let $g \mapsto X_g(\omega)$ the pullback function of type s of the section $x \mapsto T_x(\omega)$. The random field $X = (X_g)_{g \in SO(3)}$ is of type s, i.e.

$$X_{gk} = \chi_s(k^{-1}) X_g, \ g \in SO(3), k \in K \tag{3.10}$$

and satisfy

$$T_x = \theta(g, X_g), \ x = gK \in \mathbb{S}^2 \tag{3.11}$$

We shall call X the **pullback r.f.** of T.

Proposition (T and X are "equivalent")

Let T a random section of ξ_s and X its pullback random field.
a) T is a.s. square integrable if and only if X is a.s. square integrable.
b) T is second order if and only if X is second order.
c) T is isotropic if and only if X is isotropic.

Construction of Gaussian isotropic spin r.f.'s

Let S a white noise on $SO(3)$, i.e. an isometry between $L^2(SO(3))$ and $L^2(\mathbb{P})$.
- To each $f \in L^2(SO(3))$ and **bi-s-associated**, i.e.

$$f(k_1 g k_2) = \chi_s(k_1) f(g) \chi_s(k_2), \ g \in SO(3), k \in K$$

we associate the random field X^f on $SO(3)$ of type s as

$$X^f_g := S(L_g f), \quad g \in SO(3)$$

For a suitable S, X^f is complex-valued Gaussian, isotropic so that its law is completely characterized by the functions on $SO(3)$

$$\phi^f(g) := \mathbb{E}[X^f_g \overline{X^f_e}] = \langle L_g f, f \rangle_{L^2(SO(3))} \tag{3.12}$$

$$\zeta^f(g) := \mathbb{E}[X^f_g X^f_e] = \langle L_g f, \overline{f} \rangle_{L^2(SO(3))} = 0 \tag{3.13}$$

In particular X^f is complex Gaussian by (3.13).

Representation of Gaussian isotropic spin r.f.'s

Theorem

Let T a complex Gaussian isotropic spin $(-s)$ random field and X its pullback random field on $SO(3)$. Then there exists $f \in L^2(SO(3))$ bi-s-associated such that

$$X^f \overset{law}{=} X$$

Sketch of the proof.

Let ϕ the positive definite function associated to X. It is moreover continuous as X is mean square continuous (Marinucci and Peccati (2013)) therefore its Fourier series converges uniformly on $SO(3)$ (Gangolli (1967)). Hence it is possible to prove that there exists a function $f \in L^2(SO(3))$ bi-s-associated s.t.

$$\phi(g) = \langle L_g f, f \rangle_{L^2(SO(3))} \, .$$

□

The function f is not unique.

Thank you for the attention!

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