Stochastic Acceleration

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Workshop Nachwuchsforscherinnen in Stochastik
10-11 October 2013

We study the motion of a particle in a random force field.

$$\ddot{q}(t) = F\left(q(t), t\right) \quad \dot{q}(0) = v_0 \in \mathbb{R}^d \quad q(0) = q_0 \in \mathbb{R}^d,$$

where F is as follow

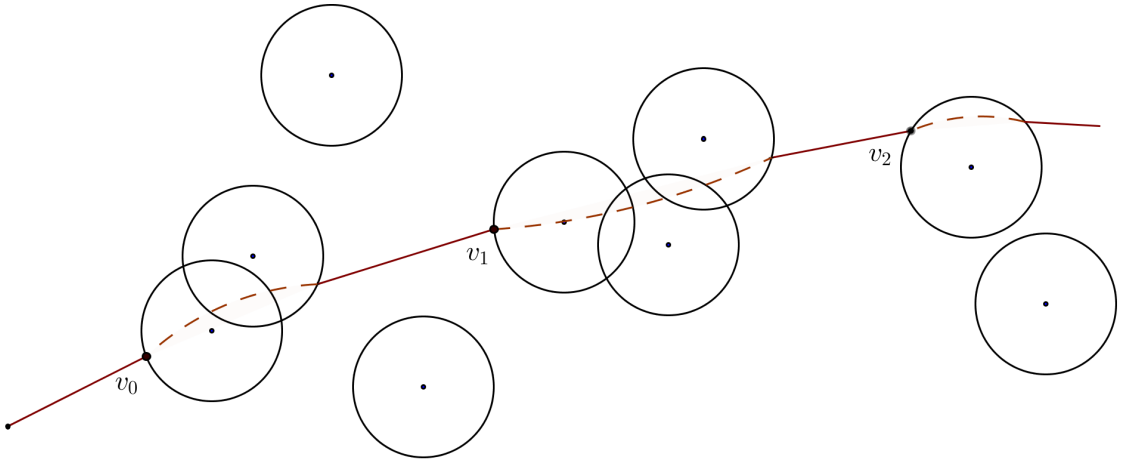
$$F\left(q(t), t\right) = -\sum_{i \geq 1} \lambda_i \, \nabla V\left(q(t) - r_i, t\right).$$

- λ_i a random coefficient (coupling constant),
- $r_i \in \mathbb{R}^d$ form a ponctuel Poisson process,
- V a potential bounded,
- $V \in C_0^\infty\left(B(0,1) \subset \mathbb{R}^d\right)$.

Question:

What is the asymptotic behavior of the velocity?

Conjecture



Conjecture [1] (based on numerical simulations):

Let $d \geq 2$,

$$\lim_{|v_0| \to +\infty} \mathbb{P}\left(\forall t \in \mathbb{R}_+, \, ||\dot{q}(t)|| \sim \left(|v_0| + t^{\frac{1}{5}}\right)\right) = 1.$$

- If the potential V was time-independant, the particule's kinetic energy ($E_k = \frac{1}{2}mv^2$) would be preserved during the scattering events and would be uniformly bounded in time.

[1] B. Aguer, S. De Bièvre, P. Lafitte & P.E. Parris, Journal of Statistical Physics (2010).

Discretization

A trajectory=periods of free motion+scattering events at instants t_n.

- t_n = input time of the $n+1$ scattering event, $v_n = \dot{q}(t_n)$,
- $v_{n+1} = v_n + R(v_n, b_n, \lambda_n)$, $t_{n+1} = t_n + \dfrac{\ell}{||v_n||}$,
- b_n impact parameter (a vector!), $b_n \cdot v_n = 0$.

Where the impulse function is

$$R(v, b, c) = -\lambda \int_0^{+\infty} dt \nabla V(q(t), t),$$

where $q(t)$ is solution

$$\ddot{q}(t) = -\lambda \nabla V(q(t), t),\ \dot{q}(t) = v, \text{ et } q(0) = b - \frac{1}{2}\frac{v}{||v||}.$$

$(v_n)_n$ is a stochastic process on a probability space generated by the (λ_n, r_n). If we ignore the possible recollisions then, $(v_n)_n$ is a Markov chain.

Description of the velocity by a Markov chain

Assuming that recollisions are not possible we consider the simplified model

- $\xi_n = \dfrac{||v_n||^3}{3D}$, $\quad D$ normalize constant,
- $\xi_{n+1} = \xi_n + \omega_n + \frac{\gamma}{\xi_n} + O_0(\xi_n^{-\frac{1}{3}}) + O(\xi_n^{-\frac{4}{3}}), \quad (\xi_n \to +\infty).$

- $D = \mathbb{E}\left(\left(\lambda \int_{-\infty}^{+\infty} dx \partial_t \nabla V(b + x\frac{v}{||v||}, t)\right)^2\right),$
- $\gamma = \dfrac{d-2}{6}$ (depends only on the dimension),
- $(\omega_n)_n$ sequence of independent and identically distributed random variables,
- $\mathbb{E}(\omega_n) = 0$ and $\mathbb{E}\left(\omega_n^2\right) = 1.$

Model

We study the behaviour of the following Markov chain

$$\xi_{k+1} = G(\xi_k, \omega_k), \quad \xi_0 > 0.$$

We cannot describe the particule's dynamic when it has a low velocity, we want to make the fewest assumptions on the behaviour of (ξ_k) when ξ_k is small.

Hypothesis

- Let $G : \mathbb{R}_+^* \times [-M, M] \to \mathbb{R}_+^*$ measurable such there exists $\xi_+ > 0$ and ,

$$G(\xi, \omega) = \xi + \omega + \frac{\gamma}{\xi} + O_0(\xi^{-\frac{1}{3}}) + O(\xi^{-\frac{4}{3}}), \quad \xi > \xi_+.$$

- Let $(\omega_k)_{k \in \mathbb{N}}$ a sequence of i.i.d random variables such that

$$\mathbb{E}(\omega_k) = 0, \quad \mathbb{E}(\omega_k^2) = 1, \quad \exists M > 1, |\omega_k| < M.$$

Principal result

We have the following result on the asymptotic behaviour of $(\xi_k)_k$:

Theorem: S.De Bièvre, E.S

Suppose $\gamma > \frac{1}{2}$. Then for all $\delta > 0$,

$$\lim_{\xi_0 \to +\infty} \mathbb{P}\left(\forall k \in \mathbb{N},\ \left(\xi_0 + \sqrt{k}\right)^{1-\delta} < \xi_k < \left(\xi_0 + \sqrt{k}\right)^{1+\delta}\right) = 1.$$

It's ok with the conjecture

- $||v_k||^3 \sim k^{\frac{1}{2}}$,
- $t_{k+1} = t_k + \dfrac{\ell}{||v_k||}$,
- $t_{k+1} - t_k \sim ||v_k||^{-1} \Rightarrow t_{k+1} - t_k \sim k^{-\frac{1}{6}} \Rightarrow t_k \sim k^{-\frac{5}{6}}$,
- $||\dot{q}(t_k)|| = ||v_k|| \sim t_k^{\frac{1}{5}}$.

Sketch of proof: Averaging

Construction of a continuous process by linear interpolation and change of scalling

- $\varepsilon = \xi_0^{-1} \ll 1$, $t_\ell = \ell\varepsilon^2$,
- $R^\varepsilon(t_\ell) = \dfrac{\xi_\ell}{\xi_0} = \varepsilon\xi_\ell$.

$$R^\varepsilon(t_{\ell+1}) = R^\varepsilon(t_\ell) + \varepsilon\omega_\ell + \varepsilon^2\frac{\gamma}{R^\varepsilon(t_\ell)}, \text{ si } R^\varepsilon(t_\ell) > \varepsilon\xi_+.$$

For all $t \in [t_\ell, t_{\ell+1}]$, $R^\varepsilon(t)$ is defined by linear interpolation between $R^\varepsilon(t_\ell)$ and $R^\varepsilon(t_{\ell+1})$.

Averaging lemma

Suppose $\gamma > 1/2$, then the family of processes $(R^\varepsilon)_{\varepsilon>0}$ converge, as $\varepsilon \to 0$, weakly to a Bessel process of dimension $2\gamma + 1$, and with initial condition 1.

Remark: For $\gamma > \frac{1}{2}$ a Bessel process of dimension $2\gamma + 1$ is transient.

Sketch of proof: Introduction of an auxiliary process

Let 0<M<L and define the intervals $J_\eta = [2^\eta - L, 2^\eta + L]$. For η large enough, the process ξ_k cannot jump across one of these intervals without visiting it.

Define the sequence of stopping time, and the process

- $\tau_0 = 0$, $\tau_1 = \inf\{k > \tau_0 | \xi_k \in J_{\eta_0+1} \cup J_{\eta_0-1}\}$,
- $\eta_1 = \begin{cases} \eta_0 + 1 & \text{si } \xi_{\tau_1} \in J_{\eta_0+1}, \\ \eta_0 - 1 & \text{si } \xi_{\tau_1} \in J_{\eta_0-1}. \end{cases}$

Recursively

- $\tau_{\ell+1} = \inf\{k > \tau_\ell | \xi_k \in J_{\eta_\ell+1} \cup J_{\eta_\ell-1}\}$,
- $\eta_{\ell+1} = \begin{cases} \eta_\ell + 1 & \text{si } \xi_{\tau_{\ell+1}} \in J_{\eta_\ell+1}, \\ \eta_\ell - 1 & \text{si } \xi_{\tau_{\ell+1}} \in J_{\eta_\ell-1}. \end{cases}$

Sketch of proof: Auxiliary Process

The proof lies on these points

- $\xi_{\tau_\ell} \sim 2^{\eta_\ell}$ (by construction of $(\tau_\ell)_{\ell\in\mathbb{N}}$ and $(\eta_\ell)_{\ell\in\mathbb{N}}$),
- $\tau_\ell \sim 2^{2\eta_\ell}$, and $\eta_\ell \sim \nu\ell$, $\nu > 0$.

Control $\tau_{\ell+1} - \tau_\ell$ in order to expand at all $k \in \mathbb{N}$.

Lemma

i) Suppose $\gamma > \frac{1}{2}$. **There exists** $\boldsymbol{p_+ > \frac{1}{2}}$ such that for all $\delta > 0$ there exists $\eta_* > \eta_+$ such that for all $\ell \in \mathbb{N}$ and for all $\eta_0, \eta_1, \cdots, \eta_\ell > \eta_*$

$$|\mathbb{P}(\eta_{\ell+1} = \eta_\ell + 1 | \eta_\ell, \cdots, \eta_0) - p_+| < \delta.$$

ii) For all $0 < p < 1$ and for all $\varepsilon > 0$, there exists $\eta_* > \eta_+$ such that **for all** $\boldsymbol{\eta_0 > \eta_*}$,

$$\mathbb{P}(|\eta_\ell - (\nu\ell + \eta_0)| \leq \varepsilon(\ell + \eta_0), \forall \ell \in \mathbb{N}) > 1 - p, \qquad \text{where } \nu = 2p_+ - 1.$$

- Porte-Manteau theorem and Burkholder-Davis-Gundy inequality.

Lemma

Suppose $\gamma > \frac{1}{2}$.

i) There exist $\eta_* > \eta_+$ and $\mathbf{0} < \boldsymbol{q_1} < \mathbf{1}$, such that for all $m \in \mathbb{N}_*$ and for all $\ell \in \mathbb{N}$,

$$\sup_{\eta > \eta_*} \sup_{n \in K_\eta} \mathbb{P}\left(\tau_{\ell+1} - \tau_\ell > m(2^\eta)^2 | \xi_{\tau_\ell} = n\right) < q_1^m.$$

ii) There exist $\eta_* > \eta_+$ and $\mathbf{0} < \boldsymbol{q_2} < \mathbf{1}$, such that for all $\ell \in \mathbb{N}$

$$\sup_{\eta > \eta_*} \sup_{n \in K_\eta} \mathbb{P}\left(\tau_{\ell+1} - \tau_\ell < (2^\eta)^2 | \xi_{\tau_\ell} = n\right) < q_2.$$

- Porte-Manteau Theorem and

$$0 < \mathbb{P}\left(\exists t \in I | \, R(t) \notin]\frac{1}{2}, 2[\right) < 1,$$

where I is a finite time-interval.

Référence

[1] D.Dolgopyat & L.Koralov, Motion in a Random Force Field, Nonlinearity, 2009,

[2] B.Aguer & S.De Bièvre & P.Lafitte & P.E.Parris, Classical motion in force fields with short range correlations, J. Stat. Phys, 2010.