On particle systems with cooperative branching and coalescence

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Joint work with Jan Swart (UTIA Prague)

Junior female researchers workshop
October 10, 2013

Outline

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The model

The **cooperative branching coalescent** is a continuous-time Markov process $X = (X_t)_{t \geq 0}$ with state space $\{0,1\}^{\mathbb{Z}}$ and transition rates

If $x(i) = 1$	$\big(x(i), x(i+1)\big)$	$\mapsto (0,1)$	at rate $\frac{1}{2}$
	$\big(x(i-1), x(i)\big)$	$\mapsto (1,0)$	at rate $\frac{1}{2}$
If $\big(x(i), x(i+1)\big) = (1,1)$	$x(i+2)$	$\mapsto 1$	at rate $\frac{1}{2}\lambda$
	$x(i-1)$	$\mapsto 1$	at rate $\frac{1}{2}\lambda$

- **Symmetric random walk with coalescence:** particles merge
- **Pairs of particles produce a new particle:** particle is placed on a neighbouring site at cooperative branching rate λ

The model - set-valued

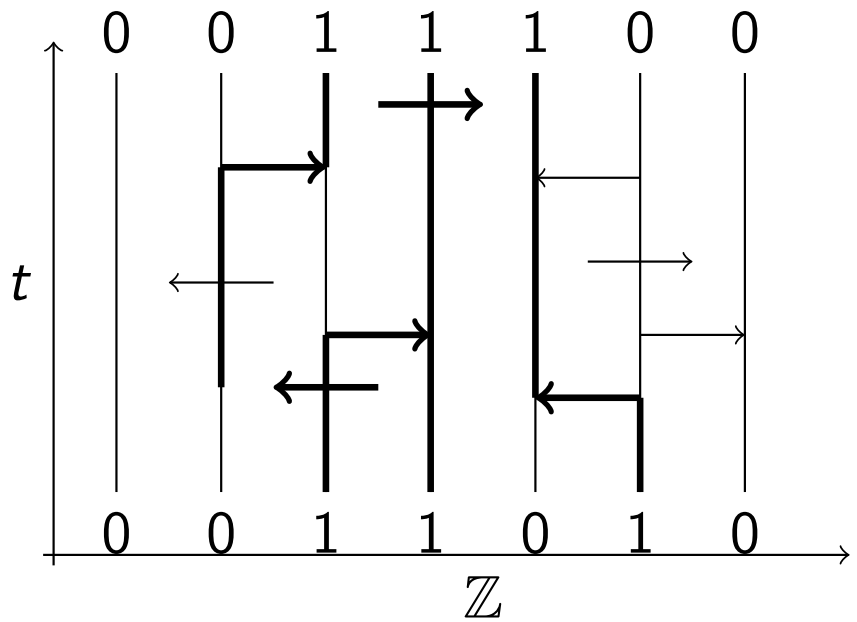
Equivalent description as a process $\eta = (\eta_t)_{t \geq 0}$ with state space $\mathcal{P}(\mathbb{Z})$, the subsets of $\mathbb{Z}$:

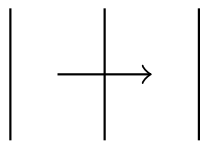
$$\eta_t := \{i \in \mathbb{Z} : X_t(i) = 1\} \qquad t \geq 0$$

Indicate initial state with superscript:

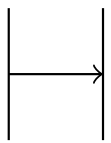
$$\begin{array}{lll} \eta^A & \text{has} & \eta_0^A = A \\ X^x & \text{has} & X_0^x = x \end{array}$$

The graphical representation




$t \in \vec{\omega}(i)$

cooperative branching


$t \in \vec{\omega}(i - \frac{1}{2})$

coalescing jump

For $i \in \mathbb{Z}$

$$\vec{\omega}(i), \overleftarrow{\omega}(i) \text{ as well as } \vec{\omega}(i - \tfrac{1}{2}), \overleftarrow{\omega}(i - \tfrac{1}{2})$$

are Poisson processes with rate $\frac{1}{2}\lambda$ and $\frac{1}{2}$.

The motivation

As a model in the biological context:

1. **Pair reproduction with migration and competition:**
   "1" is a site occupied by an individual, "0" is an empty site
   Cooperative branching: pairs of individuals reproduce
   Coalescing random walk: death due to competition
2. **Interface model of a multi type voter model:**
   "1" is an interface between "types"
   Cooperative branching: singletons give birth to a new type
   Coalescing random walk: voter dynamics and disappearance of types

As a mathematical toy model:

Tractable one dimensional model with interesting properties.

Related models

Blath and Kurt '11: ccDBARW
cooperative caring double branching annihilating random walk:

- double branching of pairs of particles
- annihilating random walk

Sturm and Swart '08: Voter models with selection
voter model in which rare types have an advantage:
dual/interface is double branching annihilating random walk
$\rightarrow$**Neuhauser and Pacalla '99**

Outline

Useful basic properties

- **Monotonicity**
  If $X_0 \leq X_0'$ then the processes can be coupled such that

$$X_t \leq X_t' \text{ for all } t \geq 0.$$

- **Subadditivity**

$$\eta_t^A \cup \eta_t^B \subset \eta_t^{A \cup B} \qquad t \geq 0,\ A, B \subset \mathbb{Z}.$$

Useful basic properties

- From monotonicity

$$\mathbb{P}\big[\eta_t^{\mathbb{Z}} \in \cdot\,\big] \underset{t\to\infty}{\Longrightarrow} \overline{\nu},$$

where $\overline{\nu}$ is called the **upper invariant law.**
Probability under $\overline{\nu}$ of finding a particle in the origin:

$$\theta(\lambda) := \int \overline{\nu}_\lambda(\mathrm{d}A)1_{\{0\in A\}}$$

$\overline{\nu}_\lambda$ is **nontrivial** if $\theta(\lambda) > 0$.

- Survival probability of pairs:

$$\psi(\lambda) := \mathbb{P}\big[|\eta_t^{\{0,1\}}| \geq 2 \ \forall t \geq 0\big]$$

The process **survives** if $\psi(\lambda) > 0$.

Existence of phase transitions

There exist phase transitions for the triviality/nontriviality of the upper invariant law as well for survival/extinction.

Theorem 1: Phase transitions for upper invariant law and survival

(a) There exists a $1 \leq \lambda_c < \infty$ such that
$\overline{\nu}_\lambda = \delta_\emptyset$ for $\lambda < \lambda_c$ but $\overline{\nu}_\lambda$ is nontrivial for $\lambda > \lambda_c$.

(b) There exists a $1 \leq \lambda'_c < \infty$ such that
the process dies out for $\lambda < \lambda'_c$ and survives for $\lambda > \lambda'_c$.

Decay rate in the subcritical regime

Theorem 2: Decay rates of the density and survival probability

(a) There exists a constant $c > 0$ such that for all $\lambda \geq 0$,

$$\mathbb{P}\big[|\eta_t^{\{0,1\}}| \geq 2\big] \geq ct^{-1/2} \text{ and } \mathbb{P}[0 \in \eta_t^{\mathbb{Z}}] \geq ct^{-1/2} \qquad t \geq 0.$$

(b) Moreover, there exists a constant $C < \infty$ such that for each $0 \leq \lambda \leq \frac{1}{2}$,

$$\mathbb{P}\big[|\eta_t^{\{0,1\}}| \geq 2\big] \leq Ct^{-1/2} \text{ and } \mathbb{P}[0 \in \eta_t^{\mathbb{Z}}] \leq Ct^{-1/2} \qquad t \geq 0.$$

Note: $\frac{1}{2} \leq \lambda_c, \lambda_c'$ (subcritical regime)

Outline

Theorem 1: Existence of phase transitions

Monotonicity implies the existence of λ_c and λ_c' if we can show

(a) $\overline{\nu} = \delta_\emptyset$ for $\lambda \leq 1$ and $\overline{\nu} \neq \delta_\emptyset$ for large λ.

(b) The process dies out for $\lambda \leq 1$ and survives for large λ.

Theorem 1: Existence of phase transitions

Triviality of the upper invariant law for $\lambda \leq 1$: $\overline{\nu} = \delta_\emptyset$

If $\lambda > 0$ and the process is started translation invariant let
$p_t(1) = \mathbb{P}(X_t(i) = 1), \quad p_t(11) = \mathbb{P}(X_t(i) = 1, X_t(i+1) = 1), \ldots$

$$\begin{aligned}\frac{\partial}{\partial t} p_t(1) &= -p_t(1) + \tfrac{1}{2} p_t(10) + \tfrac{1}{2} p_t(01) + \tfrac{1}{2}\lambda p_t(110) + \tfrac{1}{2}\lambda p_t(011) \\ &= -p_t(11) + \lambda\big(p_t(11) - p_t(111)\big) \\ &= (\lambda - 1) p_t(11) - \lambda p_t(111),\end{aligned}$$

If the process is furthermore started from an invariant law

$$0 = \frac{\partial}{\partial t} p_t(1) \leq -\lambda p_t(111) \Rightarrow p_t(111) = 0.$$

As $p_t(1) > 0$ would imply $p_t(111) > 0$ we are done.
(Case $\lambda = 0$ similar.)

Theorem 1: Existence of phase transitions

Extinction for $\lambda \leq 1 : \mathbb{P}\big[\exists T < \infty \text{ s.t. } |\eta_t^A| = 1 \ \forall t \geq T\big] = 1.$

With similar calculations

$$\frac{\partial}{\partial t}\mathbb{E}\big[|\eta_t^A|\big] = (\lambda - 1)\sum_{i\in\mathbb{Z}} \mathbb{P}[\{i, i+1\} \subset \eta_t^A] - \lambda \sum_{i\in\mathbb{Z}} \mathbb{P}[\{i, i+1, i+2\} \subset \eta_t^A]$$

So $|\eta_t^A|$ is a supermartingale for $\lambda \leq 1 : |\eta_t^A| \underset{t\to\infty}{\longrightarrow} N \quad \text{a.s.}$

Let $\rho(A) = \mathbb{P}[\exists t \geq 0 \text{ s.t. } |\eta_{t-}^A| \neq |\eta_t^A|]$.
Then one can show $\lim_{T\to\infty} \rho(\eta_T^A) = 0$ a.s.

The claim follows as ρ is uniformly bounded below on $\{A : |A| \geq 2\}$.

Theorem 1: Existence of phase transitions

Nontrivial upper invariant law and survival for large λ :
Couple pairs of particles with a contact process variant

The contact process with double deaths $\zeta = (\zeta_t)_{t\geq 0}$

- Sites infect any neighbor at rate $\frac{1}{2}\lambda$.
- Any particles on two neighboring sites die at rate 1.

Graphical representation with Poisson processes:

$$\overleftarrow{\pi}(i-\frac{1}{2}), \vec{\pi}(i-\frac{1}{2}), \quad \text{and } \pi^*(i-\frac{1}{2}), \quad i \in \mathbb{Z}.$$

Theorem 1: Existence of phase transitions

Nontrivial upper invariant law and survival for large λ :

Comparison of η with the contact process with double deaths ζ

Let

$$\eta_t^{(2)} := \{i \in \mathbb{Z} : \{i, i+1\} \subset \eta_t\} \qquad t \geq 0$$

denote the set of locations where η_t contains a pair of neighboring particles. Then $(\eta_t^{(2)})_{t\geq 0}$ and $(\zeta_t)_{t\geq 0}$ can be coupled such that

$$\zeta_0 \subset \eta_0^{(2)} \quad \text{implies} \quad \zeta_t \subset \eta_t^{(2)} \qquad t \geq 0.$$

Coupling:

$$\overleftarrow{\pi}(i-\frac{1}{2}) := \overleftarrow{\omega}(i), \quad \vec{\pi}(i-\frac{1}{2}) := \vec{\omega}(i), \quad \pi^*(i-\frac{1}{2}) := \overleftarrow{\omega}(i-\frac{1}{2}) \cup \vec{\omega}(i+\frac{1}{2})$$

Theorem 1: Existence of phase transitions

Nontrivial upper invariant law and survival for large λ :
Comparison with oriented percolation $W = (W_n)_{n\in\mathbb{N}}$

Let $\mathbb{Z}^2_{\text{even}} := \{(i,n) \in \mathbb{Z}^2 : i+n \text{ is even}\}$ with directed edges e_z^- and e_z^+ point from (i,n) to $(i-1,n+1)$ and $(i+1,n+1)$. Arrows are independently open with probability p.

W_n contains the infected sites at time n when infections follow open arrows.

Comparison with oriented percolation

For each $p < 1$, there exists $\lambda', T > 0$ such that for all $\lambda \geq \lambda'$, the processes $(\zeta_t)_{t\geq 0}$ and $(W_n)_{n\geq 0}$ with parameters λ and p can be coupled in such a way that

$$W_0 \subset \zeta_0 \quad \text{implies} \quad W_n \subset \zeta_{nT}, \qquad n \geq 0.$$

Theorem 1: Existence of phase transitions

Nontrivial upper invariant law and survival for large λ :
Coupling of ζ with W

Consider the graphical representation for ζ with rescaled time:

$$\overleftarrow{\pi}, \overrightarrow{\pi} \text{ at rate } \frac{1}{2}, \qquad \pi^* \text{ at rate } \lambda^{-1}$$

By choosing T large and then λ large the probability of the event

- infection from i to $i+1$ in the time interval $(nT, (n+1)T]$
- no double deaths affecting i or $i+1$ in $(nT, (n+1)T]$

can be made arbitrarily large (p' close to 1).

$\Rightarrow$ m-dependent open arrows for W
$\Rightarrow$ independent open arrows (with slightly smaller $p(p')$) for W

Theorem 1: Existence of phase transitions

Nontrivial upper invariant law and survival for large λ :
Completion of the proof:

Well known fact:

For large enough p the oriented percolation process has a nontrivial upper invariant law and survives.

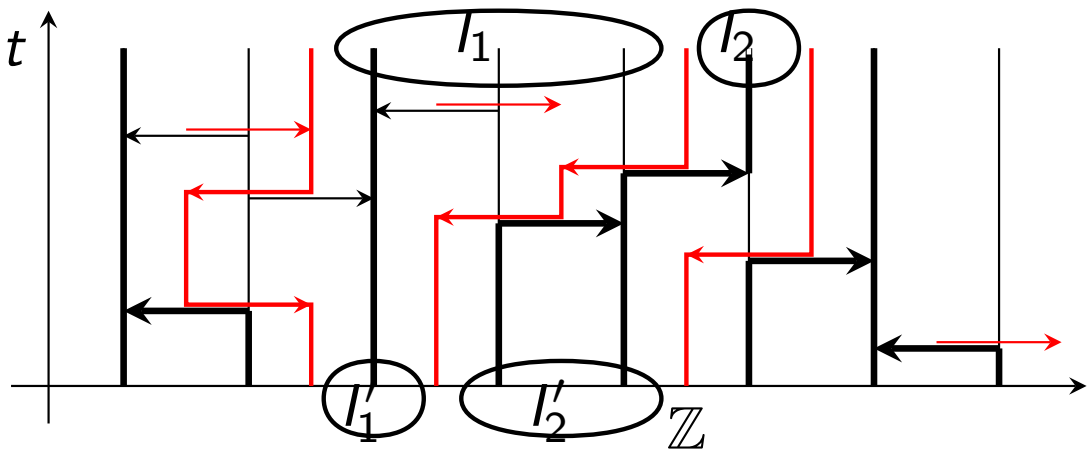
Theorem 2: Decay of the survival probability and density

Lower bound

Coalescing random walk duality to $\hat{\xi}$ for $\lambda = 0$:

For $\langle i, j\rangle := \{i + \frac{1}{2}, \ldots, j - \frac{1}{2}\}$ one has

$$\eta_t^A \cap \langle i, j\rangle \neq \emptyset \quad \text{if and only if} \quad A \cap \langle \hat{\xi}_0^{(i,t)}, \hat{\xi}_0^{(j,t)}\rangle \neq \emptyset \quad \text{a.s.}$$



In the example: $\eta_t \cap I_1 \neq \emptyset$ if and only if $\eta_0 \cap I_1' \neq \emptyset$, and $\eta_t \cap I_2 \neq \emptyset$ if and only if $\eta_0 \cap I_2' \neq \emptyset$.

Theorem 2: Decay of the survival probability and density

Lower bound

With $\tau_{\langle 2\rangle} = \inf\{t \geq 0 : \xi_t^{(i,0)} = \xi_t^{(i+1,0)}\}$ we have

$$p_t(1) = \mathbb{P}\big[i \in \eta_t^{\mathbb{Z}}\big] = \mathbb{P}\big[\hat{\xi}_0^{(i-\frac{1}{2},t)} < \hat{\xi}_0^{(i+\frac{1}{2},t)}\big] = \mathbb{P}[t < \tau_{\langle 2\rangle}].$$

We also naturally have

$$\mathbb{P}\big[|\eta_t^{\{0,1\}}| \geq 2\big] = \mathbb{P}[t \leq \tau_{\langle 2\rangle}].$$

Since one can show that $\mathbb{P}[t < \tau_{\langle 2\rangle}] \sim \frac{1}{\sqrt{\pi}} t^{-1/2}$ this provides the lower bound.

Theorem 2: Decay of the survival probability and density

Upper bound A pathwise superdual

- Define open dual paths as before.
- Define dual 3-paths $\gamma^k : [s, u) \to \mathbb{Z} + \frac{1}{2}$ $(k = 1, 2, 3)$ with

$$\gamma^1 < \gamma^2 < \gamma^3.$$

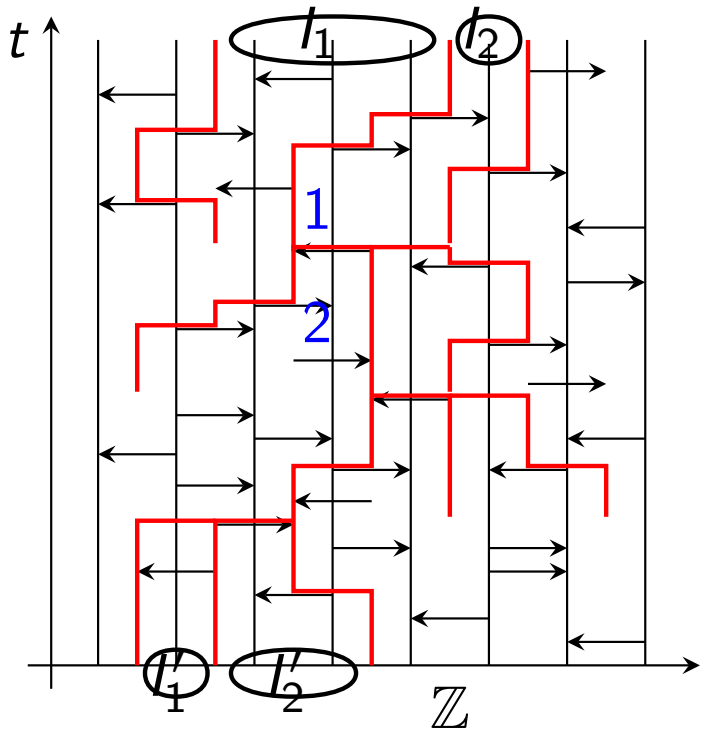
- At cooperative branching events the three path *may* **renew** itself: If either γ^2 or γ^3 (but not γ^1) hits the head of a cooperative branching arrow pointing to the left ($\overleftarrow{\omega}(i)$ for some $i \in \mathbb{Z}$), then start anew from the positions

$$i - \frac{1}{2}, i - \frac{1}{2} + 1, \text{ and } i - \frac{1}{2} + 2.$$

- A similar rule applies for cooperative branching arrows pointing to the right ($\overrightarrow{\omega}$).

Theorem 2: Decay of the survival probability and density

Upper bound A pathwise superdual



Superduality: If $\eta_t \cap I_1 \neq \emptyset$ and $\eta_t \cap I_2 \neq \emptyset$, then there must exist a backward 3-path as drawn such that $\eta_0 \cap I'_1 \neq \emptyset$ and $\eta_0 \cap I'_2 \neq \emptyset$. Cooperative branching arrows *may* ("1") but do not have to be used ("2") for renewal.

Theorem 2: Decay of the survival probability and density

Upper bound A pathwise superdual

Dual 3-paths

Let $0 \leq s < u$ and let (I_1, I_2) be a pair of adjacent intervals in $\mathbb{Z}$. Then, a.s. on the event

$$\eta_u \cap I_1 \neq \emptyset \quad \eta_u \cap I_2 \neq \emptyset,$$

there exist adjacent intervals (I_1', I_2') and a dual 3-path $(\gamma_t^1, \gamma_t^2, \gamma_t^3)_{t\in[s,u]}$ with

$$(I_1, I_2) = \left(\langle \gamma_u^1, \gamma_u^2\rangle, \langle \gamma_u^2, \gamma_u^3\rangle\right) \quad (I_1', I_2') = \left(\langle \gamma_s^1, \gamma_s^2\rangle, \langle \gamma_s^2, \gamma_s^3\rangle\right),$$

such that

$$\eta_s \cap I_1' \neq \emptyset \quad \eta_s \cap I_2' \neq \emptyset.$$

Theorem 2: Decay of the survival probability and density

Upper bound A pathwise superdual

$N_t :=$ the number of distinct dual 3-paths $(\gamma_s^1, \gamma_s^2, \gamma_s^3)_{s\in[u-t,u]}$ such that $(\gamma_u^1, \gamma_u^2, \gamma_u^3) = (-\frac{1}{2}, \frac{1}{2}, \frac{3}{2})$.

Expected number of dual 3-paths

We have for $\lambda < \frac{1}{2}$

$$\mathbb{E}[N_t] \leq K\Big(\sum_{n=1}^{\infty}(2\lambda)^n n^{5/2}\Big)t^{-3/2} \qquad t \geq 0.$$

This implies that for $\lambda < \frac{1}{2}$

$$\mathbb{P}\big[\{0,1\} \subset \eta_t\big] \leq \mathbb{P}[N_t > 0] \leq \mathbb{E}[N_t] \leq K' t^{-3/2}$$

$$\mathbb{E}\big[\big|\{i \in \mathbb{Z} : \{i, i+1\} \subset \eta_t\}\big|\big] \leq \mathbb{E}[N_t] \leq K' t^{-3/2}$$

Theorem 2: Decay of the survival probability and density

Upper bound A pathwise superdual
Particle density

$$\begin{aligned}
\mathbb{P}[0 \in \eta_t^{\mathbb{Z}}] = p_t(1) &= -\int_t^\infty \frac{\partial}{\partial s}\, p_s(1)\mathrm{d}s \\
&= \int_t^\infty \big((1-\lambda)p_s(11) + \lambda p_s(111)\big)\mathrm{d}s \\
&\leq \int_t^\infty p_s(11)\,\mathrm{d}s \\
&\leq K' \int_t^\infty s^{-3/2}\mathrm{d}s \;= 2K' t^{-1/2}.
\end{aligned}$$

Theorem 2: Decay of the survival probability and density

Upper bound A pathwise superdual

Survival probability

Since $\{|\eta_t^{\{0,1\}}| \geq 2\}$ decreases at rate 1 whenever $\eta_t^{\{0,1\}} = \{i, i+1\}$ for some $i \in \mathbb{Z}$ due to coalescence:

$$\begin{aligned} -\tfrac{\partial}{\partial t}\mathbb{P}\big[|\eta_t^{\{0,1\}}| \geq 2\big] &= \mathbb{P}\big[\eta_t^{\{0,1\}} = \{i, i+1\} \text{ for some } i \in \mathbb{Z}\big] \\ &\leq \mathbb{P}\big[\{i, i+1\} \subset \eta_t^{\{0,1\}} \text{ for some } i \in \mathbb{Z}\big] \\ &\leq K' t^{-3/2}. \end{aligned}$$

Hence,

$$\mathbb{P}\big[|\eta_t^{\{0,1\}}| \geq 2\big] \leq \int_t^\infty \mathrm{d}s\, K' s^{-3/2} = 2K' t^{-1/2} \qquad t \geq 0.$$

Outline

Open questions

- Generalize Theorems to higher dimension.
- Prove equality $\lambda_c = \lambda'_c$ of the critical parameters.
- Extend the statements in for the decay to all $\lambda < \lambda'_c$, respectively $\lambda < \lambda_c$.